Des équations de dérive-diffusion avec champ moyen aux inégalités de Hardy-Littlewood-Sobolev inversées

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Reverse HLS: joint work with J. A. Carrillo, M. G. Delgadino, R. Frank, F. Hoffmann

Outline

- **Sharp asymptotics for the subcritical Keller-Segel model**
  - ▷ An introduction to the Keller-Segel model
  - ▷ Functional framework and sharp asymptotics
- **Reverse HLS inequality**
  - ▷ The inequality and the conformally invariant case
  - ▷ A proof based on Carlson's inequality, the case $\lambda = 2$
  - ▷ Concentration and a relaxed inequality
- **Existence of minimizers and relaxation**
  - ▷ Existence minimizers if $q > 2N/(2N+\lambda)$
  - ▷ Rlaxation and measure valued minimizers
- **Regions of no concentration and regularity of measure valued minimizers**
  - ▷ No concentration results
  - ▷ Regularity issues
- **Free Energy**
  - ▷ Free energy: toy model, equivalence with reverse HLS ineq.
  - ▷ Relaxed free energy, uniqueness

Sharp asymptotics for the subcritical Keller-Segel model

1. Literature is huge
2. Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
3. Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

- do not specialize to radial solutions
- put emphasis on functional analysis
- insist on nonlinear evolution
- deal with the subcritical case

The parabolic-elliptic Keller – Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\,\nabla v) & x \in \mathbb{R}^2\,,\ t>0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t>0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,u(t,y)\,dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\,u(t,y)\,dy$$

Mass conservation: $\dfrac{d}{dt}\displaystyle\int_{\mathbb{R}^2} u(t,x)\,dx = 0$

Blow-up: the virial computation

Collapse (S. Childress, J.K. Percus 81) $M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 \, n_0 \, dx < \infty$: blow-up in finite time

A solution u of

$$\frac{\partial u}{\partial t} = \nabla \cdot (\nabla u - u \, \nabla v)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, u(t,x) \, dx$$

$$= - \underbrace{\int_{\mathbb{R}^2} 2\, x \cdot \nabla u \, dx}_{-4\, M} + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \underbrace{\tfrac{2x \cdot (y-x)}{|x-y|^2} \, u(t,x) \, u(t,y) \, dx \, dy}_{\tfrac{(x-y)\cdot(y-x)}{|x-y|^2} \, u(t,x)\, u(t,y) \, dx\, dy}$$

$$= 4\, M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

The super-critical range: regularization & life after blow-up

Regularize the Poisson kernel

$$(-\Delta)_\varepsilon^{-1} * \rho\,(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log(|x-y|+\varepsilon)\,\rho(y)\,dy$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561]

Proposition (JD, C. Schmeiser 2009)

For every $\varepsilon > 0$, the regularized problem has a global solution satisfying

$$\|\rho^\varepsilon(\cdot,t)\|_{L^1(\mathbb{R}^2)} = \|\rho_0\|_{L^1(\mathbb{R}^2)} := M$$

$$\|\rho^\varepsilon(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \le c\left(1+\frac{1}{\varepsilon^2}\right)$$

with an ε-independent constant c

The nonlinear term

$$m^\varepsilon(t,x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x-y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)dy \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x|+\varepsilon)}$$

Lemma (Poupaud)

The families $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ and $\{m^\varepsilon(t)\}_{\varepsilon>0}$ are tightly bounded locally uniformly in t, and $\{\rho^\varepsilon(t)\}_{\varepsilon>0}$ is tightly equicontinuous in t

Tight boundedness and equicontinuity of $\rho^\varepsilon(t) \Longrightarrow$ compactness

$$\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho^\varepsilon(t,x)\,\rho^\varepsilon(t,y)\,dx\,dy \to \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \varphi(x,y)\,\rho(t,x)\,\rho(t,y)\,dx\,dy$$

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m^\varepsilon(t,x)\,dx\,dt \to \int_{t_1}^{t_2}\int_{\mathbb{R}^2} \varphi(t,x)\,m(t,x)\,dx\,dt$$

for all $\varphi \in C_b([t_1,t_2]\times\mathbb{R}^2)$

Defect measure

$$\nu(t,x) = m(t,x) - \int_{\mathbb{R}^2} \mathcal{K}(x-y)\,\rho(t,x)\,\rho(t,y)\,dy\,, \quad \mathcal{K}(x) = \frac{x^{\otimes 2}}{|x|^2}$$

Atomic support

The limit is characterized by the pair (ρ, ν), the atomic support of ρ is an at most countable set

Lemma (Poupaud 2002)

ν is symmetric, nonnegative, and satisfies

$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)$$

$\mathcal{M}$: Radon measures, $\mathcal{M}_1^+$: nonnegative bounded measures

$$\mathcal{DM}^+(I;\mathbb{R}^2) = \Big\{(\rho,\nu) : \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2)\ \forall t \in I,\ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2\times 2}$$
$$\rho \text{ is tightly continuous with respect to } t$$
$$\nu \text{ is a nonnegative, symmetric, matrix valued measure}$$
$$\mathrm{tr}(\nu(t,x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x-a)\Big\}$$

Limiting problem

$$\forall \varphi \in C^1_b((0,T)\,,\times\mathbb{R}^2) \quad \int_0^T \int_{\mathbb{R}^2} \varphi(t,x)\, j[\rho,\nu](t,x)\, dx\, dt$$
$$= -\tfrac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\varphi(t,x)-\varphi(t,y))\, K(x-y)\, \rho(t,x)\, \rho(t,y)\, dx\, dy\, dt$$
$$- \tfrac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t,x) \nabla\varphi(t,x)\, dx\, dt$$

Theorem (JD, C. Schmeiser 2009)

For every $T>0$, ρ^ε converges tightly and uniformly in time to $\rho(t)$ and there exists $\nu(t)$ such that $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$ is a generalized solution of

$$\partial_t \rho + \nabla\cdot(j[\rho,\nu] - \nabla\rho) = 0$$

$\rho(t=0)=\rho_0$ holds in the sense of tight continuity

Strong formulation (formal) : an *ansatz*

- $\rho = \overline{\rho} + \hat{\rho}$, $\hat{\rho}(t,x) = \sum_{n\in N} M_n(t)\,\delta_n(t,x)$, $\delta_n(t,x) = \delta(x - x_n(t))$
- $(\rho,\nu) \in \mathcal{DM}^+((0,T);\mathbb{R}^2)$

$$\Longrightarrow \nu(t,x) = \sum_{n\in N} \nu_n(t)\,\delta_n(t,x), \;\; \mathrm{tr}(\nu_n) \le M_n^2$$

$$j[\rho,\nu] = \overline{\rho}\,\nabla S_0[\overline{\rho}+\hat{\rho}] + \sum_n M_n\,\delta_n\,\nabla S_0\left[\overline{\rho} + \sum_{m\neq n} M_m\,\delta_m\right] + \frac{1}{4\pi}\sum_n M_n\,\nu_n\,\nabla\delta_n$$

$$\begin{aligned}\partial_t\overline{\rho} + \nabla\cdot(\overline{\rho}\,\nabla S_0[\overline{\rho}] - \nabla\overline{\rho}) + \nabla\overline{\rho}\cdot\nabla S_0[\hat{\rho}]& \\ + \sum_n \delta_n(\dot{M}_n - \overline{\rho}\,M_n)& \\ - \sum_n M_n\,\nabla\delta_n\left(\dot{x}_n - \nabla S_0\left[\overline{\rho} + \textstyle\sum_{m\neq n} M_m\,\delta_m\right]\right)& \\ + \sum_n \left(\tfrac{1}{4\pi}\nu_n : \nabla^2\delta_n - M_n\,\Delta\delta_n\right) &= 0\end{aligned}$$

Keller-Segel model: the subcritical range

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence (W. Jäger, S. Luckhaus 1992), (JD, B. Perthame 2004), (A. Blanchet, JD, B. Perthame 2006)

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla \, (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla \, (\log u) - \nabla v \right|^2 \, dx$$

(log HLS) inequality (E. Carlen, M. Loss 1992):
F is bounded from below if $M \leq 8\pi$

... $M = 8\pi$ the critical case (A. Blanchet, J.A. Carrillo, N. Masmoudi 2008), (A. Blanchet et al.)

The existence setting for the subcritical regime

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \; n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \; M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\, \pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx \;\forall\, t \geq 0$

$v = -\frac{1}{2\pi} \, \log |\cdot| * u$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

(A. Blanchet, JD, B. Perthame) Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means *intermediate asymptotics* in original variables:

$$\left\|u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right)\right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry (Y. Naito)
- Uniqueness (P. Biler, G. Karch, P. Laurençot, T. Nadzieja)
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$ (A. Blanchet, JD, B. Perthame)
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

(D.D. Joseph, T.S. Lundgren) (JD, R. Stańczy)

The stationary solution when mass varies

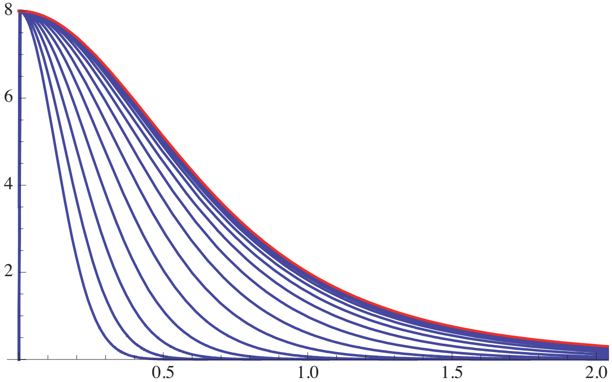


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla\Big[n\,(\log n - x + \nabla c)\Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \; dx + \int_{\mathbb{R}^2} \frac{1}{2}\,|x|^2\, n \; dx - \frac{1}{2}\int_{\mathbb{R}^2} n\, c\; dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = -\int_{\mathbb{R}^2} n\,|\nabla\,(\log n) + x - \nabla c|^2 \; dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2\, n(\lambda\, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2)\; dx + \int_{\mathbb{R}^2} \tfrac{\lambda^{-2}-1}{2}\,|x|^2\, n\; dx + \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\, \log\frac{1}{\lambda}\; dx$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi}\right)}_{>0 \text{ if } M<8\pi} \log\lambda + \frac{\lambda^{-2}-1}{2}\int_{\mathbb{R}^2} |x|^2\, n\; dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

A parametrization of the solutions and the linearized operator

(J. Campos, JD)

$$-\Delta c = M\,\frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\varphi'' - \frac{1}{r}\,\varphi' = e^{-\frac{1}{2}\,r^2+\varphi}\,,\quad r>0$$

with initial conditions $\varphi(0)=a$, $\varphi'(0)=0$ and get with $r=|x|$

$$M(a) := 2\pi\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\varphi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\varphi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\varphi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\varphi_a(r)}$$

Mass

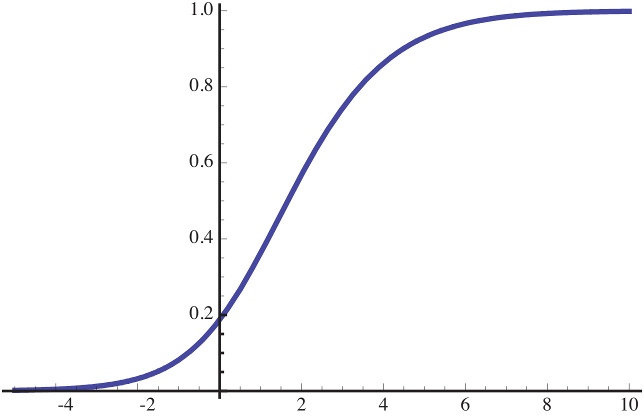


Figure: *The mass can be computed as* $M(a) = 2\pi \int_0^\infty n_a(r)\, r\, dr$. *Plot of* $a \mapsto M(a)/8\pi$

Bifurcation diagram

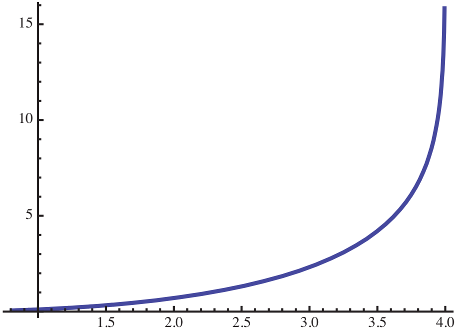


Figure: *The bifurcation diagram can be parametrized by* $a \mapsto \left(\frac{1}{2\pi} M(a), \|c_a\|_\infty\right)$ *with* $\|c_a\|_\infty = c_a(0) = a - b(a)$ *(cf. Keller-Segel system in a ball with no flux boundary conditions)*

Linearization

We can introduce two functions f and g such that

$$n = n_\infty\,(1+f) \quad \text{and} \quad c = c_\infty(1+g)$$

and rewrite the Keller-Segel model as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + \frac{1}{n_\infty}\,\nabla(f\, n_\infty\, \nabla(c_\infty\, g))$$

where the linearized operator is

$$\mathcal{L}\, f = \frac{1}{n_\infty}\nabla\cdot\big(n_\infty \nabla(f - c_\infty\, g)\big)$$

and

$$-\Delta(c_\infty\, g) = n_\infty\, f$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

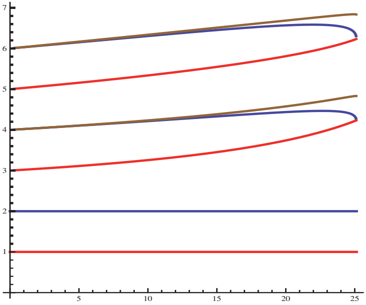


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a\, f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

(A. Blanchet, JD, M. Escobedo, J. Fernández), (J. Campos, JD), (V. Calvez, J.A. Carrillo), (J. Bedrossian, N. Masmoudi)

Functional framework and sharp asymptotics

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality (A. Blanchet, JD, B. Perthame)

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

$$\langle f, f \rangle := \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx - \int_{\mathbb{R}^2} f\, n_\infty\, (g\, c_\infty)\, dx$$

Eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_\infty\,dx - \int_{\mathbb{R}^2} f_1\,n_\infty\,(g_2\,c_\infty)\,dx$$

on the orthogonal space to f_0 in $L^2(n_\infty\,dx)$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_\infty)|^2\,n_\infty\,dx \quad \text{with} \quad g = -\frac{1}{c_\infty}\,\frac{1}{2\pi}\,\log|\cdot| * (f\,n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\, f = \frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f - c_\infty\, g) \big)$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \le \langle \mathcal{L}\, f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. Exponential decay:

$$\frac{d}{dt} \langle f, f \rangle = -\, 2 \, \langle \mathcal{L}\, f, f \rangle$$

(J. Campos, JD, 2014) (G. E. Fernández, S. Mischler, 2016)

Reverse Hardy-Littlewood-Sobolev inequality

The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2\,N - q\,(2\,N+\lambda)}{N\,(1-q)}$$

Convention: $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N} |\rho(x)|^p \, dx$ for any $p > 0$

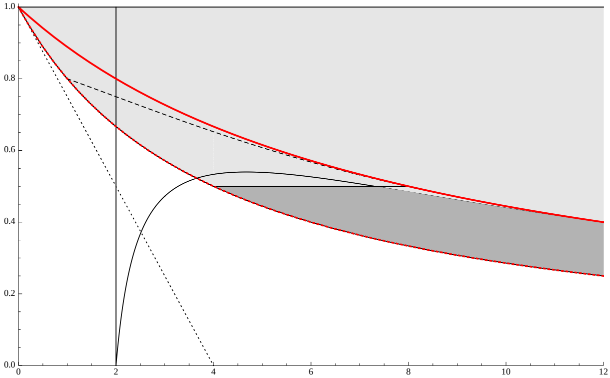
Theorem

The inequality

$$I_\lambda[\rho] \; \geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \tag{1}$$

holds for any $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ *with* $\mathcal{C}_{N,\lambda,q} > 0$ *if and only if* $q > N/(N+\lambda)$

If either $N = 1, 2$ *or if* $N \geq 3$ *and* $q \geq \min\left\{1 - 2/N \,,\, 2N/(2N+\lambda)\right\}$, *then there is a radial nonnegative optimizer* $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$
Optimal functions exist in the light grey area

The conformally invariant case $q = 2N/(2N+\lambda)$

$$I_\lambda[\rho] = \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy \ \ge \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{2/q}$$

$$2N/(2N+\lambda) \quad \Longleftrightarrow \quad \alpha = 0$$

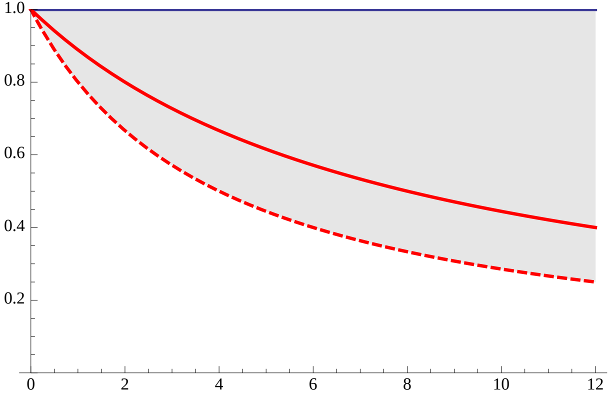
(Dou, Zhu 2015) (Ngô, Nguyen 2017)

The optimizers are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1 + |x|^2\right)^{-N/q} \quad \forall \, x \in \mathbb{R}^N$$

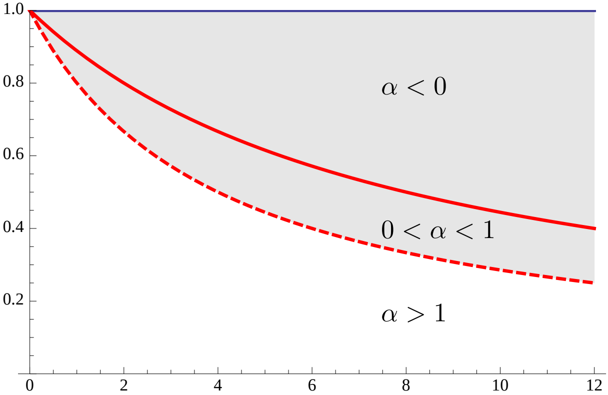
and the value of the optimal constant is

$$\mathcal{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}} \, \frac{\Gamma\left(\frac{N}{2} + \frac{\lambda}{2}\right)}{\Gamma\left(N + \frac{\lambda}{2}\right)} \left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)} \right)^{1+\frac{\lambda}{N}}$$



$N = 4$, region of the parameters (λ, q) for which $\mathcal{C}_{N,\lambda,q} > 0$
The plain, red curve is the conformally invariant case $\alpha = 0$

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \ge \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$



A Carlson type inequality

Lemma

Let $\lambda > 0$ *and* $N/(N+\lambda) < q < 1$

$$\left(\int_{\mathbb{R}^N} \rho\,dx\right)^{1-\frac{N\,(1-q)}{\lambda\,q}} \left(\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx\right)^{\frac{N\,(1-q)}{\lambda\,q}} \ge c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{1}{q}}$$

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}} \left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}$$

Equality is achieved if and only if

$$\rho(x) = \left(1+|x|^\lambda\right)^{-\frac{1}{1-q}}$$

up to dilations and constant multiples

(Carlson 1934) (Levine 1948)

An elementary proof of Carlson's inequality

$$\int_{\{|x|<R\}} \rho^q \, dx \le \left(\int_{\mathbb{R}^N} \rho \, dx \right)^q |B_R|^{1-q} = C_1 \left(\int_{\mathbb{R}^N} \rho \, dx \right)^q R^{N\,(1-q)}$$

and

$$\int_{\{|x|\ge R\}} \rho^q \, dx \le \left(\int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \right)^q \left(\int_{\{|x|\ge R\}} |x|^{-\frac{\lambda\, q}{1-q}} \, dx \right)^{1-q}$$
$$= C_2 \left(\int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \right)^q R^{-\lambda q + N\,(1-q)}$$

and optimize over $R > 0$
... existence of a radial monotone non-increasing optimal function; rearrangement; Euler-Lagrange equations

Proposition

Let $\lambda > 0$. *If* $N/(N+\lambda) < q < 1$, *then* $\mathcal{C}_{N,\lambda,q} > 0$

By rearrangement inequalities: prove the reverse HLS inequality for symmetric non-increasing ρ's so that

$$\int_{\mathbb{R}^N} |x-y|^\lambda \, \rho(y) \, dx \geq \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \quad \text{for all} \quad x \in \mathbb{R}^N$$

implies

$$I_\lambda[\rho] \geq \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \int_{\mathbb{R}^N} \rho \, dx$$

In the range $\frac{N}{N+\lambda} < q < 1$

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x) \, dx\right)^\alpha} \geq \left(\int_{\mathbb{R}^N} \rho \, dx \, dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \geq c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{\frac{2-\alpha}{q}}$$

and conclude with Carlson's inequality

The case $\lambda = 2$

Corollary

Let $\lambda = 2$ and $N/(N+2) < q < 1$. Then the optimizers for (1) are given by translations, dilations and constant multiples of

$$\rho(x) = \left(1 + |x|^2\right)^{-\frac{1}{1-q}}$$

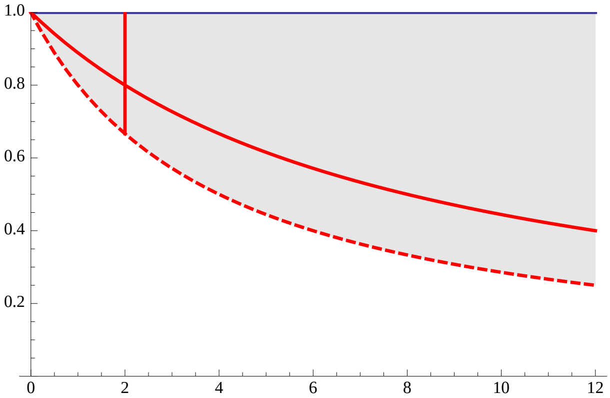
and the optimal constant is

$$\mathcal{C}_{N,2,q} = \frac{1}{2}\, c_{N,2,q}^{\frac{2\,q}{N\,(1-q)}}$$

By rearrangement inequalities it is enough to prove (7) for symmetric non-increasing ρ's, and so $\int_{\mathbb{R}^N} x\,\rho\,dx = 0$. Therefore

$$I_2[\rho] = 2\int_{\mathbb{R}^N} \rho\,dx \int_{\mathbb{R}^N} |x|^2 \rho\,dx$$

and the optimal function is optimal for Carlson's inequality



$N = 4$, *region of the parameters* (λ, q) *for which* $\mathcal{C}_{N,\lambda,q} > 0$. *The dashed, red curve is the threshold case* $q = N/(N+\lambda)$

The threshold case $q = N/(N+\lambda)$ and below

Proposition

If $0 < q \le N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0 = \lim_{q \to N/(N+\lambda)_+} \mathcal{C}_{N,\lambda,q}$

Let ρ, $\sigma \ge 0$ such that $\int_{\mathbb{R}^N} \sigma \, dx = 1$, smooth (+ compact support)

$$\rho_\varepsilon(x) := \rho(x) + M\, \varepsilon^{-N} \sigma(x/\varepsilon)$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon \, dx = \int_{\mathbb{R}^N} \rho \, dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon^q \, dx \to \int_{\mathbb{R}^N} \rho^q \, dx \quad \text{as} \quad \varepsilon \to 0_+$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \quad \text{as} \quad \varepsilon \to 0_+$$

If $0 < q < N/(N+\lambda)$, *i.e.*, $\alpha > 1$, take ρ_ε as a trial function,

$$\mathcal{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx}{\left(\int_{\mathbb{R}^N} \rho \, dx + M \right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q}} =: \mathcal{Q}[\rho, M]$$

The threshold case: If $\alpha = 1$, *i.e.*, $q = N/(N+\lambda)$, by taking the limit as $M \to +\infty$, we obtain

$$\mathcal{C}_{N,\lambda,q} \le \frac{2\int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx}{\left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{(2-\alpha)/q}}$$

For any $R > 1$, we take

$$\rho_R(x) := |x|^{-(N+\lambda)} \, \mathbb{1}_{1 \le |x| \le R}(x)$$

Then

$$\int_{\mathbb{R}^N} |x|^\lambda \, \rho_R \, dx = \int_{\mathbb{R}^N} \rho_R^q \, dx = \left|\mathbb{S}^{N-1}\right| \, \log R$$

and, as a consequence,

$$\frac{\int_{\mathbb{R}^N} |x|^\lambda \, \rho_R \, dx}{\left(\int_{\mathbb{R}^N} \rho_R^{N/(N+\lambda)} \, dx\right)^{(N+\lambda)/N}} = \left(\left|\mathbb{S}^{N-1}\right| \, \log R\right)^{-\lambda/N} \to 0 \quad \text{as} \quad R \to \infty$$

This proves that $\mathcal{C}_{N,\lambda,q} = 0$ for $q = N/(N+\lambda)$

A relaxed inequality

$$I_\lambda[\rho]+2M\int_{\mathbb{R}^N}|x|^\lambda\,\rho\,dx\;\geq\;\mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx+M\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}\tag{2}$$

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality (2) holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (1) and admits an optimizer (ρ, M)

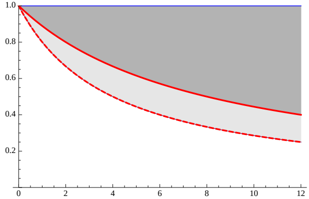
Heuristically, this is the extension of the reverse HLS inequality (1)

$$I_\lambda[\rho]\;\geq\;\mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$

to measures of the form $\rho + M\,\delta$

Existence of minimizers and relaxation

Existence of a minimizer: first case



The $\alpha < 0$ case: dark grey region

Proposition

If $\lambda > 0$ and $\frac{2N}{2N+\lambda} < q < 1$, there is a minimizer ρ for $\mathcal{C}_{N,\lambda,q}$

The limit case $\alpha = 0$, $q = \frac{2N}{2N+\lambda}$ is the *conformally invariant* case: see (Dou, Zhu 2015) and (Ngô, Nguyen 2017)

A minimizing sequence ρ_j can be taken radially symmetric non-increasing by rearrangement, and such that

$$\int_{\mathbb{R}^N} \rho_j(x)\,dx = \int_{\mathbb{R}^N} \rho_j(x)^q\,dx = 1 \quad \text{for all } j \in \mathbb{N}$$

Since $\rho_j(x) \le C \min\left\{|x|^{-N},\, |x|^{-N/q}\right\}$ by Helly's selection theorem we may assume that $\rho_j \to \rho$ a.e., so that

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] \quad \text{and} \quad 1 \ge \int_{\mathbb{R}^N} \rho(x)\,dx$$

by Fatou's lemma. Pick $p \in (N/(N+\lambda), q)$ and apply (1) with the same λ and $\alpha = \alpha(p)$:

$$I_\lambda[\rho_j] \ge \mathcal{C}_{N,\lambda,p} \left(\int_{\mathbb{R}^N} \rho_j^p\,dx\right)^{(2-\alpha(p))/p}$$

Hence the ρ_j are uniformly bounded in $\mathrm{L}^p(\mathbb{R}^N)$: $\rho_j(x) \le C'\,|x|^{-N/p}$,

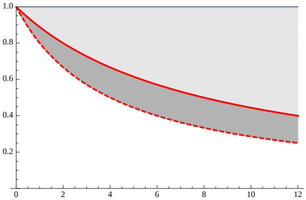
$$\int_{\mathbb{R}^N} \rho_j^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx = 1$$

by dominated convergence

Existence of a minimizer: second case

If $N/(N+\lambda) < q < 2N/(2N+\lambda)$ we consider the *relaxed inequality*

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \;\geq \mathcal{C}_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho \, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{(2-\alpha)/q}$$



The $0 < \alpha < 1$ case: dark grey region

Proposition

If $q > N/(N+\lambda)$, the relaxed inequality holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (1) and admits an optimizer (ρ, M)

Let (ρ_j, M_j) be a minimizing sequence with ρ_j radially symmetric non-increasing by rearrangement, such that

$$\int_{\mathbb{R}^N} \rho_j \, dx + M_j = \int_{\mathbb{R}^N} \rho_j^q = 1$$

Local estimates + Helly's selection theorem: $\rho_j \to \rho$ almost everywhere and $M_j \to M := L + \lim_{j\to\infty} M_j$, so that $\int_{\mathbb{R}^N} \rho \, dx + M = 1$, and $\int_{\mathbb{R}^N} \rho(x)^q \, dx = 1$
We cannot invoke Fatou's lemma because $\alpha \in (0,1)$: let $d\mu_j := \rho_j \, dx$

$$\mu_j\left(\mathbb{R}^N \setminus B_R(0)\right) = \int_{\{|x|\geq R\}} \rho_j \, dx \leq C \int_{\{|x|\geq R\}} \frac{dx}{|x|^{N/q}} = C' \, R^{-N\,(1-q)/q}$$

μ_j are tight: up to a subsequence, $\mu_j \to \mu$ weak * and $d\mu = \rho \, dx + L \, \delta$

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \geq I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \,,$$

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} |x|^\lambda \, \rho_j \, dx \geq \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx$$

Conclusion: $\liminf_{j\to\infty} \mathcal{Q}[\rho_j, M_j] \geq \mathcal{Q}[\rho, M]$

Optimizers are positive

$$\mathcal{Q}[\rho, M] := \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx}{\left(\int_{\mathbb{R}^N} \rho \, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{(2-\alpha)/q}}$$

Lemma

Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \geq 0$ is an optimal function for some $M > 0$, then ρ is radial (up to a translation), monotone non-increasing and positive a.e. on $\mathbb{R}^N$

If ρ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure, then

$$\mathcal{Q}\big[\rho, M + \varepsilon \, \mathbb{1}_E\big] = \mathcal{Q}[\rho, M] \left(1 - \frac{2-\alpha}{q} \, \frac{|E|}{\int_{\mathbb{R}^N} \rho(x)^q \, dx} \, \varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction if (ρ, M) is a minimizer of $\mathcal{Q}$

Euler–Lagrange equation

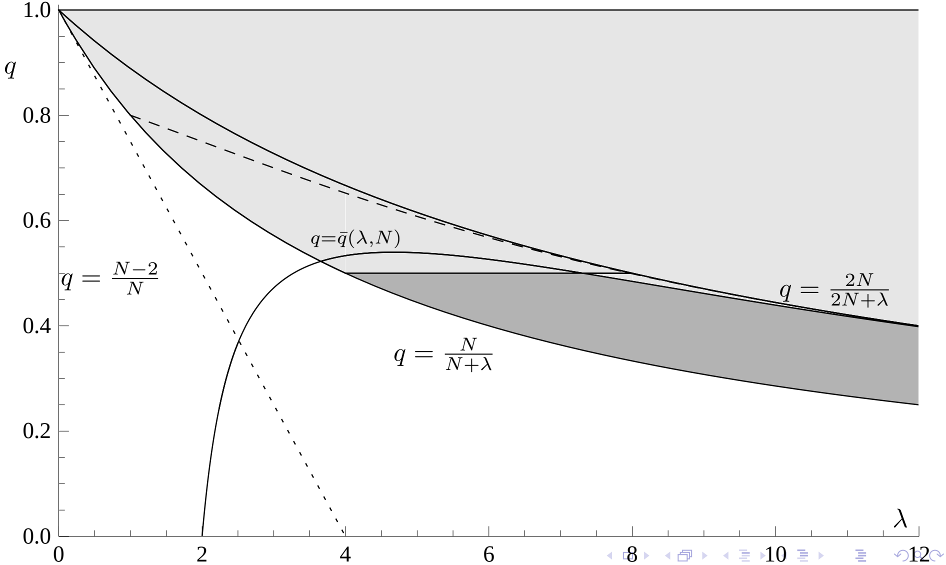
Euler–Lagrange equation for a minimizer (ρ_*, M_*)

$$\frac{2\int_{\mathbb{R}^N}|x-y|^\lambda\,\rho_*(y)\,dy+M_*|x|^\lambda}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\,\rho_*\,dy}-\frac{\alpha}{\int_{\mathbb{R}^N}\rho_*\,dy+M_*}-\frac{(2-\alpha)\,\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}=0$$

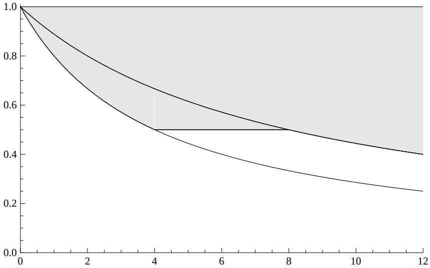
We can reformulate the question of the optimizers of (1) as: when is it true that $M_*=0$? We already know that $M_*=0$ if

$$\frac{2N}{2N+\lambda}<q<1$$

Regions of no concentration and regularity of measure valued minimizers


1.0
0.8
0.6
0.4
0.2
0.0
q
0
2
4
6
8
10
12
λ
$q=\bar{q}(\lambda,N)$
$q=\frac{N-2}{N}$
$q=\frac{2N}{2N+\lambda}$
$q=\frac{N}{N+\lambda}$

No concentration 1



Proposition

Let $N \geq 1$, $\lambda > 0$ *and* $\dfrac{N}{N+\lambda} < q < \dfrac{2N}{2N+\lambda}$

If $N \geq 3$ *and* $\lambda > 2N/(N-2)$, *assume further that* $q \geq \dfrac{N-2}{N}$

If (ρ_*, M_*) *is a minimizer, then* $M_* = 0$

Two ingredients of the proof

- Based on the Brézis–Lieb lemma

Lemma

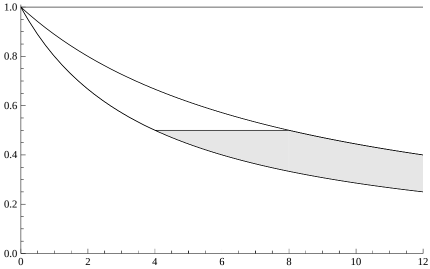
Let $0<q<p$, let $f\in \mathrm{L}^p\cap \mathrm{L}^q(\mathbb{R}^N)$ be a symmetric non-increasing function and let $g\in \mathrm{L}^q(\mathbb{R}^N)$. Then, for any $\tau>0$, as $\varepsilon\to 0_+$,

$$\int_{\mathbb{R}^N}\left|f(x)+\varepsilon^{-N/p}\,\tau\,g(x/\varepsilon)\right|^q dx=\int_{\mathbb{R}^N} f^q\,dx + \varepsilon^{N(1-q/p)}\,\tau^q\int_{\mathbb{R}^N}|g|^q\,dx+o\left(\varepsilon^{N(1-q/p)}\,\tau^q\right)$$

- $$I_\lambda\left[\rho_*+\varepsilon^{-N}\tau\,\sigma(\cdot/\varepsilon)\right]+2\,(M_*-\tau)\int_{\mathbb{R}^N}|x|^\lambda\left(\rho_*(x)+\varepsilon^{-N}\tau\,\sigma(x/\varepsilon)\right)dx$$

$$=I_\lambda[\rho_*]+2\,M_*\int_{\mathbb{R}^N}|x|^\lambda\,\rho_\star\,dx+\underbrace{\begin{cases}2\,\tau\iint_{\mathbb{R}^N\times\mathbb{R}^N}\rho_*(x)\left(|x-y|^\lambda-|x|^\lambda\right)\frac{\sigma\left(\frac{y}{\varepsilon}\right)}{\varepsilon^N}\,dx\,dy\\ +\,\varepsilon^\lambda\,\tau^2\,I_\lambda[\sigma]+2\,(M_*-\tau)\,\tau\,\varepsilon^\lambda\int_{\mathbb{R}^N}|x|^\lambda\,\sigma\,dx\end{cases}}_{=O(\varepsilon^\beta\,\tau)\quad\text{with}\quad\beta:=\min\{2,\lambda\}}$$

Regularity and concentration



Proposition

If $N\ge 3$, $\lambda>2N/(N-2)$ *and*

$$\frac{N}{N+\lambda}<q<\min\left\{\frac{N-2}{N},\,\frac{2N}{2N+\lambda}\right\},$$

and $(\rho_*,M_*)\in \mathrm{L}^{N\,(1-q)/2}(\mathbb{R}^N)\times[0,+\infty)$ *is a minimizer, then* $M_*=0$

Regularity

Proposition

Let $N \ge 1$, $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$
Let (ρ_*, M_*) be a minimizer

1. *If* $\int_{\mathbb{R}^N} \rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$*, then* $M_* = 0$ *and* ρ_**, bounded and*

$$\rho_*(0) = \left(\frac{(2-\alpha) I_\lambda[\rho_*] \int_{\mathbb{R}^N} \rho_*\,dx}{\left(\int_{\mathbb{R}^N} \rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \int_{\mathbb{R}^N} \rho_*\,dx - \alpha I_\lambda[\rho_*]\right)}\right)^{\frac{1}{1-q}}$$

2. *If* $\int_{\mathbb{R}^N} \rho_*\,dx = \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$*, then* $M_* = 0$ *and* ρ_* *is unbounded*
3. *If* $\int_{\mathbb{R}^N} \rho_*\,dx < \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$*, then* ρ_* *is unbounded and*

$$M_* = \frac{\alpha I_\lambda[\rho_*] - 2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \;\int_{\mathbb{R}^N} \rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx} > 0$$

An ingredient of the proof

Lemma

For constants $A,\,B>0$ *and* $0<\alpha<1$, *define*

$$f(M)=\frac{A+M}{(B+M)^\alpha}\quad\text{for}\quad M\ge 0$$

Then f *attains its minimum on* $[0,\infty)$ *at* $M=0$ *if* $\alpha\,A\le B$ *and at* $M=(\alpha\,A-B)/(1-\alpha)>0$ *if* $\alpha\,A>B$

No concentration 2

For any $\lambda \geq 1$ we deduce from

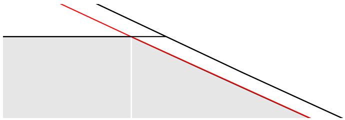
$$|x-y|^\lambda \leq \big(|x|+|y|\big)^\lambda \leq 2^{\lambda-1}\left(|x|^\lambda+|y|^\lambda\right)$$

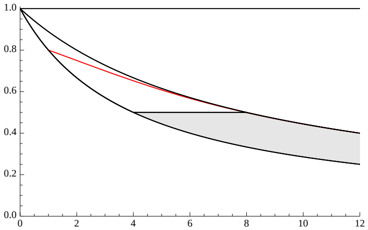
that

$$I_\lambda[\rho] < 2^\lambda \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx \int_{\mathbb{R}^N} \rho(x)\, dx$$

For all $\alpha \leq 2^{-\lambda+1}$, we infer that $M_* = 0$ if

$$q \geq \frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda}$$



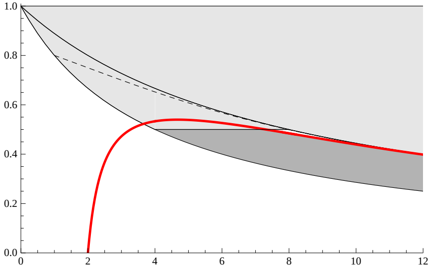


No concentration 3

Layer cake representation (superlevel sets are balls)

$$I_\lambda[\rho] \le 2\, A_{N,\lambda} \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx \int_{\mathbb{R}^N} \rho(x)\, dx$$

$$A_{N,\lambda} := \sup_{0 \le R,S < \infty} \frac{\iint_{B_R \times B_S} |x-y|^\lambda\, dx\, dy}{|B_R| \int_{B_S} |x|^\lambda\, dx + |B_S| \int_{B_R} |y|^\lambda\, dy}$$



Proposition

Assume that $N \geq 3$ *and* $\lambda > 2N/(N-2)$ *and observe that*

$$\frac{N}{N+\lambda} < \bar{q}(\lambda, N) \leq \frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda} < \frac{2\,N}{2\,N+\lambda}$$

for $\lambda > 2$ *large enough. If*

$$\max\left\{\bar{q}(\lambda, N), \frac{N}{N+\lambda}\right\} < q < \frac{N-2}{N}$$

and if (ρ_*, M_*) *is a minimizer, then* $M_* = 0$ *and* $\rho_* \in \mathrm{L}^\infty(\mathbb{R}^N)$

More on regularity

Lemma

Assume that ρ_* is an unbounded minimizer

- *if $\lambda < 2$, there is a constant $c > 0$ such that*

$$\rho_*(x) \geq c\,|x|^{-\lambda/(1-q)} \quad \textit{as} \quad x \to 0$$

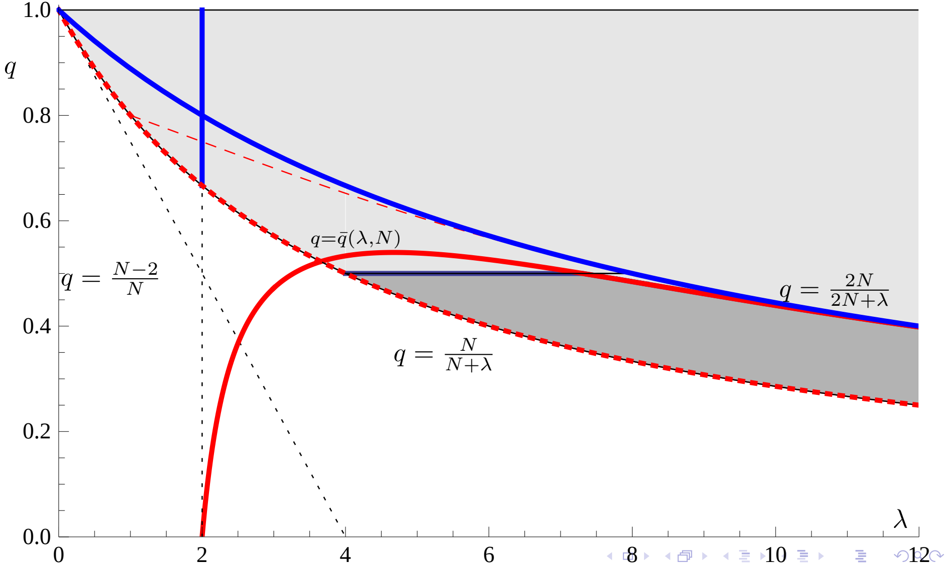
- *if $\lambda \geq 2$, there is a constant $C > 0$ such that*

$$\rho_*(x) = C\,|x|^{-2/(1-q)}\,\big(1 + o(1)\big) \quad \textit{as} \quad x \to 0$$

Corollary

$$q \neq \frac{2N}{2N+\lambda}\,, \quad \frac{N}{N+\lambda} < q < 1 \quad \textit{and} \quad q \geq \frac{N-2}{N} \textit{ if } N \geq 3$$

If ρ_* is a minimizer for $\mathcal{C}_{N,\lambda,q}$, then $\rho_* \in \mathrm{L}^\infty(\mathbb{R}^N)$


1.0
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12
λ
$q = \frac{N-2}{N}$
$q=\bar{q}(\lambda,N)$
$q = \frac{2N}{2N+\lambda}$
$q = \frac{N}{N+\lambda}$

Free energy point of view

A toy model

Assume that u solves the *fast diffusion with external drift* V given by

$$\frac{\partial u}{\partial t} = \Delta u^q + \nabla \cdot \big(u \, \nabla V\big)$$

To fix ideas: $V(x) = 1 + \frac{1}{2}\,|x|^2 + \frac{1}{\lambda}\,|x|^\lambda$. *Free energy* functional

$$\mathcal{F}[u] := \int_{\mathbb{R}^N} V\,u\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} u^q\,dx$$

- Under the mass constraint $M = \int_{\mathbb{R}^N} u\,dx$, smooth minimizers are

$$u_\mu(x) = \big(\mu + V(x)\big)^{-\frac{1}{1-q}}$$

- The equation can be seen as a gradient flow

$$\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\int_{\mathbb{R}^N} u\left|\tfrac{q}{1-q}\nabla u^{q-1} - \nabla V\right|^2\,dx$$

A toy model (continued)

If $\lambda = 2$, the so-called *Barenblatt profile* u_μ has finite mass if and only if

$$q > q_c := \frac{N-2}{N}$$

- For $\lambda > 2$, the integrability condition is $q > 1 - \lambda/N$ but $q = q_c$ is a threshold for the regularity: the mass of $u_\mu = (\mu + V)^{1/(1-q)}$ is

$$M(\mu) := \int_{\mathbb{R}^N} u_\mu \, dx \le M_\star = \int_{\mathbb{R}^N} \left(\tfrac{1}{2}\,|x|^2 + \tfrac{1}{\lambda}\,|x|^\lambda\right)^{-\frac{1}{1-q}} dx$$

- If one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u\, dx = M$ for an arbitrary $M > M_\star$, the limit of a minimizing sequence is the measure

$$\big(M - M_\star\big)\,\delta + u_{-1}$$

A model for nonlinear springs: heuristics

$$V = \rho * W_\lambda \,, \quad W_\lambda(x) := \tfrac{1}{\lambda}\,|x|^\lambda$$

is motivated by the study of the nonnegative solutions of the evolution equation

$$\frac{\partial \rho}{\partial t} = \Delta \rho^q + \nabla \cdot (\rho\,\nabla W_\lambda * \rho)$$

Optimal functions for (1) are energy minimizers (eventually measure valued) for the *free energy* functional

$$\mathcal{F}[\rho] := \frac{1}{2}\int_{\mathbb{R}^N} \rho\,(W_\lambda{*}\rho)\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx = \frac{1}{2\lambda}\,I_\lambda[\rho] - \frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx$$

under a *mass* constraint $M = \int_{\mathbb{R}^N} \rho\,dx$ while smooth solutions obey to

$$\frac{d}{dt}\mathcal{F}[\rho(t,\cdot)] = -\int_{\mathbb{R}^N} \rho\,\left|\tfrac{q}{1-q}\nabla \rho^{q-1} - \nabla W_\lambda * \rho\right|^2\,dx$$

Free energy or minimization of the quotient

$$\mathcal{F}[\rho] = -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx + \frac{1}{2\,\lambda} I_\lambda[\rho]$$

- If $0 < q \leq N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0$: take test functions $\rho_n \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ such that $\|\rho_n\|_{\mathrm{L}^1(\mathbb{R}^N)} = I_\lambda[\rho_n] = 1$ and $\int_{\mathbb{R}^N} \rho_n^q\,dx = n \in \mathbb{N}$

$$\lim_{n\to+\infty} \mathcal{F}[\rho_n] = -\infty$$

- If $N/(N+\lambda) < q < 1$, $\rho_\ell(x) := \ell^{-N}\,\rho(x/\ell)/\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}$

$$\mathcal{F}[\rho_\ell] = -\,\ell^{(1-q)\,N}\,\mathsf{A} + \ell^\lambda\,\mathsf{B}$$

has a minimum at $\ell = \ell_\star$ and

$$\mathcal{F}[\rho] \geq \mathcal{F}[\rho_{\ell_\star}] = -\,\kappa_\star\left(\mathsf{Q}_{q,\lambda}[\rho]\right)^{-\frac{N\,(1-q)}{\lambda - N\,(1-q)}}$$

Proposition

$\mathcal{F}$ is bounded from below if and only if $\mathcal{C}_{N,\lambda,q} > 0$

Relaxed free energy

$$\mathcal{F}^{\rm rel}[\rho, M] := -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx + \frac{1}{2\,\lambda}\,I_\lambda[\rho] + \frac{M}{\lambda}\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx$$

Corollary

Let $q \in (0,1)$ *and* $N/(N+\lambda) < q < 1$

$$\inf\left\{\mathcal{F}^{\rm rel}[\rho, M]\,:\,0 \le \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ M \ge 0\,,\ \int_{\mathbb{R}^N} \rho\,dx + M = 1\right\}$$

is achieved by a minimizer of (2) *such that* $\int_{\mathbb{R}^N} \rho_*\,dx + M_* = 1$ *and*

$$I_\lambda[\rho_*] + 2\,M_*\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx = 2\,N\int_{\mathbb{R}^N} \rho_*^q\,dx$$

Uniqueness

Proposition

Let $N/(N+\lambda) < q < 1$ *and assume either that* $(N-1)/N < q < 1$ *and* $\lambda \geq 1$*, or* $2 \leq \lambda \leq 4$*. Then the minimizer of*

$$\mathcal{F}^{\mathrm{rel}}[\rho, M] := \frac{1}{2\,\lambda}\, I_\lambda[\rho] + \frac{M}{\lambda} \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx - \frac{1}{1-q} \int_{\mathbb{R}^N} \rho^q\, dx$$

is unique up to translation, dilation and multiplication by a positive constant

- If $(N-1)/N < q < 1$ and $\lambda \ge 1$, the lower semi-continuous extension of $\mathcal{F}$ to probability measures is strictly geodesically convex in the Wasserstein-p metric for $p \in (1,2)$

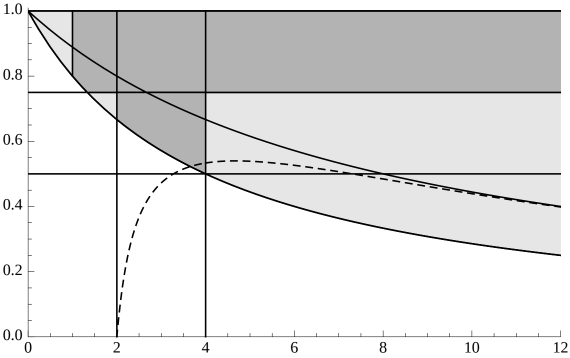
- By strict rearrangement inequalities a minimizer (ρ, M) such that $M \in [0,1)$ of the relaxed free energy $\mathcal{F}^{\rm rel}$ is (up to a translation) such that ρ is radially symmetric and $\int_{\mathbb{R}^N} x\,\rho\,dx = 0$
Let (ρ, M) and (ρ', M') be two minimizers and

$$[0,1] \ni t \mapsto f(t) := \mathcal{F}^{\rm rel}\big[(1-t)\,\rho + t\,\rho', (1-t)\,M + t\,M'\big]$$

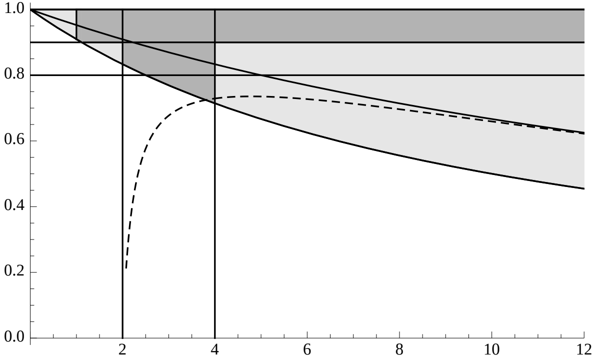
$$f''(t) = \frac{1}{\lambda}\,I_\lambda[\rho'-\rho] + \frac{2}{\lambda}\,(M'-M)\int_{\mathbb{R}^N} |x|^\lambda\,(\rho'-\rho)\,dx \\ + q\int_{\mathbb{R}^N} \big((1-t)\,\rho + t\,\rho'\big)^{q-2}(\rho'-\rho)^2\,dx$$

(Lopes, 2017) *$I_\lambda[h] \ge 0$ if $2 \le \lambda \le 4$, for all h such that $\int_{\mathbb{R}^N}\big(1+|x|^\lambda\big)\,|h|\,dx < \infty$ with $\int_{\mathbb{R}^N} h\,dx = 0$ and $\int_{\mathbb{R}^N} x\,h\,dx = 0$*

$N = 4$



$N = 10$



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Thank you for your attention !