A weighted Moser-Trudinger inequality and its relation to the Caffarelli-Kohn-Nirenberg inequalities in two space dimensions

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(A JOINT WORK WITH M.J. ESTEBAN AND G.TARANTELLO)

WORKSHOP IFO

Toulouse, 31 octobre, 2008

http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/

Outline

- Introduction : critical Sobolev exponent, dimension, interpolation, S^2, Onofri's inequality
- Generalized Onofri inequalities
- Caffarelli-Kohn-Nirenberg inequalities
- Symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities
- A symmetry result for Caffarelli-Kohn-Nirenberg inequalities

Goals of this talk:

- emphasize the structure of a family of inequalities in the “critical” case $N = 2$
- relate families of inequalities : Hardy-Sobolev / Caffarelli-Kohn-Nirenberg / Moser-Trudinger-Onofri
- identify extremal functions and optimal constants
- understand symmetry breaking

1. Critical Sobolev exponent, dimension, interpolation, S^2, Onofri's inequality

Some naive remarks about Sobolev's embeddings

In the euclidean space $\mathbb{R}^N$, with $N \geq 3$:

$$\left(\int_{\mathbb{R}^N} |u|^{2*} \, dx \right)^{2/2^*} \leq S(N) \int_{\mathbb{R}^N} |\nabla u|^2 \, dx$$

Here $2^* = 2^*(N) = \frac{2\,N}{N-2}$. Optimal functions are (up to the invariances):
$u(x) = (1 + r^2)^{-(N-2)/2}$, $r = |x|$

$$S(N) = \frac{\left(\int_0^\infty \frac{r^{N-1}}{(1+r^2)^N} \, dr \right)^{1-\frac{2}{N}}}{(N-2)^2 \, |S^{N-1}|^{\frac{2}{N}} \int_0^\infty \frac{r^{N+1}}{(1+r^2)^N} \, dr} = \frac{1}{\pi \, N \, (N-2)} \left[\frac{\Gamma\,(N)}{\Gamma\left(\frac{N}{2}\right)} \right]^{\frac{2}{N}}$$

For radial functions, N can be considered as real. With
$s(N) = S(N) \, |S^{N-1}|^{\frac{2}{N}}$

$$\left(\int_0^\infty |u|^{2*} \, r^{N-1} \, dr \right)^{2/2^*} \leq s(N) \int_0^\infty |\nabla u|^2 \, r^{N-1} \, dr$$

The case of $N = 2$

For radial functions, we can consider the limit case corresponding to $N \to 2$.

[J. Moser. A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J., 20: 1077-1092, 1970/71]
[N. S. Trudinger. On imbeddings into Orlicz spaces and some applications. J. Math. Mech., 17: 473-483, 1967]

Result 1: $\exists\, C_2 > 0$ such that, if $u \in H^1(S^2)$ is s. t. $\int_{S^2} |\nabla u|^2 \, d\sigma \leq 1$ and $\int_{S^2} u \, d\sigma = 0$

$$\int_{S^2} e^{u^2} \, d\sigma \leq C_2$$

Result 2: $\exists\, C_1 > 0$ such that, if $u \in H^1(S^2)$, then

$$\int_{S^2} e^{2u - 2\int_{S^2} u \, d\sigma} \, d\sigma \leq C_1 \, e^{\int_{S^2} |\nabla u|^2 \, d\sigma}$$

σ is induced by Lebesgue's measure $\mathbb{R}^3 \supset S^2$, such that $\sigma(S^2) = 1$

Onofri's inequality

[E. Onofri. On the positivity of the effective action in a theory of random surfaces. Comm. Math. Phys., 86 (3): 321-326, 1982]

$$\int_{S^2} e^{2\,u-2\int_{S^2} u\,d\sigma}\;d\sigma\;\le\;e^{\|\nabla u\|^2_{L^2(S^2,d\sigma)}}$$

for all $u \in \mathcal{E} = \{u \in L^1(S^2, d\sigma)\;:\;|\nabla u| \in L^2(S^2, d\sigma)\}$

By the stereographic projection from S^2 onto $\mathbb{R}^2$, we get an Onofri type inequality in $\mathbb{R}^2$

$$\int_{\mathbb{R}^2} e^{v-\int_{\mathbb{R}^2} v\,d\mu}\;d\mu\;\le\;e^{\frac{1}{16\,\pi}\,\|\nabla v\|^2_{L^2(\mathbb{R}^2,dx)}}$$

for all $v \in \mathcal{D} = \{v \in L^1(\mathbb{R}^2, d\mu)\;:\;|\nabla v| \in L^2(\mathbb{R}^2, dx)\}$ and

$$d\mu = \frac{dx}{\pi\,(1+|x|^2)^2}$$

Sobolev's inequality on the sphere

[E. H. Lieb. Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities. Ann. of Math. (2), 118 (2): 349-374, 1983]

$$\frac{4}{N(N-2)}\,\|\nabla u\|^2_{L^2(S^N,d\sigma)}+\|u\|^2_{L^2(S^N,d\sigma)}\geq\|u\|^2_{L^{\frac{2N}{N-2}}(S^N,d\sigma)}\quad\forall u\in H^1(S^N)$$

if $N\geq 3$ and equality is achieved by constants

Stereographic projection

Coordinates on S^N : $(\rho\,\omega, z) \in \mathbb{R}^N \times (-1,1)$, $z = \sin\theta$, $\rho = \cos\theta$

Coordinates on $\mathbb{R}^N$: $x \in \mathbb{R}^N$, $r = |x|$, $\omega = \frac{x}{|x|}$, $z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}$, $\rho = \frac{2\,r}{r^2+1}$

$$u \circ \Sigma_0^{-1}(x) = h(r)\, v(x)\ , \quad h(r) = \left(\frac{1+r^2}{2}\right)^{\frac{N-2}{2}}$$

... elementary computations

$$|S^N| \left[\|\nabla u\|^2_{L^2(S^N,d\sigma)} + \frac{N(N-2)}{4}\,\|u\|^2_{L^2(S^N,d\sigma)}\right] = \int_{\mathbb{R}^N} |\nabla v|^2\, dx$$

$$|S^N| \int_{S^N} |u|^{\frac{2N}{N-2}}\, d\mu = \int_{\mathbb{R}^N} |v|^{\frac{2N}{N-2}}\, dx$$

Equality case is achieved for $u = Const$, i.e. $v = 1/h$

Optimal constant in Sobolev's inequality is : S $= \frac{4}{N(N-2)}\,|S^N|^{\frac{N-2}{N}-1}$

Interpolation on the sphere

[W. Beckner. Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality. Ann. of Math. (2), 138 (1): 213-242, 1993]

For any $N \geq 3$, if $q \in [2, \frac{2N}{N-2}]$

$$\frac{q-2}{N} \, \|\nabla u\|^2_{L^2(S^N, d\sigma)} + \|u\|^2_{L^2(S^N, d\sigma)} \geq \|u\|^2_{L^q(S^N, d\sigma)} \quad \forall u \in H^1(S^N)$$

Also true for any $q \in (2, +\infty)$ if $N = 2$

$$\frac{q-2}{2} \, \|\nabla u\|^2_{L^2(S^2, d\sigma)} + \|u\|^2_{L^2(S^2, d\sigma)} \geq \|u\|^2_{L^q(S^2, d\sigma)} \quad \forall u \in H^1(S^2)$$

Limit cases: logarithmic Sobolev and Onofri's inequalities

$$\frac{q-2}{N}\,\|\nabla u\|^2_{L^2(S^N,d\sigma)}+\|u\|^2_{L^2(S^N,d\sigma)}\geq\|u\|^2_{L^q(S^N,d\sigma)}\quad\forall u\in H^1(S^N)$$

A derivation at $q=2$ gives a logarithmic Sobolev inequality on S^N

$$\int_{S^N}|u|^2\log\left(\frac{|u|^2}{\int_{S^N}|u|^2\,d\sigma}\right)\,d\sigma\leq\frac{N}{2}\int_{S^N}|u|^2\,d\sigma\log\left(\frac{2}{\pi\,N\,e}\,\frac{\int_{S^N}|\nabla u|^2\,d\sigma}{\int_{S^N}|u|^2\,d\sigma}\right)$$

Let $N=2$, $q=2\,(1+t)$, $t\to+\infty$, $u=1+\frac{1}{t}\,F$ s.t. $\int_{S^2}F\,d\sigma=0$,
$d\nu=4\pi\,d\sigma$,

$$\left(2^{1+\varepsilon}\,\frac{\Gamma(1+\varepsilon/2)}{\Gamma(2+\varepsilon)}\right)^t\int_{S^2}\left|1+\tfrac{1}{t}\,F\right|^{2(1+t)}d\nu\leq\left(\frac{\int_{S^2}|\nabla F|^2\,d\nu}{1+t}+1+\tfrac{1}{t^2}\int_{S^2}|F|^2\,d\nu\right)^{1+t}$$

gives $\int_{S^2}e^{2F}\,d\sigma\leq e^{\int_{S^2}|\nabla F|^2\,d\sigma}$

2. Generalized Onofri inequalities

A first result: generalized Onofri inequalities

[J.D., M. Esteban, G. Tarantello. The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions. To appear in Annali SNS Pisa]

On $\mathbb{R}^2$ for $\alpha > -1$, consider the family of probability measures

$$d\mu_\alpha = \frac{\alpha+1}{\pi} \, \frac{|x|^{2\alpha}\, dx}{\left(1+|x|^{2\,(\alpha+1)}\right)^2}$$

Theorem 1. *Theorem [J.D., M. Esteban, G. Tarantello]*

$$\int_{\mathbb{R}^2} e^{v - \int_{\mathbb{R}^2} v\, d\mu_\alpha}\, d\mu_\alpha \;\le\; e^{\frac{1}{16\,\pi\,(\alpha+1)}\, \|\nabla v\|^2_{L^2(\mathbb{R}^2,\, dx)}}$$

holds in the space $\mathcal{E}_\alpha = \left\{ v \in L^1(\mathbb{R}^2, d\mu_\alpha) \,:\, |\nabla v| \in L^2(\mathbb{R}^2, dx) \right\}$ *restricted to radially symmetric functions* $\forall\, \alpha > -1$*, and without restriction iff* $\alpha \in (-1, 0]$

Proof (1/2)

Proposition 2. *Let $\alpha > -1$. For all $v \in \mathcal{E}_\alpha$, there holds*

$$\int_{\mathbb{R}^2} e^{v - \int_{\mathbb{R}^2} v \, d\mu_\alpha} \, d\mu_\alpha \le e^{\frac{1}{16\,\pi\,(\alpha+1)} \left(\|\nabla v\|_2^2 + \alpha\,(\alpha+2)\,\| \frac{1}{r}\,\partial_\theta v \,\|_2^2 \right)}$$

$\mathbb{C} \approx \mathbb{R}^2 \ni x = r\,e^{i\theta}$, $r \ge 0$, $\theta \in [0, 2\pi)$. Stereographic projection: Σ_0
Let $\alpha > -1$ and define the inverse of a dilated stereographic projection

$$\Sigma_\alpha^{-1}\big(r\,e^{i\theta}\big) := \left(\frac{2\,r^{\alpha+1}\,e^{i\theta}}{1 + r^{2(\alpha+1)}} \,,\, \frac{r^{2(\alpha+1)} - 1}{1 + r^{2(\alpha+1)}} \right) = \Sigma_0^{-1}\big(r^{1+\alpha}\,e^{i\theta}\big)$$

If $f \in C(\mathbb{R})$, $f(u)$, $|\nabla u|^2 \in L^1(S^2)$ and $v = u \circ \Sigma_\alpha^{-1}$, then

- $\int_{S^2} f(u)\,d\sigma = \int_{\mathbb{R}^2} f(v)\,d\mu_\alpha$
- $4\pi \int_{S^2} |\nabla u|^2\,d\sigma = \frac{1}{\alpha+1} \int_{\mathbb{R}^2} \left(|\nabla v|^2 + \alpha\,(\alpha+2) \left| \frac{1}{r}\,\partial_\theta v \right|^2 \right) dx$

The result follows from Onofri's inequality

Proof (2/2)

Corollary 3. *If $\alpha \in (-1, 0]$, then the inequality holds true for any $v \in \mathcal{E}_\alpha$*

$\alpha \in (-1, 0] \Longrightarrow \alpha\,(\alpha + 2) \le 0$

$$\|\nabla v\|_2^2 + \alpha\,(\alpha + 2)\,\|\,\frac{1}{r}\,\partial_\theta v\,\|_2^2 \le \|\nabla v\|_2^2$$

Proposition 4. *If $\alpha > 0$, then the inequality fails to hold in $\mathcal{E}_\alpha$*

Let $\alpha > 0$, $\varepsilon \in (0, 1)$, $\bar{x} = (1, 0)$

$$2\,v_\varepsilon = \begin{cases} \log\left(\frac{\varepsilon}{(\varepsilon + \pi\,|x - \bar{x}|^2)^2}\right) & \text{if} \quad |x - \bar{x}| \le 1 \\ \log\left(\frac{\varepsilon}{(\varepsilon + \pi)^2}\right) & \text{if} \quad |x - \bar{x}| > 1 \end{cases}$$

- $\lim_{\varepsilon \to 0} \mu_\alpha(e^{2 v_\varepsilon}) = \frac{\alpha + 1}{4\pi}$
- $\frac{1}{4\pi\,(\alpha + 1)}\,\|\nabla v_\varepsilon\|_2^2 + 2\,\mu_\alpha(v_\varepsilon) = \frac{\alpha}{1 + \alpha}\,\log\varepsilon + O(1)$ as $\varepsilon \to 0$

Generalized Onofri inequality on the cylinder

$$\mathfrak{E}_\alpha = \Big\{ w = w(t,\theta) \in L^1(\mathcal{C}, d\nu_\alpha) \;:\; |\nabla w| \in L^2(\mathcal{C}, dx) \Big\}$$

$$\mathcal{C} = \mathbb{R} \times S^1\,, \quad d\nu_\alpha := \frac{\alpha+1}{2}\, \frac{dt\, d\theta}{\big[\cosh\big((\alpha+1)\,t\big)\big]^2}$$

Proposition 5. *If $\alpha > -1$, then for any $w \in \mathfrak{E}_\alpha$,*

$$\int_{\mathcal{C}} e^{w - \int_{\mathcal{C}} w\, d\nu_\alpha}\, d\nu_\alpha \;\le\; e^{\frac{1}{16\,\pi\,(\alpha+1)}\left(\|\nabla w\|^2_{L^2(\mathcal{C})} + \alpha\,(\alpha+2)\,\|\,\partial_\theta w\,\|^2_{L^2(\mathcal{C})}\right)}$$

If $-1 < \alpha \le 0$, then for any $w \in \mathfrak{E}_\alpha$,

$$\int_{\mathcal{C}} e^{w - \int_{\mathcal{C}} w\, d\nu_\alpha}\, d\nu_\alpha \;\le\; e^{\frac{1}{16\,\pi\,(\alpha+1)}\,\|\nabla w\|^2_{L^2(\mathcal{C})}}$$

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3. Caffarelli-Kohn-Nirenberg inequalities

Caffarelli-Kohn-Nirenberg (Hardy-Sobolev) inequalities

[L. Caffarelli, R. Kohn, and L. Nirenberg. First order interpolation inequalities with weights. Compositio Math., 53 (3): 259-275, 1984]
[F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. Comm. Pure Appl. Math., 54 (2): 229-258, 2001]

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{bp}}\,dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2a}}\,dx \quad \forall\, u \in \mathcal{D}_{a,b}$$

$$a < b \le a+1\,, \quad p = \frac{2\,N}{N-2+2\,(b-a)]}\,, \quad a \le \frac{N-2}{2}\,, \quad N \ge 3$$

$$\mathcal{D}_{a,b} = \{|x|^{-b}\,u \in L^p(\mathbb{R}^N, dx)\,:\,|x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^N, dx)\}$$

The space $\mathcal{D}_{a,b}$ is obtained as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|u\|^2 = \|\,|x|^{-b}\,u\,\|_p^2 + \|\,|x|^{-a}\,\nabla u\,\|_2^2$

If $N=2$, same inequality for $a<0$

Inequalities on the cylinder

Emden-Fowler transformations

$$t = \log|x|\;, \quad \theta = \frac{x}{|x|} \in S^{N-1}\;, \quad w(t,\theta) = |x|^{\frac{N-2-2a}{2}}\,u(x)$$

CKN inequality for u is equivalent to a Sobolev inequality for w on $\mathbb{R} \times S^{N-1} =: \mathcal{C}$

$$\|w\|^2_{L^p(\mathbb{R}\times S^{N-1})} \leq C_{a,b}\left[\|\nabla w\|^2_{L^2(\mathbb{R}\times S^{N-1})} + \tfrac{1}{4}(N-2-2a)^2\|w\|^2_{L^2(\mathbb{R}\times S^{N-1})}\right]$$

for $w \in H^1(\mathbb{R} \times S^{N-1})$, with $p = 2\,N/[(N-2) + 2\,(b-a)]$, $a \neq (N-2)/2$

For $N = 2$, the inequality holds for functions $w = w(t,\theta)$ defined over the two-dimensional cylinder $\mathcal{C} = \mathbb{R} \times S^1 \approx \mathbb{R} \times (\mathbb{R}/2\pi\mathbb{Z})$

$$\|w\|^2_{L^p(\mathcal{C})} \leq C_{a,a+2/p}\left(\|\nabla w\|^2_{L^2(\mathcal{C})} + a^2\,\|w\|^2_{L^2(\mathcal{C})}\right) \quad \forall\, w \in H^1(\mathcal{C})$$

for all $a \neq 0$ and $p > 2$, with $b = a + 2/p$

An extended Caffarelli-Kohn-Nirenberg inequality

The "modified inversion symmetry" transforms extremal points into solutions of the same equation: $u(x) \mapsto \left|\frac{x}{\tau}\right|^{-(N-2-2a)} u\left(\tau^2 \frac{x}{|x|^2}\right) = v(x)$

Lemma 6. *[J.D., M. Esteban, G. Tarantello] If $N = 2$, then the CKN inequality holds for any $a \neq 0$ and b such that $a < b \leq a + 1$. If $N \geq 3$, then the CKN inequality holds for any $a \neq (N-2)/2$ and b such that $a \leq b \leq a + 1$.*

Let $N = 2$, $a > 0$, $a' = -a$, $b' = b - 2a \in (-a, -a+1]$

$$\int_{\mathbb{R}^2} \left(\frac{|v|^p}{|x|^{b'p}}\,dx\right)^{2/p} \leq C_{a',b'} \int_{\mathbb{R}^2} \frac{|\nabla v|^2}{|x|^{2a'}}\,dx \quad \text{in } \mathcal{D}_{a',b'}$$

$C_{a,b} = C_{a',b'}$, $4 - b'p = bp$, $-2a' = 2a$, $p = 2/(b'-a') = 2/(b-a)$

$$\int_{\mathbb{R}^2} \left(\frac{|u|^p}{|y|^{4-b'p}}\,dy\right)^{2/p} \leq C_{a',b'} \int_{\mathbb{R}^2} \frac{|\nabla u|^2}{|y|^{-2a'}}\,dy \quad \text{in } \mathcal{D}_{a,b}$$

$N \geq 3$: $a = N - 2 - a'$, $b\,p = 2N - b'\,p$, $p = \frac{2N}{N-2-2(b'-a')} = \frac{2N}{N-2-2(b-a)}$

Which space of functions ?

$\mathcal{D}^N_{a,b}$ is given by the completion with respect to $\|\cdot\|$ of the set

$$\{u \in C^\infty_c(\mathbb{R}^2) \,:\, \text{supp}(u) \subset \mathbb{R}^2 \setminus \{0\}\}$$

Norm: $\|u\|^2 = \|\,|x|^{-b}\,u\,\|_p^2 + \|\,|x|^{-a}\,\nabla u\,\|_2^2$

Radial symmetry of extremal functions...

[K. S. Chou and C. W. Chu. On the best constant for a weighted Sobolev-Hardy inequality. J. London Math. Soc. (2), 48 (1): 137-151, 1993]

If $N \geq 3$, $0 \leq a < (N-2)/2$, extremal functions are radially symmetric given up to scalar multiplication and dilation by

$$u_{a,b}^{\mathrm{rad}}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

For radial functions: Hardy-Sobolev inequalities in dimension "$N-2a$"

$$\left(\int_0^\infty \frac{|u|^p}{r^{bp}}\, r^{N-1}\, dr\right)^{2/p} \leq\, C_{a,b}\, |S^{N-1}|^{1-2/p} \int_0^\infty |u'|^2\, r^{N-2a-1}\, dr$$

With $w(t) = r^{\frac{N-2-2a}{2}}\, u(r)$, $t = \log r$, up to a scaling and a multiplication by a constant, w is a positive "Aubin-Talenti" type solution of

$$-w'' + w = w^{p-1}\,, \qquad \lim_{t \to \pm\infty} w(t) = 0$$

. . . vs. symmetry breaking

Extremals are known to be non-radially symmetric for a certain range of parameters (a, b) if $N \geq 3$

[F. Catrina and Z.-Q. Wang. On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions. Comm. Pure Appl. Math., 54 (2): 229-258, 2001]

[V. Felli and M. Schneider. Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type. J. Differential Equations, 191 (1): 121-142, 2003]

The case of dimension $N = 2$ will be considered later

The Onofri ineq. as a limit case of the CKN inequalities (1/2)

For $N=2$, $\alpha>-1$, $\varepsilon\in(0,1)$, $a<0$, let

$$a=-\frac{\varepsilon}{1-\varepsilon}\,(\alpha+1)\;,\quad b=a+\varepsilon\quad\text{and}\quad p=\frac{2}{\varepsilon}$$

Let $u_\varepsilon(x)=\Big(1+|x|^{2(\alpha+1)}\Big)^{-\frac{\varepsilon}{1-\varepsilon}}$ be a radial extremal function

κ_ε, λ_ε are numerical coefficients and f_ε is a weight

Lemma 7. *Let* $\alpha_0>-1$, $v\in C_c^\infty(\mathbb{R}^2)$, $w_\varepsilon=(1+\varepsilon\,v)\,u_\varepsilon$

$$\frac{1}{\kappa_\varepsilon}\int_{\mathbb{R}^2}\frac{|w_\varepsilon|^p}{|x|^{bp}}\,dx=\int_{\mathbb{R}^2}|1+\varepsilon\,v|^{\frac{2}{\varepsilon}}\;\frac{f_\varepsilon\,dx}{\int_{\mathbb{R}^2}f_\varepsilon\,dx}$$

and, as $\varepsilon\to0$*, uniformly with respect to* $\alpha\geq\alpha_0$*,*

$$\int_{\mathbb{R}^2}\frac{|\nabla w_\varepsilon|^2}{|x|^{2a}}\,dx=\lambda_\varepsilon+\varepsilon^2\left[\frac{8(1+\alpha)^2}{(1-\varepsilon)^2}\int_{\mathbb{R}^2}\frac{u_\varepsilon^{2/\varepsilon}\,v}{|x|^{2(a-\alpha)}}\,dx+\int_{\mathbb{R}^2}|\nabla v|^2\,\frac{u_\varepsilon^2}{|x|^{2a}}\,dx+O(a^2\varepsilon)\right]$$

The Onofri ineq. as a limit case of the CKN inequalities (1/2)

$$\frac{|x|^{-bp}\, f_\varepsilon\; dx}{\int_{\mathbb{R}^2} f_\varepsilon\; dx} \sim \frac{\alpha+1}{\pi}\,|x|^{2\alpha}\, u_\varepsilon^{2/\varepsilon}\, dx \sim d\mu_\alpha(x) = \frac{\alpha+1}{\pi}\,\frac{|x|^{2\alpha}\, dx}{(1+|x|^{2\,(\alpha+1)})^2}$$

as $\varepsilon \to 0_+$. With $w_\varepsilon = (1+\varepsilon\, v)\, u_\varepsilon$, we have, up to $O(a^2\varepsilon^2)$ terms,

$$\int_{\mathbb{R}^2} |1+\varepsilon\, v|^{\frac{2}{\varepsilon}}\; \frac{f_\varepsilon\; dx}{\int_{\mathbb{R}^2} f_\varepsilon\; dx} \lesssim \left(1+\frac{\varepsilon^2}{\lambda_\varepsilon}\left[\frac{8(1+\alpha)^2}{(1-\varepsilon)^2}\int_{\mathbb{R}^2}\frac{u_\varepsilon^{2/\varepsilon}\; v}{|x|^{2(a-\alpha)}}\; dx + \int_{\mathbb{R}^2}\frac{|\nabla v|^2\; u_\varepsilon^2}{|x|^{2a}}\; dx\right]\right)^{1/\varepsilon}$$

Proposition 8. *Let $\alpha > -1$, $\varepsilon_n \to 0$ such that the radial extremal function u_{ε_n} is also extremal for CKN with $p_n = \frac{2}{\varepsilon_n}$, $a_n = -\frac{\varepsilon_n}{1-\varepsilon_n}\,(\alpha+1)$, $b_n = a_n + \varepsilon_n$. Then the generalized Onofri inequality holds true in $\mathcal{E}_\alpha$*

$$\int_{\mathbb{R}^2} e^{v-\int_{\mathbb{R}^2} v\, d\mu_\alpha}\; d\mu_\alpha \;\le\; e^{\frac{1}{16\,\pi\,(\alpha+1)}\;\|\nabla v\|^2_{L^2(\mathbb{R}^2,\,dx)}}$$

NB. This gives a proof of the Onofri inequality in $\mathbb{R}^2$

4. Symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

A first result of symmetry breaking

Take $a_n < 0$, $0 < b_n - a_n = \varepsilon_n \to 0_+$, $\alpha_n = -1 - a_n\,(1-\varepsilon_n)/\varepsilon_n$

Corollary 9. *Let* $N = 2$*. For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that if* $|a| \in (0,\delta)$*,* $b \in (a, a+1)$*, if one of the following conditions holds*

(i) $a > 0$ *and* $b/a < 2 - \varepsilon$

(ii) $a < 0$ *and* $b/a > \varepsilon$

then CKN inequalities cannot admit a radially symmetric extremal

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- If α_n converges to some α_0, then the generalized Onofri inequality would be true without radial symmetry
- $\alpha_n \to \infty$ means $b_n/a_n \to 1$ and requires a special analysis

A second result of symmetry breaking

Adaptation to the case $N = 2$ of

[V. Felli and M. Schneider. Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type. J. Differential Equations, 191 (1): 121-142, 2003]

Theorem 10. *Let* $a \neq 0$ *and* $N = 2$*. If* $a < b < h(a) = a + \frac{|a|}{\sqrt{1+a^2}}$*, then CKN inequalities admit only* non radially symmetric *extremals*

Reformulation in the cylinder

Emden-Fowler transformations

$$t = \log|x|\,, \quad \theta = \frac{x}{|x|} \in S^{N-1}\,, \quad w(t,\theta) = |x|^{\frac{N-2-2a}{2}}\,u(x)$$

For $N=2$, CKN inequality for u is equivalent to a Sobolev inequality for $w = w(t,\theta)$ defined over the cylinder $\mathcal{C} = \mathbb{R}\times S^1 \approx \mathbb{R}\times(\mathbb{R}/2\pi\mathbb{Z})$

$$\|w\|^2_{L^p(\mathcal{C})} \le C_{a,a+2/p}\left(\|\nabla w\|^2_{L^2(\mathcal{C})} + a^2\,\|w\|^2_{L^2(\mathcal{C})}\right) \quad \forall\, w\in H^1(\mathcal{C})$$

for all $a\neq 0$ and $p>2$, with $b = a+2/p$. Equality is achieved if

$$\begin{cases} -(w_{tt}+w_{\theta\theta}) + a^2\,w = w^{p-1} \quad \text{in} \quad \mathbb{R}\times[-\pi,\pi] \\ w>0\,, \quad w(t,\cdot) \quad \text{is } 2\pi\text{-periodic} \quad \forall\, t\in\mathbb{R} \end{cases}$$

The radial solution $w^*_{a,p}(t) = \left(\frac{a^2\,p}{2}\right)^{\frac{1}{p-2}}\left[\cosh\left(\frac{p-2}{2}\,a\,t\right)\right]^{-\frac{2}{p-2}}$ is unique

a proof by linearization

Theorem 11. *Let $a \neq 0$, $p > 2$. If $|a|\,p > 2\sqrt{1+a^2}$, extremal functions are not radial*

Let $Q(\psi) := \|\nabla\psi\|^2_{L^2(\mathcal{C})} + a^2\,\|\psi\|^2_{L^2(\mathcal{C})} - (p-1)\int_{\mathcal{C}} |w^*_{a,p}|^{p-2}\,|\psi|^2\,dx$. On the set of functions $\psi \in H^1(\mathcal{C})$ such that $\int_{-\pi}^{\pi} \psi(t,\theta)\,d\theta = 0,\ t \in \mathbb{R}$ a.e.,

$$\inf \frac{Q(\psi)}{\|\psi\|^2_{L^2(\mathcal{C})}} = a^2 + 1 - \left(\frac{a\,p}{2}\right)^2$$

is achieved by $\psi(t,\theta) = \big(\cosh((\alpha+1)\,t)\big)^{-\frac{p}{p-2}}\cos\theta$, with $\alpha = (p-2)\,\frac{a}{2} - 1$

- decompose on Fourier modes
- derive the equation for $w^*_{a,p}$ and scale

5. A symmetry result for Caffarelli-Kohn-Nirenberg inequalities

Statement of the result

[J.D., M. Esteban, G. Tarantello. The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions. To appear in Annali SNS Pisa]

Theorem 12. *Let* $a \neq 0$ *and* $N = 2$*. For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for* $|a| \in (0, \delta)$, $b \in (a, a+1)$*, if one of the following conditions holds*

(i) $a > 0$ *and* $b/a > 2 + \varepsilon$

(ii) $a < 0$ *and* $b/a < -\varepsilon$

then the extremals of CKN inequalities are radially symmetric, *and given, up to scalar multiplication and dilation, by* $u_{a,b}^{\mathrm{rad}}$

This can be rewritten in the cylinder as

Theorem 13. *Let* $a \neq 0$, $p > 2$*. For every* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that, if* $0 < |a| < \delta$ *and* $|a|\, p < 2 - \varepsilon$*, then* $w_{a,p}^*$ *is an extremal function*

Previously known results

Moving planes: [K. S. Chou and C. W. Chu. On the best constant for a weighted Sobolev-Hardy inequality. J. London Math. Soc. (2), 48 (1): 137-151, 1993]

Symmetrization: [T. Horiuchi. Best constant in weighted Sobolev inequality with weights being powers of distance from the origin. J. Inequal. Appl., 1 (3): 275-292, 1997]

Partial symmetry: [C.-S. Lin and Z.-Q. Wang. Symmetry of extremal functions for the Caffarrelli-Kohn-Nirenberg inequalities. Proc. Amer. Math. Soc., 132 (6): 1685-1691 (electronic), 2004]

Schwartz foliated symmetry, symmetry close to $a = 0$, $N \geq 3$: [D. Smets and M. Willem. Partial symmetry and asymptotic behavior for some elliptic variational problems. Calc. Var. Partial Differential Equations, 18 (1): 57-75, 2003]

A preliminary result: a Pohozaev type identity

On $H^1(\mathcal{C}) \setminus \{0\}$ consider the functional

$$\mathcal{F}(w) = \frac{\|\nabla w\|^2_{L^2(\mathcal{C})} + a^2 \|w\|^2_{L^2(\mathcal{C})}}{\|w\|^2_{L^p(\mathcal{C})}}$$

It has a minimizer with symmetry and monotonicity (sliding method) properties

$$\begin{cases} w_{a,p}(t,\theta) = w_{a,p}(-t,\theta) \quad \forall\, t \in \mathbb{R}\,, \quad \theta \in [-\pi,\pi) \\ \frac{\partial w_{a,p}}{\partial t}(t,\theta) < 0 \quad \forall\, t > 0 \quad \forall\, \theta \in [-\pi,\pi) \\ \max_{\mathbb{R}\times[-\pi,\pi)} w_{a,p} = w_{a,p}(0,0) \end{cases}$$

Lemma 14.

$$\int_{-\pi}^{\pi} \left(\frac{\partial w}{\partial \theta}\right)^2 d\theta = \int_{-\pi}^{\pi} \left(\frac{\partial w}{\partial t}\right)^2 d\theta - a^2 \int_{-\pi}^{\pi} w^2\, d\theta + \frac{2}{p} \int_{-\pi}^{\pi} w^p\, d\theta$$

Proof. Multiply the equation by $\frac{\partial w}{\partial t}$ $(\approx r\,\partial(r^a\,u(r,\theta))/\partial r)$ and integrate over $[-\pi,\pi]$

$$\int_{-\pi}^{\pi}\left(-\frac{\partial^2 w}{\partial t^2}\,\frac{\partial w}{\partial t}-\frac{\partial^2 w}{\partial \theta^2}\,\frac{\partial w}{\partial t}+a^2\,\frac{\partial w}{\partial t}\,w\right)\,d\theta=\int_{-\pi}^{\pi} w^{p-1}\,\frac{\partial w}{\partial t}\,d\theta$$

$$\int_{-\pi}^{\pi}\left\{-\tfrac{\partial}{\partial\theta}\left(\tfrac{\partial w}{\partial\theta}\,\tfrac{\partial w}{\partial t}\right)+\tfrac{1}{2}\,\tfrac{d}{dt}\left[\left(\tfrac{\partial w}{\partial\theta}\right)^2-\left(\tfrac{\partial w}{\partial t}\right)^2+a^2\,w^2\right]\right\}d\theta=\frac{1}{p}\int_{-\pi}^{\pi}\frac{d\,(w^p)}{dt}\,d\theta$$

Since $\int_{-\pi}^{\pi}\frac{\partial}{\partial\theta}\left(\frac{\partial w}{\partial\theta}\,\frac{\partial w}{\partial t}\right)\,d\theta=0$, we get

$$\frac{d}{dt}\int_{-\pi}^{\pi}\left[\left(\frac{\partial w}{\partial t}\right)^2-\left(\frac{\partial w}{\partial\theta}\right)^2-a^2\,w^2+\frac{2}{p}\,w^p\right]d\theta=0$$

for all $t\in\mathbb{R}$. Hence as a function of t, the above integral must be a constant. Since it is also integrable over $\mathbb{R}\ni t$, then it must vanish identically $\square$

Proof of the symmetry result

The method is based on the strong convergence properties of a suitable rescaling of the minimizer $w_{a,p}$ of $\mathcal{F}$ towards a solution of a Liouville equation. We argue by contradiction and suppose that there exists $\varepsilon_0 \in (0,1)$ and, for all $n \in \mathbb{N}$, $a_n > 0$, $p_n > 2$, such that

$$\lim_{n\to+\infty} a_n = 0\,,\quad a_n\,p_n < 2-\varepsilon_0 \quad \text{and} \quad \mathcal{F}(w_{a_n,\,p_n}) < \mathcal{F}(w^*_{a_n,\,p_n})$$

For simplicity, set $w_n = w_{a_n,\,p_n}$ and $w_n^* = w^*_{a_n,\,p_n}$. By the previous identity and using symmetry,

$$\frac{p_n^2\,a_n^2}{2}\int_{-\pi}^{\pi} w_n^2(0,\theta)\,d\theta \le p_n\int_{-\pi}^{\pi} w_n^{p_n}(0,\theta)\,d\theta \le p_n\,\|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})}\int_{-\pi}^{\pi} w_n^2(0,\theta)\,d\theta$$

Lemma 15. $p_n\,\|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})} \ge \frac{1}{2}\,p_n^2\,a_n^2$

Lemma 16. $\liminf_{n\to+\infty} p_n\,\|w_n\|^{p_n-2}_{L^\infty(\mathcal{C})} \ge 1$

Next we introduce the new parameters

$$\varepsilon_n = \frac{2}{p_n} \quad \text{and} \quad \alpha_n = -1 + (1-\varepsilon_n)\,\frac{a_n}{\varepsilon_n} = -1 + \frac{1}{2}\,(1-\varepsilon_n)\,a_n\,p_n$$

Lemma 17. $\lim_{n\to+\infty} \alpha_n = \alpha \in [-1,0)$, $\lim_{n\to+\infty} p_n = +\infty$ *and* $\lim_{n\to+\infty} \varepsilon_n = 0$

If $\liminf_{n\to+\infty} w_n(0,0) < 1$, then $\liminf_{n\to+\infty} p_n\,\|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2} = 0$

Lemma 18. $\liminf_{n\to+\infty} w_n(0,0) \geq 1$

Lemma 19. $\limsup_{n\to+\infty} p_n \, \|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2} < +\infty$

By contradiction: if $\delta_n = \left(p_n \, \|w_n\|_{L^\infty(\mathcal{C})}^{p_n-2}\right)^{-1/2} \to 0$, then

$W_n(t,\theta) = p_n \left(\frac{w_n(\delta_n \, t, \delta_n \, \theta)}{w_n(0,0)} - 1\right)$ on $\mathcal{C}_n = \mathbb{R} \times [-\pi/\delta_n, \, \pi/\delta_n]$ satisfies

$$\begin{cases} -\Delta W_n = \left(1 + \frac{W_n}{p_n}\right)^{p_n-1} - \, a_n^2 \, p_n \, \delta_n^2 \, \left(1 + \frac{W_n}{p_n}\right) \quad \text{in} \quad \mathcal{C}_n \\ \\ W_n \leq 0 = W_n(0,0) \end{cases}$$

$$\lim_{n\to+\infty} \|1 + W_n/p_n\|_{L^{p_n}(\mathcal{C}_n)}^{p_n} \leq \lim_{n\to+\infty} \frac{1}{w_n(0,0)^2} \, p_n \int_{\mathcal{C}} |w_n^*|^{p_n} \, dx \leq 8\pi \, (1+\alpha)$$

Harnack's inequality, elliptic regularity theory: W_n converges pointwise to W (change $\mathcal{C}$ to $\mathbb{R}^2$) which satisfies

$$-\Delta W = e^W \quad \text{in} \quad \mathbb{R}^2$$

[W. X. Chen and C. Li. Classification of solutions of some nonlinear elliptic equations. Duke Math. J., 63 (3): 615-622, 1991]

Every solution W of $-\Delta W = e^W$ in $\mathbb{R}^2$ with $e^W \in L^1(\mathbb{R}^2)$, must satisfy $\int_{\mathbb{R}^2} e^W \, dx = 8\pi$

By Fatou's Lemma,

$$\int_{\mathbb{R}^2} e^W \, dx \leq \lim_{n\to+\infty} \int_{\mathcal{C}_n} \left(1 + \frac{W_n}{p_n}\right)^{p_n} dx \leq 8\pi\,(1+\alpha) < 8\pi$$

a contradiction, as $\alpha \in [-1, 0)$

Some technical consequences

- $\lim_{n\to+\infty} w_n(0,0) = 1$
- $\lim_{n\to+\infty} \left[w_n(0,0)\right]^{p_n} = 0$
- $\lim_{n\to+\infty} p_n \left[w_n(0,0)\right]^{p_n - 2} = \mu \in [1, +\infty)$

Define the function $V_n(t,\theta) := p_n \left(\frac{w_n(t,\theta)}{w_n(0,0)} - 1 \right) \quad \forall\, (t,\theta) \in \mathcal{C}$

Lemma 20. *Up to a subsequence, V_n converges to a function V pointwise and C^2-uniformly in any compact set in $\mathbb{R} \times [-\pi,\pi]$, and*

$$\begin{cases} -\Delta V = \mu\, e^V \quad \textit{in} \quad \mathcal{C} \\ \\ \max_{\mathcal{C}} V \le 0 = V(0,0)\ , \quad V(t,\cdot) \quad \textit{is}\ 2\pi\textit{-periodic} \quad \forall\, t \in \mathbb{R} \\ \\ \mu \int_{\mathcal{C}} e^V\, dx \ \le\ 8\pi\,(1+\alpha) \end{cases}$$

$$V(t,\theta) = V(-t,\theta)\ , \quad \frac{\partial V}{\partial t}(t,\theta) < 0 \quad \forall\, t > 0\ , \quad \forall\, \theta \in [-\pi,\pi]$$

$$\int_{-\pi}^{\pi} \left(\frac{\partial V}{\partial \theta}\right)^2 d\theta = \int_{-\pi}^{\pi} \left(\frac{\partial V}{\partial t}\right)^2 d\theta - 8\pi\,(1+\alpha)^2 + 2\mu \int_{-\pi}^{\pi} e^V\, d\theta \quad \forall\, t \in \mathbb{R}$$

Lemma 21. $\mu = 2\,(\alpha+1)^2, V(t) = -2\log\big[\cosh((\alpha+1)\,t)\big]$ *and*

$$\lim_{n\to+\infty} p_n\left(\|w_n\|^{p_n}_{L^{p_n}(\mathcal{C})} - \|w_n^*\|^{p_n}_{L^{p_n}(\mathcal{C})}\right) = 0$$

$$\int_{\mathcal{C}} e^V\,dx = \lim_{n\to+\infty}\int_{\mathcal{C}}\left(1+\frac{V_n}{p_n}\right)^{p_n} dx = \frac{4\pi}{\alpha+1}$$

$$\lim_{n\to+\infty}\sup_{\mathcal{C}}\left|\left(\tfrac{w_n}{w_n(0,0)}\right)^{p_n-2} - e^V\right| = 0$$

Proof. The function $\varphi(r,\theta) := V\,(-\log r\,,\,\theta) - 2\log r + \log\mu$ satisfies

$$-\Delta\varphi = -r^{-2}\,(V_{tt}+V_{\theta\theta})\,(-\log r\,,\,\theta) = e^\varphi \quad \text{in } \mathbb{R}^2\backslash\{0\}$$

$$\int_{\mathbb{R}^2} e^\varphi\,dx \le 8\pi\,(1+\alpha)\,, \quad \varphi\left(r^{-1}\,\theta\right) = \varphi(r,\theta) + 4\log r$$

[K. S. Chou and T. Y.-H. Wan. Asymptotic radial symmetry for solutions of $\Delta u + e^u = 0$ in a punctured disc. Pacific J. Math., 163 (2): 269-276, 1994]

Lemma 22. *For n large enough, we have $w_n = w_n^*$*

Proof. $\chi_n := \frac{\partial w_n}{\partial \theta} \in H^1(\mathcal{C})$ satisfies $\int_{-\pi}^{\pi} \chi_n(t,\theta)\, d\theta = 0$ and

$$-\Delta\chi_n + a_n^2\, \chi_n = (p_n - 1)\, \big(w_n(t,\theta)\big)^{p_n-2}\, \chi_n$$

$$\|\nabla\chi_n\|_{L^2}^2 + a_n^2\, \|\chi_n\|_{L^2}^2 = (p_n - 1)\, \int_{\mathcal{C}} \left(\frac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2} \chi_n^2\, dx$$

$$\|\nabla\chi_n\|_{L^2}^2 + a_n^2\,\|\chi_n\|_{L^2}^2 = (p_n - 1)\,\int_{\mathcal{C}} \left(\frac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2} \chi_n^2\,dx$$

$$\begin{aligned} 0 = & \quad \|\nabla\chi_n\|_2^2 + a_n^2\,\|\chi_n\|_{L^2}^2 - (p_n-1)\,\textstyle\int_{\mathcal{C}} \big(w_n(t,\theta)\big)^{p_n-2}\chi_n^2\,dx \\ \geq & \;\Big[1 + a_n^2 - (\alpha+1)^2 - (p_n-1)\,\big(w_n(0,0)\big)^{p_n-2}\,r_n\Big]\,\|\chi_n\|_{L^2(\mathcal{C})}^2 \\ & \quad +\Big[2\,(\alpha+1)^2 - (p_n-1)\,(w_n(0,0))^{p_n-2}\Big]\int_{\mathcal{C}} \frac{\chi_n^2}{\big(\cosh((\alpha+1)\,t)\big)^2}\,dx \end{aligned}$$

- $r_n := \sup_{\mathcal{C}} \left|\big(w_n(t,\theta)/w_n(0,0)\big)^{p_n-2} - e^V\right| \to 0$
- $\lim_{n\to+\infty}(p_n-1)(w_n(0,0))^{p_n-2} = \mu = 2\,(\alpha+1)^2$
- $a_n \to 0$
- $(1+\alpha)^2 < 1$

... a contradiction for large n, unless $\chi_n \equiv 0$: $w_n = w_n^*$ $\square$