Stability in Sobolev and related inequalities

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Outline

The stability issue in critical Sobolev and related inequalities

- **Sobolev and Hardy-Littlewood-Sobolev inequalities**
  Joint work with G. Jankowiak
- **Subcritical interpolation inequalities**
  ▷ On the Euclidean space: *joint work with G. Toscani*
  ▷ On the sphere: *joint work with M.J. Esteban and M. Loss*
- **Reverse HLS inequality**
  ▷ A quick introduction to a new family of inequalities for mean-field diffusion equations

A question by H. Brezis and E. Lieb

(Brezis, Lieb (1985)) *Is there a natural way to bound*

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

from below in terms of the"distance" off from the set of optimal [Aubin-Talenti] functions when $d \geq 3$?

- (Bianchi, Egnell 1990) There is a positive constant α such that

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \alpha \inf_{\varphi \in \mathcal{M}} \|\nabla u - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- (Cianchi, Fusco, Maggi, Pratelli 2009) (also a version for $\|\nabla u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}$) There are constants α and κ such that

$$\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \left(1 + \kappa \, \lambda(u)^\alpha\right) \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

where $\lambda(u) = \inf_{\varphi \in \mathcal{M}} \left\{ \frac{\|u-\varphi\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|u\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} \, : \, \|u\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|\varphi\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right\}$

Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm but with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Improved Sobolev inequality by duality

Theorem

(JD, G. Jankowiak) *Assume that* $d \geq 3$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathcal{C} \leq 1$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1}\, v|^2\, dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\, v\, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1}\, v\, dx = \int_{\mathbb{R}^d} u\, v\, dx = \int_{\mathbb{R}^d} u^{2^*}\, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d\, \|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\, \nabla u - \nabla(-\Delta)^{-1}\, v \right|^2 dx$$

shows that

$$0 \le \mathsf{S}_d\, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] - \left[\mathsf{S}_d\, \|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-1}\, u^q\, dx \right]$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

A preliminary observation

Proposition

(JD) *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}} \left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

(JD) *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

(M. del Pino, M. Saez), (J. L. Vázquez, J. R. Esteban, A. Rodriguez)
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

The function $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the Cauchy-Schwarz inequality, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$

$$\le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\,\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\;, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

$d = 2$: Onofri's and log HLS inequalities

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)dx$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(v/\mu\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition

If $v = \mu\, e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\;dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\;d\mu$$
$$\geq \tfrac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx + \int_{\mathbb{R}^2} u\;d\mu - \log\left(\int_{\mathbb{R}^2} e^u\;d\mu\right) \geq 0$$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem (J.D., G. Jankowiak)

Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$

$$\forall\, u \in \mathcal{D}\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ *for any* $x \ge 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))\ \forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

... but $\mathcal{C} = 1$ is not optimal

Theorem

(JD, G. Jankowiak) *In the inequality*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx$$
$$\le \mathsf{C}_d \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

we have

$$\frac{d}{d+4} \le \mathsf{C}_d < 1$$

based on a (painful) linearization

Extensions:

- fractional Laplacian operator (Jankowiak, Nguyen)
- Moser-Trudinger-Onofri inequality

Subcritical interpolation inequalities

▷ Euclidean space: fast diffusion, entropies and improved asymptotic expansions
Based on papers with A. Blanchet, M. Bonforte, G. Grillo, J.L. Vázquez and papers with G. Toscani

▷ Sphere: explicit remainder terms based on nonlinear diffusions
Joint work with MJ. Esteban and M. Loss

Higher order matching asymptotics

(J.D., G. Toscani) For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{4}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(v - B_\sigma)\right] dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt}\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v]\right)_{|\sigma=\sigma(t)}}_{\text{choose it }=0} + \frac{m}{m-1}\int_{\mathbb{R}^d}\left(v^{m-1} - B_{\sigma(t)}^{m-1}\right)\frac{\partial v}{\partial t}\,dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

The entropy / entropy production estimate

According to the definition of B_σ, we know that
$2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right] B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

Theorem (J.D., G. Toscani)

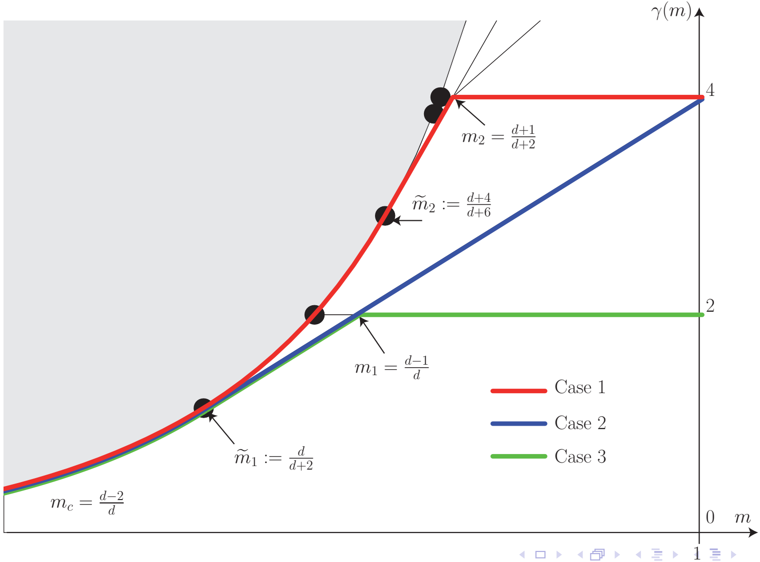
Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[u \left(\sigma^{\frac{d}{2}(m-m_c)} \nabla u^{m-1} - 2\,x \right) \right] = 0 \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, u(x,t)\, dx\,, \quad K_M := \int_{\mathbb{R}^d} |x|^2\, B_1(x)\, dx$$

where

$$B_\lambda(x) := \lambda^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\lambda}\, |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\lambda^m - m\, B_\lambda^{m-1}\, (u - B_\lambda) \right] dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x,t)] = \mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)} \frac{m}{1-m} \int_{\mathbb{R}^d} u \left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2 dx$$

2. In the *Bakry-Emery method*, there is *an additional (good) term*

$$4\left[1 + 2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\,\sigma_0^{\frac{d}{2}\,(1-m)}}\right]\frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \geq \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \geq \frac{m}{8\int_{\mathbb{R}^d} B_1^m\,dx} C_M^2 \|u - B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

improved decay for the relative entropy

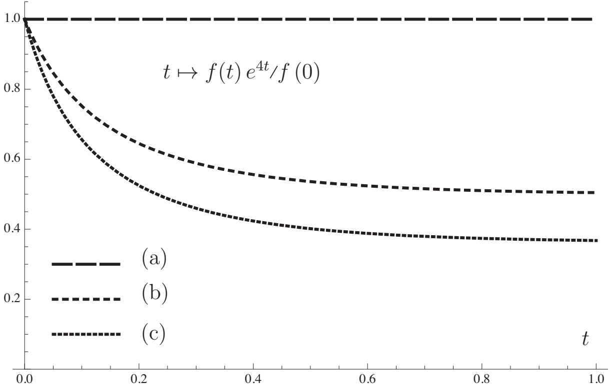


Figure: Upper bounds on the decay of the relative entropy:
$t \mapsto f(t)\, e^{4t}/f(0)$ (a): estimate given by the entropy-entropy production method
(b): exact solution of a simplified equation
(c): numerical solution (found by a shooting method)

A Csiszár-Kullback(-Pinsker) inequality

Let $m \in (\widetilde{m}_1, 1)$ with $\widetilde{m}_1 = \frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m\, B_\sigma^{m-1} \, (u - B_\sigma) \right] \, dx$$

Theorem

Let $d \geq 1$, $m \in (\widetilde{m}_1, 1)$ and assume that u is a nonnegative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2 \, u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{\mathrm{L}^1(\mathbb{R}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2 \, u \, dx = \int_{\mathbb{R}^d} |x|^2 \, B_\sigma \, dx$, then

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m \, dx} \left(C_M \|u - B_\sigma\|_{\mathrm{L}^1(\mathbb{R}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2 \, |u - B_\sigma| \, dx \right)^2$$

An improved Gagliardo-Nirenberg inequality: setting

The inequality

$$\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

with $\theta=\theta(p):=\frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $1<p\le\frac{d}{d-2}$ if $d\ge3$ and $1<p<\infty$ if $d=2$, can be rewritten, in a non-scale invariant form, as

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx+\int_{\mathbb{R}^d}|f|^{p+1}\,dx\ge\mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d}|f|^{2\,p}\,dx\right)^{\gamma}$$

with $\gamma=\gamma(p,d):=\frac{d+2-p\,(d-2)}{d-p\,(d-4)}$. Optimal function are given by

$$f_{M,y,\sigma}(x)=\frac{1}{\sigma^{\frac{d}{2}}}\left(C_M+\frac{|x-y|^2}{\sigma}\right)^{-\frac{1}{p-1}}\quad\forall\,x\in\mathbb{R}^d$$

where C_M is determined by $\int_{\mathbb{R}^d}f^{2\,p}_{M,y,\sigma}\,dx=M$

$$\mathfrak{M}_d:=\left\{f_{M,y,\sigma}\;:\;(M,y,\sigma)\in\mathcal{M}_d:=(0,\infty)\times\mathbb{R}^d\times(0,\infty)\right\}$$

An improved Gagliardo-Nirenberg inequality

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g\in\mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \Big[g^{1-p}\left(|f|^{2p}-g^{2p}\right)-\tfrac{2p}{p+1}\left(|f|^{p+1}-g^{p+1}\right)\Big]\,dx$$

Theorem

Let $d\ge 2$, $p>1$ and assume that $p<d/(d-2)$ if $d\ge 3$. If

$$\frac{\int_{\mathbb{R}^d}|x|^2\,|f|^{2p}\,dx}{\left(\int_{\mathbb{R}^d}|f|^{2p}\,dx\right)^\gamma}=\frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\quad \sigma_*(p):=\left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)}\right)^{\frac{4p}{d-p\,(d-4)}}$$

for any $f\in\mathrm{L}^{p+1}\cap\mathcal{D}^{1,2}(\mathbb{R}^d)$, then we have

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx+\int_{\mathbb{R}^d}|f|^{p+1}\,dx-\mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d}|f|^{2p}\,dx\right)^\gamma\ge\mathsf{C}_{p,d}\,\frac{\left(\mathcal{R}^{(p)}[f]\right)^2}{\left(\int_{\mathbb{R}^d}|f|^{2p}\,dx\right)^\gamma}$$

By Csiszár-Kullback: control of $\big\||f|^{2p}-g^{2p}\big\|_{\mathrm{L}^1(\mathbb{R}^d)}^4$

Best matching Barenblatt profiles are delayed

Let u be such that

$$v(\tau,x)=\frac{\mu^d}{R(D\,\tau)^d}\,u\left(\frac{1}{2}\log R(D\,\tau),\frac{\mu\,x}{R(D\,\tau)}\right)$$

with $\tau\mapsto R(\tau)$ given as the solution to

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left(\frac{\mu^2}{K_M}\int_{\mathbb{R}^d}|x|^2\,v(\tau,x)\,dx\right)^{-\frac{d}{2}(m-m_c)}\,,\quad R(0)=1$$

Then

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left[R^2(\tau)\,\sigma\left(\frac{1}{2}\log R(D\,\tau)\right)\right]^{-\frac{d}{2}(m-m_c)}$$

that is $R(\tau)=R_0(\tau)\le R_0(\tau)$ where $\frac{1}{R}\,\frac{dR_0}{d\tau}=\left(R_0^2(\tau)\,\sigma\,(0)\right)^{-\frac{d}{2}(m-m_c)}$ and asymptotically as $\tau\to\infty$, $R(\tau)=R_0(\tau-\delta)$ for some delay $\delta>0$

The interpolation inequalities on $\mathbb{S}^d$

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \ne 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

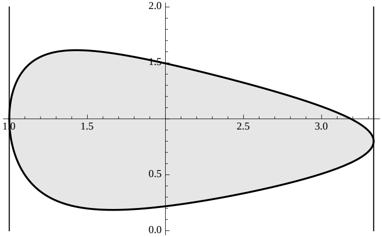
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{5}$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) admissible region, $d = 5$

Improved interpolation inequalities in the sphere

Let

$$\lambda^\star \quad := \inf_{\substack{v \in \mathrm{H}^1_+(\mathbb{S}^d, d\mu) \\ \int_{\mathbb{S}^d} v \, d\mu = 1 \\ \int_{\mathbb{S}^d} x \, |v|^p \, d\mu = 0}} \frac{\int_{\mathbb{S}^d} (\Delta v)^2 \, d\mu}{\int_{\mathbb{S}^d} |\nabla v|^2 \, \nu \, d\mu} > d$$

and consider the inequality

$$\int_{\mathbb{S}^d} |\nabla f|^2 \, \nu \, d\mu + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \text{ s.t. } \int_{\mathbb{S}^d} x \, |f|^p \, d\mu = 0$$

Proposition

For any $p \in (2, 2^\#)$, the inequality holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

$p = 2$: the logarithmic Sobolev case

$$\lambda^\star = d + \frac{2\,(d+2)}{2\,(d+3) + \sqrt{2\,(d+3)\,(2\,d+3)}}$$

Proposition

Let $d \geq 2$. For any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$ such that $\int_{\mathbb{S}^d} x\,|u|^2\,d\mu = 0$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu \geq \frac{\delta}{2} \int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|_2^2}\right) d\mu$$

with $\delta := d + \frac{2}{d}\,\frac{4\,d-1}{2\,(d+3)+\sqrt{2\,(d+3)\,(2\,d+3)}}$

Stability under antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

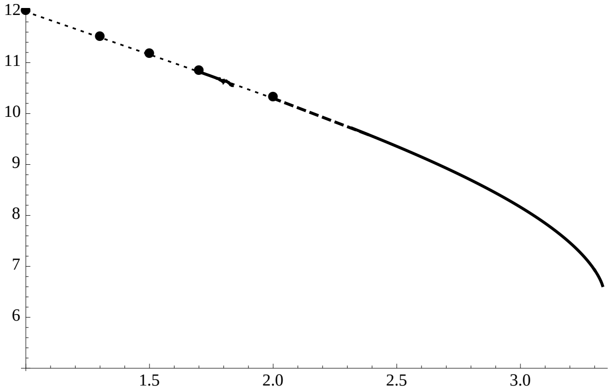
If $p \in (1,2) \cup (2, 2^*)$, *we have*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ *with antipodal symmetry. The limit case* $p = 2$ *corresponds to the improved logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ and $1 \leq p \leq 10/3 \approx 3.33$. The limiting value of the constant is numerically found to be equal to $\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ with $d = 5$ and $p = 10/3$

Reverse Hardy-Littewood-Sobolev inequality

Joint work with J. A. Carrillo, M. G. Delgadino, R. Frank, F. Hoffmann

▷ A family of inequalities

▷ Existence of minimizers and relaxation

▷ No concentration and regularity of measure valued minimizers

▷ Free Energy

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1\,, \quad 0 < q < 1\,, \quad \alpha := \frac{2\,N - q\,(2\,N + \lambda)}{N\,(1-q)}$$

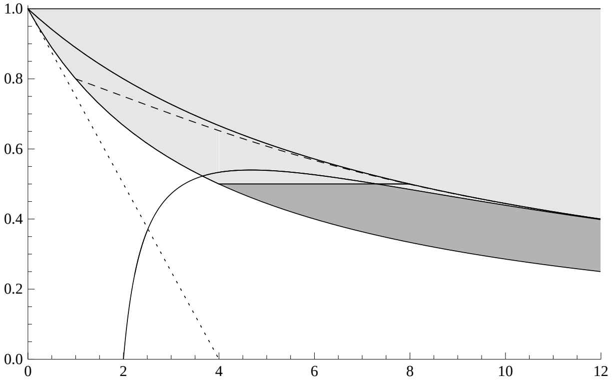
Convention: $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N} |\rho(x)|^p \, dx$ for any $p > 0$

Theorem

$$I_\lambda[\rho] \, \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho(x) \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q \, dx \right)^{(2-\alpha)/q} \qquad (6)$$

holds for any $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ *with* $\mathcal{C}_{N,\lambda,q} > 0$ *if and only if* $q > N/(N+\lambda)$

If either $N = 1,\, 2$ *or if* $N \geq 3$ *and* $q \geq \min\left\{1 - 2/N\,,\, 2N/(2N+\lambda)\right\}$, *then there is a radial nonnegative optimizer* $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$

The conformally invariant case $q = 2N/(2N+\lambda)$

$$I_\lambda[\rho] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \ge \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{2/q}$$

$$2N/(2N+\lambda) \quad\Longleftrightarrow\quad \alpha = 0$$

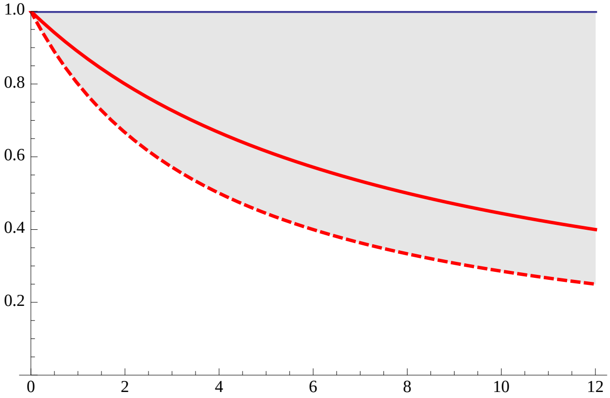
(Dou, Zhu 2015) (Ngô, Nguyen 2017)

The optimizers are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1+|x|^2\right)^{-N/q} \quad \forall\, x\in\mathbb{R}^N$$

and the value of the optimal constant is

$$\mathcal{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}}\,\frac{\Gamma\left(\frac{N}{2}+\frac{\lambda}{2}\right)}{\Gamma\left(N+\frac{\lambda}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1+\frac{\lambda}{N}}$$



$N=4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$. The plain, red curve is the conformally invariant case

A Carlson type inequality

Lemma

Let $\lambda > 0$ *and* $N/(N+\lambda) < q < 1$

$$\left(\int_{\mathbb{R}^N} \rho\,dx\right)^{1-\frac{N\,(1-q)}{\lambda\,q}} \left(\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx\right)^{\frac{N\,(1-q)}{\lambda\,q}} \geq c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{1}{q}}$$

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}} \left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}$$

Equality is achieved if and only if

$$\rho(x) = \left(1+|x|^\lambda\right)^{-\frac{1}{1-q}}$$

up to translations, dilations and constant multiples)

(Carlson 1934) (Levine 1948)

An elementary proof of Carlson's inequality

$$\int_{\{|x|<R\}} \rho^q\,dx \le \left(\int_{\mathbb{R}^N} \rho\,dx\right)^q |B_R|^{1-q} = C_1 \left(\int_{\mathbb{R}^N} \rho\,dx\right)^q R^{N\,(1-q)}$$

and

$$\int_{\{|x|\ge R\}} \rho^q\,dx \le \left(\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx\right)^q \left(\int_{\{|x|\ge R\}} |x|^{-\frac{\lambda\,q}{1-q}}\,dx\right)^{1-q}$$
$$= C_2 \left(\int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx\right)^q R^{-\lambda q+N\,(1-q)}$$

and optimize over $R>0$
... existence of a radial monotone non-increasing optimal function; rearrangement; Euler-Lagrange equations

Proposition

Let $\lambda > 0$. *If* $N/(N+\lambda) < q < 1$, *then* $\mathcal{C}_{N,\lambda,q} > 0$

By rearrangement inequalities: prove the reverse HLS inequality for symmetric non-increasing ρ's so that

$$\int_{\mathbb{R}^N} |x-y|^\lambda \rho(y)\,dx \ge \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \quad \text{for all} \quad x \in \mathbb{R}^N$$

implies

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \int_{\mathbb{R}^N} \rho\,dx$$

In the range $\frac{N}{N+\lambda} < q < 1$

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N} \rho\,dx\,dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx$$

$$\ge c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{2-\alpha}{q}}$$

and conclude with Carlson's inequality

The case $q = 2$

Corollary

Let $\lambda = 2$ and $N/(N+2) < q < 1$. Then the optimizers for (6) are given by translations, dilations and constant multiples of

$$\rho(x) = \left(1 + |x|^2\right)^{-\frac{1}{1-q}}$$

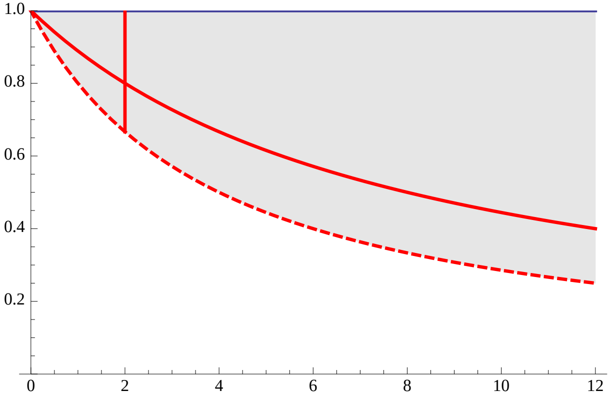
and the optimal constant is

$$\mathcal{C}_{N,2,q} = \frac{1}{2}\, c_{N,2,q}^{\frac{2\,q}{N\,(1-q)}}$$

By rearrangement inequalities it is enough to prove (7) for symmetric non-increasing ρ's, and so $\int_{\mathbb{R}^N} x\rho(x)\,dx = 0$. Therefore

$$I_2[\rho] = 2\int_{\mathbb{R}^N} \rho(x)\,dx \int_{\mathbb{R}^N} |x|^2 \rho(x)\,dx$$

and the optimal function is optimal for Carlson's inequality



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$. The dashed, red curve is the threshold case $q = N/(N+\lambda)$

The threshold case $q = N/(N+\lambda)$ and below

Proposition

Let $\lambda > 0$. If $0 < q \le N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0$

Let $\rho \ge 0$ be bounded with compact support, $\sigma \ge 0$ a smooth function with $\int_{\mathbb{R}^N} \sigma(x)\,dx = 1$ and

$$\rho_\varepsilon(x) := \rho(x) + M\,\varepsilon^{-N}\,\sigma(x/\varepsilon)$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon(x)\,dx = \int_{\mathbb{R}^N} \rho(x)\,dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon(x)^q\,dx \to \int_{\mathbb{R}^N} \rho(x)^q\,dx \quad \text{as} \quad \varepsilon \to 0_+ \tag{7}$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx \quad \text{as} \quad \varepsilon \to 0_+$$

If $0 < q < N/(N+\lambda)$, *i.e.*, $\alpha > 1$, take ρ_ε as a trial function,

$$\mathcal{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho(x)\,dx}{\left(\int_{\mathbb{R}^N} \rho(x)\,dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho(x)^q\,dx\right)^{(2-\alpha)/q}} =: \mathcal{Q}[\rho, M] \tag{8}$$

The threshold case: If $\alpha = 1$, *i.e.*, $q = N/(N+\lambda)$, by taking the limit as $M \to +\infty$, we obtain

$$\mathcal{C}_{N,\lambda,q} \le \frac{2\int_{\mathbb{R}^N} |x|^\lambda \, \rho(x)\, dx}{\left(\int_{\mathbb{R}^N} \rho(x)^q \, dx\right)^{(2-\alpha)/q}}$$

For any $R > 1$, we take

$$\rho_R(x) := |x|^{-(N+\lambda)} \, \mathbb{1}_{1 \le |x| \le R}(x)$$

Then

$$\int_{\mathbb{R}^N} |x|^\lambda \, \rho_R \, dx = \int_{\mathbb{R}^N} \rho_R^q \, dx = \left|\mathbb{S}^{N-1}\right| \, \log R$$

and, as a consequence,

$$\frac{\int_{\mathbb{R}^N} |x|^\lambda \, \rho_R(x)\, dx}{\left(\int_{\mathbb{R}^N} \rho_R^{N/(N+\lambda)} \, dx\right)^{(N+\lambda)/N}} = \left(\left|\mathbb{S}^{N-1}\right| \, \log R\right)^{-\lambda/N} \to 0 \quad \text{as} \quad R \to \infty$$

This proves that $\mathcal{C}_{N,\lambda,q} = 0$ for $q = N/(N+\lambda)$

A relaxed inequality

$$I_\lambda[\rho]+2M\int_{\mathbb{R}^N}|x|^\lambda\,\rho(x)\,dx\;\geq\mathcal{C}^{\rm rel}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho(x)\,dx+M\right)^\alpha\left(\int_{\mathbb{R}^N}\rho(x)^q\,dx\right)^{\frac{2-\alpha}{q}}$$

with $q>N/(N+\lambda)$. Let

$$\mathcal{C}^{\rm rel}_{N,\lambda,q}:=\inf\left\{\mathcal{Q}[\rho,M]\;:\;0\le\rho\in\mathrm{L}^1\cap\mathrm{L}^q(\mathbb{R}^N)\,,\;\rho\not\equiv0\,,\;M\ge0\right\}$$

We know that $\mathcal{C}^{\rm rel}_{N,\lambda,q}\le\mathcal{C}_{N,\lambda,q}$ by restricting the minimization to $M=0$. On the other hand, $\mathcal{C}^{\rm rel}_{N,\lambda,q}\ge\mathcal{C}_{N,\lambda,q}$ with appropriate test functions:

$$\mathcal{C}^{\rm rel}_{N,\lambda,q}=\mathcal{C}_{N,\lambda,q}$$

Lemma

Let $\lambda>0$ and $N/(N+\lambda)<q<1$. If $\rho\ge0$ is an optimal function for either $\mathcal{C}^{\rm rel}_{N,\lambda,q}$ (for an $M\ge0$) or $\mathcal{C}_{N,\lambda,q}$ (with $M=0$), then ρ is radial (up to a translation), monotone non-increasing and positive almost everywhere on $\mathbb{R}^N$

Regularity of the minimizers

Proposition

Let $N \geq 1$, $\lambda > 0$ *and* $N/(N+\lambda) < q < 2N/(2N+\lambda)$*Let* (ρ_*, M_*) *be a minimizer for* $\mathcal{C}^{\rm rel}_{N,\lambda,q}$*. Then the following holds:*

1. *If* $\int_{\mathbb{R}^N} \rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$*, then* $M_* = 0$*, and* ρ_* *is bounded*

$$\rho_*(0) = \left(\frac{(2-\alpha) I_\lambda[\rho_*] \int_{\mathbb{R}^N} \rho_*\,dx}{\left(\int_{\mathbb{R}^N} \rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx \int_{\mathbb{R}^N}\rho_*\,dx - \alpha I_\lambda[\rho_*]\right)}\right)^{\frac{1}{1-q}}$$

2. *If* $\int_{\mathbb{R}^N} \rho_*\,dx = \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$*, then* $M_* = 0$ *and* $\rho_*(0) = \infty$
3. *If* $\int_{\mathbb{R}^N} \rho_*\,dx < \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx}$*, then* $\rho_*(0) = \infty$ *and*

$$M_* = \frac{\alpha\, I_\lambda[\rho_*] - 2\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx \quad \int_{\mathbb{R}^N}\rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N}|x|^\lambda\,\rho_*(x)\,dx} > 0$$

Regularity of the measure valued minimizers

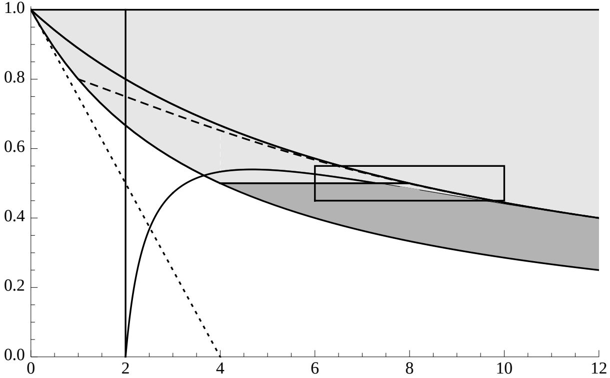
Lemma

Let $N \geq 1$, $\lambda > 0$ and $N/(N+\lambda) < q < 1$. Let (ρ_*, M_*) be a minimizer for $\mathcal{C}^{\rm rel}_{N,\lambda,q}$ if $q < 2N/(2N+\lambda)$ or let ρ_* be a minimizer for $\mathcal{C}_{N,\lambda,q}$ if $q \geq 2N/(2N+\lambda)$. Assume that ρ_* is unbounded. If $\lambda < 2$, there is a $c > 0$ such that for all sufficiently small $x \in \mathbb{R}^N$,

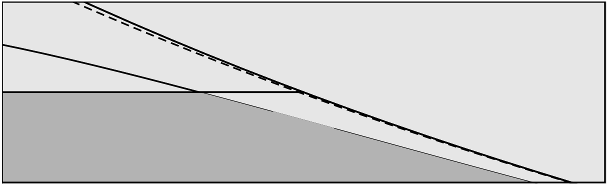
$$\rho_*(x) \geq c\,|x|^{-\lambda/(1-q)}$$

and if $\lambda \geq 2$, there is a $C > 0$ such that

$$\rho_*(x) = C\,|x|^{-2/(1-q)}\,\big(1 + o(1)\big) \quad \text{as} \quad x \to 0$$



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$ has a bounded optimizer



A mean-field evolution equation and the free energy

Let us consider

$$\partial_t \rho = \Delta \rho^q + \nabla \cdot (\rho \, \nabla W_\lambda * \rho)$$

where kernel $W_\lambda(x) := \frac{1}{\lambda} \, |x|^\lambda$ and the *free energy* functional

$$\mathcal{F}[\rho] := -\frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q \, dx + \frac{1}{2\lambda} \, I_\lambda[\rho]$$

the equation conserves the *mass* and

$$\frac{d}{dt} \mathcal{F}[\rho(t,\cdot)] = -\int_{\mathbb{R}^N} \rho \left| \tfrac{q}{1-q} \nabla \rho^{q-1} - \nabla W_\lambda * \rho \right|^2 dx$$

Boundedness of the free energy

Theorem

The free energy $\mathcal{F}$ *is bounded from below on* $\mathcal{P}(\mathbb{R}^N)$ *if and only if* $q > N/(N+\lambda)$*. If* $q > N/(N+\lambda)$*, then there exists a global minimizer* $\mu_* \in \mathcal{P}(\mathbb{R}^N)$ *and, modulo translations, it has the form*

$$\mu_* = (1-a)\,\delta_0 + a\,\rho_*$$

for some $a \in (0,1]$*. Moreover* $\rho_* \in \mathcal{P}(\mathbb{R}^N) \cap \mathrm{L}^q(\mathbb{R}^N)$ *is radially symmetric, non-increasing modulo translations and such that* $\int_{\mathbb{R}^N} \rho_*(x)\,dx = 1$

If $a = 1$*, then* ρ_* *is an optimizer of* (6)*. Conversely, if* $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ *is an optimizer of* (6) *with mass* $M > 0$*, then* ρ/M *is a global minimizer of* $\mathcal{F}$ *on* $\mathcal{P}(\mathbb{R}^N)$

Finally, if either $\max\left\{N/(N+\lambda), (N-1)/N\right\} < q < 1$ *and* $\lambda \geq 1$*, or* $N/(N+\lambda) < q < 1$ *and* $2 \leq \lambda \leq 4$*, then the global minimizer* μ_* *is unique up to translation*

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