Improved Sobolev inequalities using nonlinear flows

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A question by H. Brezis and E. Lieb

[Brezis, Lieb (1985)] *Is there a natural way to bound*

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

from below in terms of the "distance" off from the set of optimal [Aubin-Talenti] functions when $d\ge 3$?

- [Bianchi-Egnell (1990)] There is a positive constant α such that

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\ge\alpha\inf_{\varphi\in\mathcal{M}}\|\nabla u-\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- [Cianchi, Fusco, Maggi, Pratelli (2009)] (also a version for $\|\nabla u\|^p_{\mathrm{L}^p(\mathbb{R}^d)}$) There are constants α and κ such that

$$\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\ge\big(1+\kappa\,\lambda(u)^\alpha\big)\,\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

where $\lambda(u)=\inf_{\varphi\in\mathcal{M}}\left\{\frac{\|u-\varphi\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|u\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}\;:\;\|u\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=\|\varphi\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right\}$

A – Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Outline

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- A result motivated by [Carrillo, Carlen and Loss]
- Sobolev and HLS inequalities can be related using a nonlinear flow *compatible with Legendre's duality*
- The asymptotic behaviour close to the *vanishing time* is determined by a solution with *separation of variables* based on the Aubin-Talenti solution
- The entropy H (to be defined) is negative, concave, and we can relate $\mathsf{H}(0)$ with $\mathsf{H}'(0)$ by integrating estimates on $(0, T)$, which provides *a first improvement of Sobolev's inequality* if $d \geq 5$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\; dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\; dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\; dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

A first statement

Proposition

[J.D.] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\;dx\right)^{\frac{2}{d}} \left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[J.D.] *Assume that* $d \geq 5$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^{\alpha}\,(F(x))^{\frac{d+2}{d-2}} \quad \forall\,(t,x) \in (0,T) \times \mathbb{R}^d$$

where $\alpha = (d+2)/4$, $c^{1-m} = 4\,m\,d$, $m = \frac{d-2}{d+2}$, $p = d/(d-2)$ and F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v of (3) with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Such an estimate makes sense if $v = \overline{v}_T$. This is also true for any solution v as can be seen by rewriting the problem on $\mathbb{S}^d$: integrability conditions for v are exactly the same as for $\overline{v}_T$ $\square$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the Cauchy-Schwarz inequality, we have

$$\frac{\mathsf{J}'^2}{m+1} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$
$$\le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\,\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q}-1)\,,\quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1-e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

$d = 2$: Onofri's and log HLS inequalities

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(\frac{v}{\mu}\right)\quad t>0\,,\quad x\in\mathbb{R}^2$$

Proposition

If $v = \mu\,e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\begin{aligned}\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] &= \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\;d\mu\\ &\ge \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx + \int_{\mathbb{R}^2} u\;d\mu - \log\left(\int_{\mathbb{R}^2} e^u\;d\mu\right)\ge 0\end{aligned}$$

Fast diffusion equations

1. entropy methods
2. linearization of the entropy
3. improved Gagliardo-Nirenberg inequalities

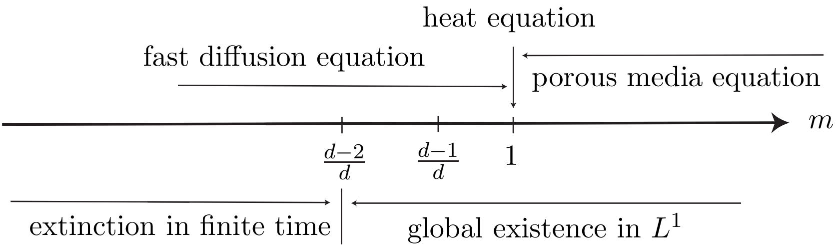
B1 – Fast diffusion equations: entropy methods

Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Time-dependent rescaling, Free energy

- **Time-dependent rescaling**: Take $u(\tau, y) = R^{-d}(t)\, v\,(t, y/R(\tau))$ where

$$\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

$$\Sigma[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \Sigma_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt}\Sigma[v] = -I[v]\;, \quad I[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- **Stationary solution:** choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form: for $m \neq 1$,

$$\Sigma[v] = \int_{\mathbb{R}^d} \psi\left(\frac{v}{v_\infty}\right)\,v_\infty^m\,dx \quad \textit{with } \psi(t) = \frac{t^m - 1 - m\,(t-1)}{m-1}$$

- **Entropy – entropy production inequality**

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\Sigma[v(t,\cdot)] \leq \Sigma[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\Sigma[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2} \left(\frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \leq \frac{d}{d-2}$ *(fast diffusion case) and* $d \geq 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A \, \|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \, \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d} \right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y} \right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})} \right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}$$

...the Bakry-Emery method

Consider the generalized Fisher information

$$I[v] := \int_{\mathbb{R}^d} v\,|Z|^2\,dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\,I[v(t,\cdot)] + 2\,I[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\operatorname{div} Z)^2\,dx - 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i Z^j)^2\,dx$$

- the Fisher information decays exponentially:

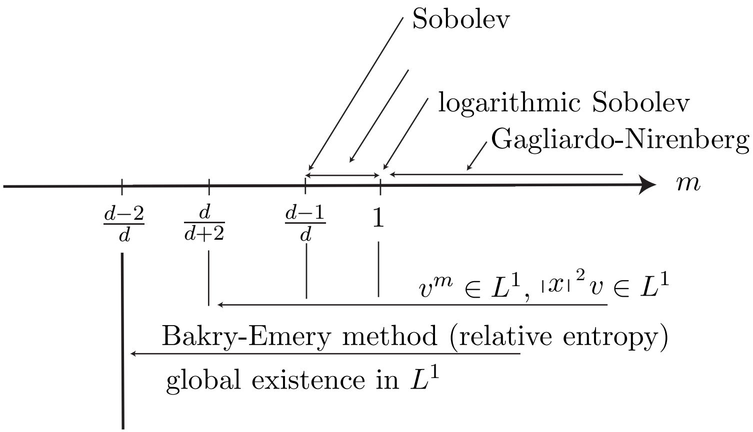
$$I[v(t,\cdot)] \le I[u_0]\,e^{-\,2\,t}$$

- $\lim_{t\to\infty} I[v(t,\cdot)] = 0$ and $\lim_{t\to\infty} \Sigma[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(I[v(t,\cdot)] - 2\,\Sigma[v(t,\cdot)]\Big) \le 0$ means $I[v] \ge 2\,\Sigma[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

Fast diffusion: finite mass regime

Inequalities...



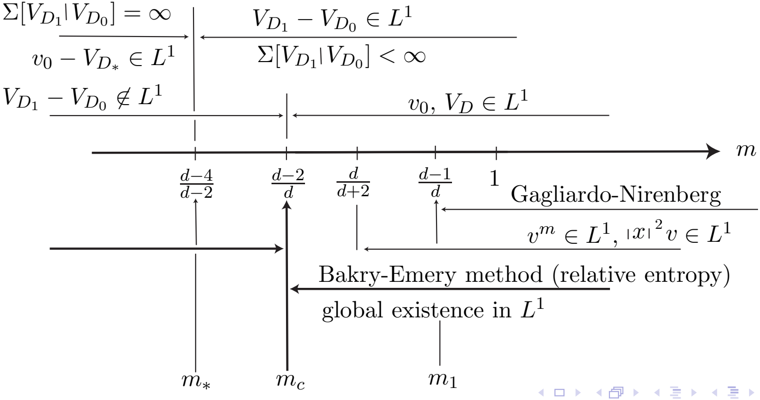
... existence of solutions of $u_t = \Delta u^m$

B2 – Fast diffusion equations: the infinite mass regime
Linearization of the entropy

Extension to the infinite mass regime, finite time vanishing

- If $m > m_c := \frac{d-2}{d} \le m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass.
- For $m \le m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \le m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez]

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \nabla u^{m-1}) = \frac{1-m}{m} \Delta u^m \tag{4}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$

$$R(\tau) := (T + \tau)^{\frac{1}{d\,(m-m_c)}}$$

- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

$$R(\tau) := (T - \tau)^{-\frac{1}{d\,(m_c-m)}}$$

Self-similar *Barenblatt type solutions* exists for any m
Rescaling: time-dependent change of variables

$$t := \frac{1-m}{2} \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

Sharp rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

[Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

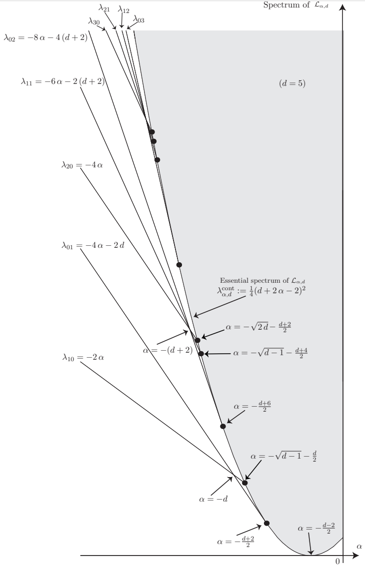
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

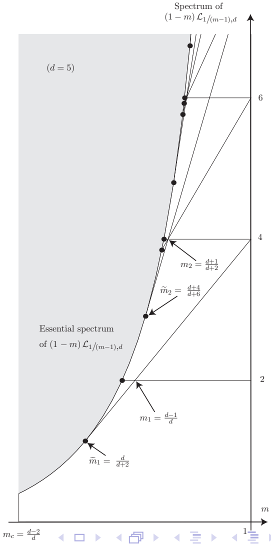
where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$

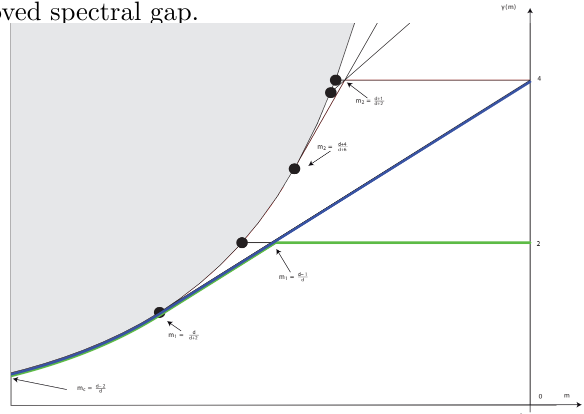
Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d=5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d+2\,\alpha-2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d=5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of the fast diffusion equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$, then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap.



Higher order matching asymptotics

[J.D., G. Toscani] For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{5}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} (v - B_\sigma) \right] dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

The entropy / entropy production estimate

According to the definition of B_σ, we know that
$2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right]B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \dfrac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\; t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

Theorem (J.D., G. Toscani)

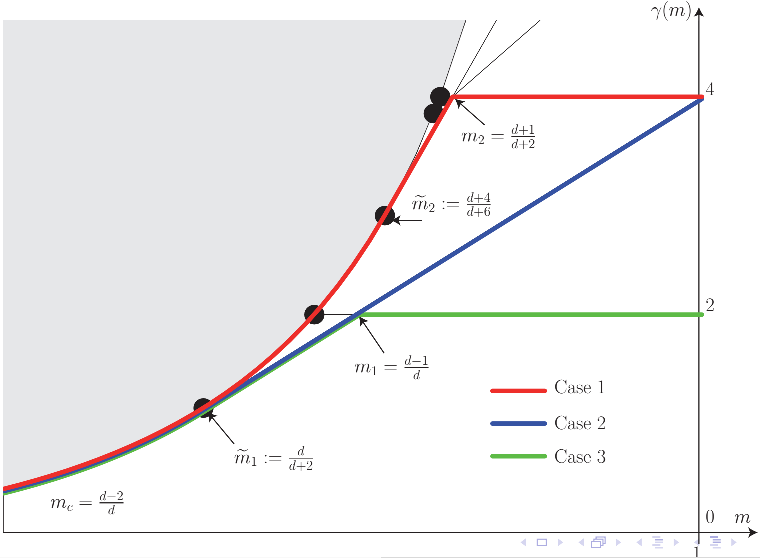
Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\; t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



Gagliardo-Nirenberg and Sobolev inequalities : improvements

[J.D., G. Toscani]

Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t}+\nabla\cdot\left[u\left(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1}-2\,x\right)\right]=0\quad t>0\,,\quad x\in\mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\;dx\;,\quad K_M:=\int_{\mathbb{R}^d}|x|^2\,B_1(x)\;dx$$

where

$$B_\lambda(x):=\lambda^{-\frac{d}{2}}\left(C_M+\tfrac{1}{\lambda}\,|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\lambda^m-m\,B_\lambda^{m-1}\,(u-B_\lambda)\right]\;dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0}\mathcal{F}_\lambda[u(x,t)]=\mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)]=-\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u]:=\lambda^{\frac{d}{2}(m-m_c)}\,\frac{m}{1-m}\int_{\mathbb{R}^d}u\,\left|\nabla u^{m-1}-\nabla B_\lambda^{m-1}\right|^2\,dx$$

2. In the *Bakry-Emery method*, there is *an additional (good) term*

$$4\left[1+2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\,\sigma_0^{\frac{d}{2}\,(1-m)}}\right]\frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right)\geq\frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u]\geq\frac{m}{8\int_{\mathbb{R}^d}B_1^m\,dx}\,C_M^2\|u-B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

improved decay for the relative entropy

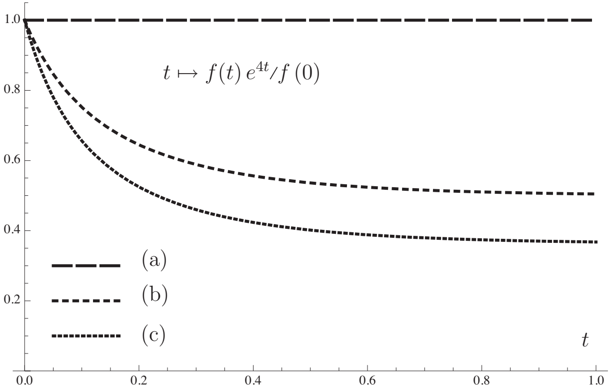


Figure: Upper bounds on the decay of the relative entropy: $t \mapsto f(t)\, e^{4t}/f(0)$
(a): estimate given by the entropy-entropy production method
(b): exact solution of a simplified equation
(c): numerical solution (found by a shooting method)

A Csiszár-Kullback(-Pinsker) inequality

Let $m \in (\widetilde{m}_1, 1)$ with $\widetilde{m}_1 = \frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m\, B_\sigma^{m-1} (u - B_\sigma) \right] \, dx$$

Theorem

Let $d \geq 1$, $m \in (\widetilde{m}_1, 1)$ and assume that u is a nonnegative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2\, u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{\mathrm{L}^1(\mathbb{R}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2\, u \, dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma \, dx$, then

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m \, dx} \left(C_M \|u - B_\sigma\|_{\mathrm{L}^1(\mathbb{R}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2\, |u - B_\sigma| \, dx \right)^2$$

Csiszár-Kullback(-Pinsker): proof (1/2)

Let $v := u/B_\sigma$ and $d\mu_\sigma := B_\sigma^m\,dx$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma=\int_{\mathbb{R}^d}B_\sigma^{m-1}\,(u-B_\sigma)\,dx$$

$$=\sigma^{\frac{d}{2}\,(1-m)}\,C_M\int_{\mathbb{R}^d}(u-B_\sigma)\,dx+\sigma^{\frac{d}{2}\,(m_c-m)}\int_{\mathbb{R}^d}|x|^2\,(u-B_\sigma)\,dx=0$$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma=\int_{v>1}(v-1)\,d\mu_\sigma-\int_{v<1}(1-v)\,d\mu_\sigma=0$$

$$\int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma=\int_{v>1}(v-1)\,d\mu_\sigma+\int_{v<1}(1-v)\,d\mu_\sigma$$

$$\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx=\int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma=2\int_{v<1}|v-1|\,d\mu_\sigma$$

Csiszár-Kullback(-Pinsker): proof (2/2)

A Taylor expansion shows that

$$\mathcal{F}_\sigma[u] = \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - 1 - m\,(v-1) \right] d\mu_\sigma = \frac{m}{2} \int_{\mathbb{R}^d} \xi^{m-2}\,|v-1|^2\,d\mu_\sigma$$

$$\geq \frac{m}{2} \int_{v<1} |v-1|^2\,d\mu_\sigma$$

Using the Cauchy-Schwarz inequality, we get

$$\left(\int_{v<1} |v-1|\;d\mu_\sigma \right)^2 = \left(\int_{v<1} |v-1|\;B_\sigma^{\frac{m}{2}}\;B_\sigma^{\frac{m}{2}}\;dx \right)^2 \leq \int_{v<1} |v-1|^2\,d\mu_\sigma \int_{\mathbb{R}^d} B_\sigma^m\;dx$$

and finally obtain that

$$\mathcal{F}_\sigma[u] \geq \frac{m}{2}\,\frac{\left(\int_{v<1} |v-1|\;d\mu_\sigma \right)^2}{\int_{\mathbb{R}^d} B_\sigma^m\;dx} = \frac{m}{8}\,\frac{\left(\int_{\mathbb{R}^d} |u-B_\sigma|\;B_\sigma^{m-1}\;dx \right)^2}{\int_{\mathbb{R}^d} B_\sigma^m\;dx}$$

An improved Gagliardo-Nirenberg inequality: the setting

The inequality

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\mathrm{GN}} \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta} \, \|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

with $\theta = \theta(p) := \frac{p-1}{p} \frac{d}{d+2-p\,(d-2)}$, $1 < p \le \frac{d}{d-2}$ if $d \ge 3$ and $1 < p < \infty$ if $d = 2$, can be rewritten, in a non-scale invariant form, as

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^{p+1} \, dx \ge \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p} \, dx \right)^{\gamma}$$

with $\gamma = \gamma(p,d) := \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$. Optimal function are given by

$$f_{M,y,\sigma}(x) = \frac{1}{\sigma^{\frac{d}{2}}} \left(C_M + \frac{|x-y|^2}{\sigma} \right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

where C_M is determined by $\int_{\mathbb{R}^d} f_{M,y,\sigma}^{2p} \, dx = M$

$$\mathfrak{M}_d := \left\{ f_{M,y,\sigma} \;:\; (M,y,\sigma) \in \mathcal{M}_d := (0,\infty) \times \mathbb{R}^d \times (0,\infty) \right\}$$

An improved Gagliardo-Nirenberg inequality (1/2)

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g \in \mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p} \left(|f|^{2\,p} - g^{2\,p} \right) - \tfrac{2\,p}{p+1} \left(|f|^{p+1} - g^{p+1} \right) \right] dx$$

Theorem

Let $d \geq 2$, $p > 1$ and assume that $p < d/(d-2)$ if $d \geq 3$. If

$$\frac{\int_{\mathbb{R}^d} |x|^2\,|f|^{2\,p}\,dx}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx \right)^\gamma} = \frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\quad \sigma_*(p) := \left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)} \right)^{\frac{4\,p}{d-p\,(d-4)}}$$

for any $f \in \mathrm{L}^{p+1} \cap \mathcal{D}^{1,2}(\mathbb{R}^d)$, then we have

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx \right)^\gamma \geq \mathsf{C}_{p,d}\,\frac{\left(\mathcal{R}^{(p)}[f] \right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2\,p}\,dx \right)^\gamma}$$

An improved Gagliardo-Nirenberg inequality (2/2)

A Csiszár-Kullback inequality

$$\mathcal{R}^{(p)}[f] \geq \mathsf{C}_{\mathrm{CK}} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-2)} \inf_{g \in \mathfrak{M}_d^{(p)}} \||f|^{2p} - g^{2p}\|_{\mathrm{L}^1(\mathbb{R}^d)}^2$$

with $\mathsf{C}_{\mathrm{CK}} = \frac{p-1}{p+1} \, \frac{d+2-p(d-2)}{32\,p} \, \sigma_*^{d \, \frac{p-1}{4p}} \, M_*^{1-\gamma}$. Let

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p} \, {\mathsf{C}_{\mathrm{CK}}}^2$$

Corollary

Under previous assumptions, we have

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^{p+1} \, dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p} \, dx \right)^{\gamma} \geq \mathfrak{C}_{p,d} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-4)} \inf_{g \in \mathfrak{M}_d(p)} \||f|^{2p} - g^{2p}\|_{\mathrm{L}^1(\mathbb{R}^d)}^4$$

Conclusion 1: improved inequalities

- We have found an improvement of an optimal Gagliardo-Nirenberg inequality, which provides an explicit measure of the distance to the manifold of optimal functions.
- The method is based on the nonlinear flow
- The explicit improvement gives (is equivalent to) an improved entropy – entropy production inequality

Conclusion 2: improved rates

If $m \in (m_1, 1)$, with

$$f(t) := \mathcal{F}_{\sigma(t)}[u(\cdot, t)]$$

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2 \, u(x, t) \, dx$$

$$j(t) := \mathcal{J}_{\sigma(t)}[u(\cdot, t)]$$

$$\mathcal{J}_\sigma[u] := \frac{m \, \sigma^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} u \left|\nabla u^{m-1} - \nabla \mathfrak{B}_\sigma^{m-1}\right|^2 dx$$

we can write a system of coupled ODEs

$$\begin{cases} f' = -j \le 0 \\ \sigma' = -2\, d \, \frac{(1-m)^2}{m\, K_M} \, \sigma^{\frac{d}{2}(m-m_c)} \, f \le 0 \\ j' + 4\, j = \frac{d}{2}\,(m - m_c) \left[\frac{j}{\sigma} + 4\, d\,(1-m)\,\frac{f}{\sigma}\right] \sigma' - \mathrm{r} \end{cases} \tag{6}$$

In the rescaled variables, we have found an *improved decay* (algebraic rate) of the relative entropy. This is a new nonlinear effect, which matters for the initial time layer

Conclusion 3: Best matching Barenblatt profiles are delayed

Let u be such that

$$v(\tau,x)=\frac{\mu^d}{R(D\,\tau)^d}\,u\left(\frac{1}{2}\log R(D\,\tau),\frac{\mu\,x}{R(D\,\tau)}\right)$$

with $\tau\mapsto R(\tau)$ given as the solution to

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left(\frac{\mu^2}{K_M}\int_{\mathbb{R}^d}|x|^2\,v(\tau,x)\,dx\right)^{-\frac{d}{2}(m-m_c)}\,,\quad R(0)=1$$

Then

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left[R^2(\tau)\,\sigma\left(\frac{1}{2}\log R(D\,\tau)\right)\right]^{-\frac{d}{2}(m-m_c)}$$

that is $R(\tau)=R_0(\tau)\le R_0(\tau)$ where $\frac{1}{R}\,\frac{dR_0}{d\tau}=\left(R_0^2(\tau)\,\sigma\,(0)\right)^{-\frac{d}{2}(m-m_c)}$
and asymptotically as $\tau\to\infty$, $R(\tau)=R_0(\tau-\delta)$ for some delay $\delta>0$

Thank you for your attention !