Direct entropy methods

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I — Entropy

The logarithmic Sobolev inequality

Convex Sobolev inequalities — Applications

Entropy - Entropy production method
Intermediate asymptotics : heat equation

• *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...

• *linear diffusions:* [Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Otto, Kinderlehrer, Jordan]

• *nonlinear diffusions:* [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Markowich, Lederman], [Carrillo, Vazquez]

Heat equation:
$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{1}$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$. What is the (optimal) rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)} v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t} \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log\left(\frac{v}{v_\infty}\right)\, dx$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbf{R}^n} v \log v \, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \leq \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v}\, dx$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2}\, e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \leq \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \leq e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \log\left(\frac{u_0}{v_\infty}\right)\, dx$$

- Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v \, dx = \int_{\mathbf{R}^n} \bar{v} \, dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log\left(\frac{v}{\bar{v}}\right) \, dx \geq \frac{1}{4\,M} \|v - \bar{v}\|^2_{L^1(\mathbf{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbf{R}^n)} \leq 4 M \Sigma[u_0] \, e^{-2\tau}$$

$\tau(t) = \log \sqrt{1 + 2t}$

$$\|u(\cdot, t) - u_\infty(\cdot, t)\|^2_{L^1(\mathbf{R}^n)} \leq \frac{4}{1 + 2t} \Sigma[u_0]$$

$$u_\infty(x, t) = \frac{1}{R^n(t)} v_\infty\left(\frac{x}{R(t)}, \tau(t)\right)$$

The proof of the Csiszár-Kullback inequality is given by a Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension: gaussian form [Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M\, v_\infty}}$ and optimize for $w_\lambda(x) = \lambda^{n/2} w(\lambda\, x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2\, dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2\, dx$$
$$= \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2\, dx - \int_{\mathbb{R}^n} w^2 \log w^2\, dx - n \log \lambda \int_{\mathbb{R}^n} w^2\, dx$$

$\square$

Entropy-entropy production method: a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)]\right) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a\, \frac{w_i w_j}{w} + b\, w\, \delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \geq 0$ for $\tau > 0$

Entropy-entropy production method: improvements of convex Sobolev inequalities

goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \mathrm{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] = \mathrm{div}[e^{-A}\nabla(ve^{A})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^n \\ v(x,t=0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \quad (3)$$

$A(x)\ldots$ given 'potential'
$v_\infty(x) = e^{-A(x)} \in L^1 \ldots$ (unique) steady state
mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ? [Bakry-Emery '84, Gross '75, Toscani '96, AMTU...]
[Anton Arnold, J.D.]

Entropy-entropy production method

[Bakry, Emery, 84]

[Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condtion: $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''\left(\frac{v}{v_\infty}\right) |\underbrace{\nabla\left(\frac{v}{v_\infty}\right)}_{=:u}|^2 v_\infty\, dx \le 0$$

Assume: $D \equiv 1$, $\underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \ge \lambda_1 Id$, $\lambda_1 > 0$ ($A(x)\ldots$ unif. convex)

entropy production rate:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u\, v_\infty\, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, v_\infty\, dx}_{\ge 0} \\ &\ge -2\lambda_1 I \end{aligned}$$

with

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad |I(t)| \leq e^{-2\lambda_1 t}\,|I(t=0)| \qquad t > 0$$

$\forall v_0$ with $|I(v_0|v_\infty)| < \infty$

Exponential decay of relative entropy

$$
\begin{array}{llll}
\text{known:} & I' & \geq & -2\lambda_1 \underbrace{I}_{=\Sigma'} \qquad , \qquad \int_t^\infty ...dt \\
\Rightarrow & \Sigma' = I & \leq & -2\lambda_1 \Sigma
\end{array}
\tag{4}
$$

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id$ (*"Bakry–Emery condition"*), $\Sigma[v_0|v_\infty] < \infty$

$$\Rightarrow \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty]\, e^{-2\lambda_1 t}, \qquad t > 0$$

$\|v(t) - v_\infty\|_{L^1}^2 \leq C\,\Sigma[v(t)|v_\infty] \ldots$ Csiszár-Kullback

Convex Sobolev inequalities

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln\left(\frac{v}{v_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int v \left|\nabla \ln\left(\frac{v}{v_\infty}\right)\right|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^n),\ \int v\, dx = \int v_\infty\, dx = 1$$

logarithmic Sobolev inequality – “entropy version”

Set $f^2 = \frac{v}{v_\infty} \Rightarrow$

$$\int f^2 \ln f^2 dv_\infty \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty),\ \int f^2 dv_\infty = 1$$

logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

Example 2: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \leq 2$

$$(B_p)\ \ \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^p\right] \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \leq 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

[Arnold, J.D.]: Observation for $\psi_p(w) = w^p - p(w-1) - 1,\ 1 < p < 2$:

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}, \qquad \kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{v}{v_\infty}\right)|\nabla \frac{v}{v_\infty}|^2 dv_\infty$$

"refined convex Sobolev inequality" with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$
$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty)$$

"refined Beckner inequality" [Arnold, J.D. '00]

$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

Long Time Behavior of the Quantum Fokker-Planck equation

[A. Arnold, J. L. Lopez, P. A. Markowich, J. Soler], [C. Sparber, J. A. Carrillo, J. D., P. A. Markowich]

Heat equation with a source term

[G. Karch, J.D.]

An example of application: the flashing ratchet. Long time behavior and dynamical systems interpretation

[M. Chipot, D. Heath, D. Kinderlehrer, M. Kowalczyk, N. Walkington,...]

[J.D., David Kinderlehrer, Michał Kowalczyk]

Flashing ratchet: a simple model for a molecular motor (Brownian motors, molecular ratchets, or Brownian ratchets)

Diffusion tends to spread and dissipate density / transport concentrates density at specific sites determined by the energy landscape: unidirectional transport of mass.

Fokker-Planck type problem

$$\begin{array}{ll} u_t = (u_x + \psi_x u)_x & (x,t) \in \Omega \times (0,\infty) \\ u_x + \psi_x u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega \end{array} \qquad (5)$$

$u_0 > 0$, $\int_\Omega u_0 = 1$, $\psi = \psi(x,t)$

Periodic state and asymptotic behaviour

Theorem 1 *Let $\psi \in L^\infty([0,T)\times\Omega)$ be a T-periodic potential and assume that there exists a finite partition of $[0,T)$ into intervals $[T_i,T_{i+1})$, $i=0,...,n$ with $T_0=0$, $T_n=T$ such that $\psi_{[T_i,T_{i+1})} \in L^\infty([T_i,T_{i+1}),W^{1,\infty}(\Omega))$. Then there exists a unique nonnegative T-periodic solution U to (5) such that $\int_\Omega U(x,t)\,dx = 1$ for any $t\in[0,T)$.*

Entropy and entropy production : $\sigma_q(u) = \begin{cases} \frac{u^q-1}{q-1} & \text{if } q>1, \\ u\ln u & \text{if } q=1. \end{cases}$

$$\Sigma_q[u|v] = \int_\Omega \left[\sigma_q\left(\frac{u}{v}\right) - \sigma_q'(1)\left(\frac{u}{v}-1\right)\right] v\ dx$$

$$I_q[u|v] = \int_\Omega \sigma_q''\left(\frac{u}{v}\right)\left|\nabla\left(\frac{u}{v}\right)\right|^2 v\ dx,$$

Theorem 2 *Let u_1, u_2 be any two solutions to (5).*

$$\Sigma_q[u_1(t)|u_2(t)] \leq e^{-C_q t}\, \Sigma_q[u_1(0)|u_2(0)]$$

Proposition 3 *Ω is a bounded domain in $\mathbb{R}^d$ with C^1 boundary. Let u and v be two nonnegative functions in $L^1 \cap L^q(\Omega)$ if $q \in (1,2]$ and in $L^1(\Omega)$ with $u \log u$ and $u \log v$ in $L^1(\Omega)$ ($q = 1$).*

$$\Sigma_q[u|v] \geq 2^{-2/q} q \left[\max\left(\|u\|^{2-q}_{L^q(\Omega)}, \|v\|^{2-q}_{L^q(\Omega)}\right)\right]^{-1} \|u - v\|^2_{L^q(\Omega)}$$

Corollary 4 *Let $q \in [1,2]$. Any solution of (5) with initial data $u_0 \in L^1 \cap L^q(0,1)$ $u_0 \log u_0 \in L^1(0,1)$ if $q = 1$, converges to $\|u_0\|_{L^1} U(x,t)$, (periodic solution):*

$$\left\| u(x,t) - \|u_0\|_{L^1}\, U(x,t) \right\|_{L^q(0,1;dx)} \leq K\, e^{-C_{q,\psi}\, t} \quad \forall\, t \geq 0$$

Let $u_\psi := \|u_0\|_{L^1} \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}$.

$$\begin{aligned} \frac{d}{dt}\Sigma_1[u|u_\psi] &= \int_\Omega \left[1 + \log\left(\frac{u}{u_\psi}\right)\right] u_t\,dx - \int_\Omega \frac{u}{u_\psi}\,u_{\psi,t}\,dx \\ &= -I_1[u|u_\psi] - \int_\Omega \frac{u}{u_\psi}\,u_{\psi,t}\,dx \end{aligned}$$

Lemma 5 *Let $u \geq 0$ be a solution to (5) such that $\int_\Omega u\,dx = 1$. With the above notations, the following estimate holds:*

$$\frac{d}{dt}\Sigma_1[u|u_\psi] \leq -C_\psi\,\Sigma_1[u|u_\psi] + K_\psi.$$

Fixed-point for the map $\mathcal{T}(u(\cdot,0)) = u(\cdot,T)$ in

$\mathcal{Y} = \{u \in H^1(\Omega) \mid u \geq 0\,,\ \|u\|_{L^1(\Omega)} = 1\,,\ \Sigma_1[u|u_0(\cdot,0)] \leq K_\psi/C_\psi\}$.

Flashing potentials: same on each time interval.

Let $\mathcal{X}$ be the set of bounded nonnegative functions u in $L^1 \cap L^q(\Omega)$ (resp. in $L^1(\Omega)$ with $u \log u$ in in $L^1(\Omega)$) if $q \in (1,2]$ (resp. if $q = 1$) such that $\int_\Omega u\,dx = 1$.

Theorem 6 *Assume that $v \in \mathcal{X}$ with $0 < m := \inf_\Omega v \leq v \leq \sup_\Omega v =: M < \infty$. For any $q \in [1,2]$*

$$\mathcal{J} = \frac{q}{q-1} \inf_{\substack{u \in \mathcal{X} \\ u \neq v \; a.e.}} \frac{I_q[u|v]}{\Sigma_q[u|v]} \quad \textit{if } q > 1, \quad \mathcal{J} = \inf_{\substack{u \in \mathcal{X} \\ u \neq v \; a.e.}} \frac{I_1[u|v]}{\Sigma_1[u|v]} \quad \textit{if } q = 1 \tag{6}$$

can be estimated by

$$\mathcal{J} \geq 4\,\lambda_1(\Omega)\,\frac{m}{M} \tag{7}$$

where $\lambda_1(\Omega)$ is Poincaré's constant of Ω (with weight 1).

Relation between entropy and entropy production: exponential decay of the relative entropy.

II — Entropy methods for nonlinear models involving diffusions

A model for traffic flow

[R. Illner, A. Klar, T. Materne, 2002], [Reinhard Illner, J.D., 2002] $f = f(t,v)$ is an homogeneous distribution function, with velocities ranging in $(0,1)$:

$$f_t = (-B(t,v)\, f + D(t,v)\, f')' \,, \quad (t,v) \in \mathbb{R}^+ \times (0,1)$$

where $f_t = \partial f/\partial t$, $f' = \partial f/\partial v$. Let $C(t,v) := -\int_0^v \frac{B(t,w)}{D(t,w)}\, dw$

$$g(t,v) = \rho\, \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,w)}\, dw} \quad \text{is a } \textit{local equilibrium}$$

Zero flux: $-B(t,v)\, g + D(t,v)\, g' = 0$ but $g_t \equiv 0$ is not granted. Relative entropy:

$$\begin{aligned} e[t,f] := \int_0^1 & \left(f \log f - g \log g + C(t,v)(f-g)\right) dv \\ & - \iint_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)\, dv\, ds \end{aligned}$$

Then $\frac{d}{dt} e[t, f(t,.)] = -\int_0^1 D(t,v)\, f \left|\frac{f'}{f} - \frac{g'}{g}\right|^2 dv$ ... but we dont have a lower bound for $e[t, f(t,.)]$.

Density: $\rho = \int_0^1 f(t,v)\, dv$ does not depend on t
Mean velocity: $u(t) = \frac{1}{\rho} \int_0^1 v\, f(t,v)\, dv$
Braking term:

$$B(t,v) = \begin{cases} -C_B\, |v - u(t)|^2 \rho \left(1 - \left|\frac{v-u(t)}{1-u(t)}\right|^\delta\right) & \text{if } v > u(t) \\ \\ C_A\, |v - u(t)|^2 (1-\rho) & \text{if } v \le u(t) \end{cases}$$

Diffusion term: $D(t,v) = \sigma\, m_1(\rho)\, m_2(u(t))\, |v - u(t)|^\gamma$

Proposition 7 [Illner-Klar-Materne02] *Any stationary solution is uniquely determined by ρ and its average velocity u. The set $(\rho, u[\rho])$ is in general multivalued. For any $\rho \in (0,1]$.*

Example. *The Maxwellian case.*

Convex entropies

Relative entropy of f *w.r.t.* g by $E[\,f\,|\,g\,] = \int_0^1 \Phi\left(\frac{f}{g}\right) g\,dv$

"Standard" example: $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ for some $\alpha > 1$,
$\Phi(x) = x \log x$ if "$\alpha = 1$"

$$\begin{cases} f_t = \left[D(t,v)\, f \left(\frac{f'}{f} - \frac{g'}{g}\right)\right]' = \left[D(t,v)\, g \left(\frac{f}{g}\right)'\right]' & \forall\,(t,v) \in \mathbb{R}^+ \times (0,1) \\ \left(\frac{f}{g}\right)'(t,v) = 0 & \forall\, t \in \mathbb{R}^+,\ v = 0,\ 1 \end{cases}$$

$$g(t,v) := \kappa(t)\, e^{-C(t,v)} \text{ for some } \kappa(t) \neq 0.$$

$$\frac{d}{dt} E[\,f(t,\cdot)\,|\,g(t,\cdot)\,] = \int_0^1 \Phi'\left(\frac{f}{g}\right) f_t\,dv + \underbrace{\int_0^1 \left[\Phi\left(\frac{f}{g}\right) - \frac{f}{g}\,\Phi'(\frac{f}{g})\right] g_t\,dv}$$

$$= 0 \quad \text{if} \quad \dot{\kappa} = \kappa\, \frac{\int_0^1 \Psi(\frac{f}{g})\, g\, C_t(t,v)\, dv}{\int_0^1 \Psi(\frac{f}{g})\, g\, dv}, \quad \kappa(0) = 1$$

with $\Psi(x) := \Phi(x) - x\Phi'(x) < 0$

Convergence to a stationary solution

$$\limsup_{t\to+\infty} \kappa(t) < +\infty \ .$$

Theorem 8 *Let* $\Phi = \Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$, f *be a smooth global in time solution and assume that* $E[\, f \,|\, g\,]$ *is well defined and* C^1 *in* t. *If* $\exists\, \varepsilon \in (0, \frac{1}{2})$ *s.t.* $\varepsilon < u(t) = \frac{1}{\rho} \int_0^1 v\, f(t,v)\, dv < 1 - \varepsilon$ $\forall\ t > 0$, *then, as* $t \to +\infty$, $f(t,\cdot)$ *converges a.e. to a stationary solution* f_∞.

Coupling with a Poisson equation

[AMT], [P. Biler, J.D.] Nernst-Planck / Debye-Hückel drift-diffusion

$$u_t = \nabla \cdot (\nabla u + u\nabla\phi)$$
$$v_t = \nabla \cdot (\nabla v - v\nabla\phi)$$
$$\Delta\phi = v - u$$

in a bounded domain $\Omega \subset \mathbb{R}^d$, $d \leq 3$.
No-flux boundary conditions: $\frac{\partial u}{\partial \nu} + u\frac{\partial \phi}{\partial \nu} = 0$, $\frac{\partial v}{\partial \nu} - v\frac{\partial \phi}{\partial \nu} = 0$ on $\partial\Omega$ and either conducting boundary conditions

$$\phi = 0 \quad \text{on} \quad \partial\Omega$$

or "free" boundary conditions (this corresponds to a container immersed in a medium with the same dielectric constant as the solute)

$$\phi = E_d * (v - u)$$

where E_d is the fundamental solution of the Laplacian in $\mathbb{R}^d$.

Intermediate asymptotics: $\Omega = \mathbb{R}^d$

$$\frac{1}{M_u} u_{as}(x,t) = \frac{1}{M_v} v_{as}(x,t) = (2\pi(2t+1))^{d/2} \exp\left(-\frac{|x|^2}{2(2t+1)}\right)$$

Entropy functional :

$$\Sigma = \int u \log\left(\frac{u}{u_{as}}\right)\, dx + \int v \log\left(\frac{v}{v_{as}}\right)\, dx + \frac{1}{2}\|\nabla\phi(t)\|_2^2$$

Theorem 9 *There exists a constant C such that*

$$L(t)\,, \quad \|u(t)-u_{as}(t)\|_1^2 + \|v(t)-v_{as}(t)\|_1^2 + \|\nabla\phi(t)\|_2^2 \le C\; H(t)$$

$$\textit{where} \quad H(t) = \begin{cases} (2t+1)^{-1/2}\,, & d=3 \\ (2t+1)^{-1}(\log(2t+1)+1)\,, & d=4 \\ (2t+1)^{-1}\,, & d>4 \end{cases}$$

Moreover if $M_u = M_v$, then $H(t) = (2t+1)^{-1}$ for any $d \ge 3$.

Convergence to a stationary solution Assume that Ω is bounded. Entropy functional

$$\begin{aligned} W(t) &= \int u \log u \, dx - \int U(x) \log U(x) \, dx \\ &+ \int v \log v \, dx - \int V(x) \log V(x) \, dx \\ &+ \frac{1}{2} \int (u-v)\phi \, dx - \frac{1}{2} \int (U-V)\Phi \, dx \, , \end{aligned}$$

where U, V, Φ is the unique stationary solution such that $\int U \, dx = \int u \, dx$, $\int V \, dx = \int v \, dx$.

Theorem 10 *If $d \geq 2$, then there exist two positive constants λ and C such that*

$$W(t) \leq W(0) \, e^{-\lambda t}$$

$$\|u(t) - U\|_1^2 + \|v(t) - V\|_1^2 + \|\nabla(\phi - \Phi)\|_2^2 \leq C \, e^{-\lambda t}$$

Streater's model

[P. Biler, J.D., M. Esteban, G. Karch]

Nonlinear diffusions coupled with a Poisson equation

[P. Biler, J.D., P. Markowich]

Entropy methods for nonlinear diffusions

[1. Comparison techniques]

2. Entropy-Entropy production methods

3. Variational approach: [Gross], [Aubin], [Talenti], [Weissler], [Carlen], [Carlen-Loss], [Beckner], [Del Pino, JD]

[4. Mass transportation methods] [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 11 $1<p<n$, $1<a\le\frac{p(n-1)}{n-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \textit{if } a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \textit{if } a<p$$

Equality if $w(x)=A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a>p$: $\theta=\frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a<p$: $\theta=\frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

Proof based on [Serrin, Tang]

The optimal L^p-Euclidean logarithmic Sobolev inequality

[Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 12 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n \frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

Remark: The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \le \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \le \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \le \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}$$

Case $a = p$ $(q = 1)$: it is more convenient to use this inequality in a non homogeneous form:

$$\inf_{\mu>0} \left[\frac{n}{p} \log\left(\frac{n}{p\mu}\right) + \mu \frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right] = n \log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right) + \frac{n}{p}\,.$$

Corollary 13 *For any* $w \in W^{1,p}(\mathbb{R}^n)$, $w \neq 0$, *for any* $\mu > 0$,

$$p \int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right) dx + \frac{n}{p} \log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right) \int |w|^p\,dx \leq \mu \int |\nabla w|^p\,dx\,.$$

Case $a \neq p$ $(q \neq 1)$: ... $\Sigma \leq C\,I$

INTERMEDIATE ASYMPTOTICS FOR NONLINEAR DIFFUSIONS

[Manuel Del Pino, J.D.]

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x\, v)$$

Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$.

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\, dx$$

$\sigma(s) = \frac{s^q - s}{q-1}$ if $q \neq 1$
$\sigma(s) = s \log s$ if $q = 1$ $(p = 2)$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 14 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\textit{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-\left(\frac{\alpha}{2}+n\left(1-\frac{1}{q}\right)\right)}$$

$$\textit{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii)*: $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$
$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$
$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$
$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla\cdot(x\, v)$
$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

Intermediate asymptotics in L^1 for general nonlinear diffusion equations

[P. Biler, J.D., M. Esteban]

$$u_t = \Delta f(u) \tag{8}$$

If $f(u) = u^m$, Equation (8) can be transformed via a suitable space-time rescaling into an autonomous equation

$$v_\tau = \Delta f(v) + \nabla \cdot (v \nabla V)\,, \tag{9}$$

with $V(x) = \frac{1}{2}|x|^2$. Here we simply assume that f *behaves like a power law* in a neighborhood of 0.

General case: Time-dependent rescaling $u(t,x) = R^{-d}(t)\, v\Big(\tau(t)\,, \frac{x}{R(t)}\Big)$, $\tau(t) = \log(R(t))$, $\frac{dR}{dt} = R^{(1-m)d-1}$

$$v_\tau = e^{md\tau} \Delta f\left(e^{-d\tau} v\right) + \nabla \cdot (xv) \tag{10}$$

"Local" stationary solution: $v_\infty^R(x) = R^d g\left(R^{-(m-1)d}(\alpha_\infty^R(M) - \frac{1}{2}|x|^2)\right)$
such that $\|v_\infty^R\|_{L^1(\mathbb{R}^d)} = \|v\|_{L^1(\mathbb{R}^d)}$.

Example: $f(u) = u^m$, $m \neq 1$, $v_\infty(x) = g(\alpha_\infty(M) - \frac{1}{2}|x|^2)$,

$$g(x) = ((m-1)\,x/m)_+^{1/(m-1)}$$

Assumption:

(i) $f(s) = s^m \varphi(s)$ with $\varphi \in C^0([0,+\infty)) \cap C^1(0,+\infty)$, $\varphi > 0$, $\varphi(0) = 1$, $\varphi'(s) = \mathcal{O}(s^k)\ (k > -1)$ near $s = 0^+$

(ii) there exists a function H with $H''(s) = \frac{f'(s)}{s}$, $H(0) = 0$, such that $(m-1)H(s) \leq f(s) \quad \forall\, s > 0$

(iii) $f \in C^3(0,+\infty) \cap C^0[0,+\infty)$, $f' > 0$, $f'/s \in L^1_{loc}(0,+\infty)$ *and* $f(u) \leq \frac{d}{d-1} u f'(u)$ *for any* $u > 0$ *($d \geq 3$). If* $m \leq 1$, $\exists s_0 > 0$ *such that* $f''_{|[0,s_0)} \geq 0$ *and* $g(\alpha - \frac{1}{2}|\cdot|^2) \in L^1(\mathbb{R}^d)$.

Relative entropy :

$$\Sigma[\tau, v] = \int_{\mathbb{R}^d} \left(e^{md\tau} H(e^{-d\tau} v) - e^{md\tau} H(e^{-d\tau} v_\infty^R) + \frac{1}{2}|x|^2 (v - v_\infty^R) \right) dx$$

Theorem 15 *Under these assumptions, then for any solution* v *of (10), there exist two constants* $\beta := \min\{2, d(k+1)\} > 0$, $K > 0$ *such that for all* $\tau > 0$,

$$0 \leq \Sigma[\tau, v] \leq K\, e^{-\beta\,\tau}\ .$$

Rates are as good as in the power-law case as soon as $k \geq -(d-2)/d$ (if the function φ is C^1 at the origin, then $\beta = 2$).

Theorem 16 *If moreover* $\ell_1 := \liminf_{s \to +\infty} f'(s) s^{m-1} > 0$ *if* $m \in (1,2)$; $\ell_2 := \liminf_{s \to +\infty} \frac{s\, H''(s)}{|H'(s)|^3} > 0$ *if* $m \in \left(\frac{d-1}{d}, 1\right)$, *then as* $t \to +\infty$,

$$\|u(t,.) - u_\infty(t,.)\|_{L^1(\mathbf{R}^d,\, u_\infty^{m-1}(t,x)\, dx)} \leq C\, t^{-\frac{d(m-1)+\beta/2}{2+(m-1)d}} \quad \textit{if} \quad 1 < m \leq 2$$

$$\|H(u)(t,.) - H(u_\infty)(t,.)\|_{L^1(\mathbf{R}^d,\, dx)} \leq C\, t^{-\frac{d(m-1)+\beta/2}{2+(m-1)d}} \quad \textit{if} \quad (d-1)/d \leq m$$

III — L^1 *Intermediate asymptotics for scalar conservation laws*

J.D., Miguel Escobedo

Introduction

Consider for some $q > 1$ a nonnegative entropy solution of

$$\begin{cases} U_\tau + (U^q)_\xi = 0\,, \quad \xi \in \mathbb{R}\,, \quad \tau > 0\,, \\ \\ U(\tau = 0, \cdot) = U_0\,. \end{cases} \tag{11}$$

T.-P. Liu & M. Pierre (1984): for every $p \in [1, +\infty)$,

$$\lim_{\tau \to \infty} \tau^{\frac{1}{q}(1-\frac{1}{p})} \|U(\tau) - U_\infty(\tau)\|_p = 0\,.$$

U_∞ is the self-similar solution defined by

$$U_\infty(\tau, \xi) = \begin{cases} \left(\frac{|\xi|}{q\,\tau}\right)^{\frac{1}{q-1}} & 0 \le \xi \le c(\tau) := q\ (\|u_0\|_1/(q-1))^{(q-1)/q}\ \tau^{1/q} \\ \\ 0 & \text{elsewhere} \end{cases}$$

P. Lax (1957): if U is the unique entropy solution to

$$U_\tau + f(U)_\xi = 0\,, \quad U(0,\xi) = U_0(\xi)$$

with $f \in C^2$, $f(0) = f'(0) = 0$ and $f'' > 0$, and if $U_0 \geq 0$ has compact support in (s_-, s_+), then the following estimate holds:

$$\|U(\tau,\cdot) - W_\infty(\tau, \cdot - s_-)\|_1 = O(\tau^{-1/2}) \quad \text{as} \quad \tau \to \infty\,,$$

where $W_\infty(\tau,\xi) = \frac{\xi}{f''(0)}\,\tau^{-1}$ if $0 < \xi < -s_- + s_+ + \sqrt{2\,\|u_0\|_1\, f''(0)}\,\tau^{-1/2}$, and 0 elsewhere.

Y.-J. Kim (2003): an alternative approach is based on a detailed study of special self-similar solutions. Same kind of results.

Main results

Theorem 17 [Entropy estimate] *Let U be a global, piecewise C^1 entropy solution of (11) corresponding to a nonnegative initial data U_0 in $L^1 \cap L^\infty(\mathbb{R})$ which is compactly supported in $(\xi_0, +\infty)$ for some $\xi_0 \in \mathbb{R}$ and such that*

$$\liminf_{\substack{\xi \to \xi_0 \\ \xi > \xi_0}} \frac{U_0(\xi)}{|\xi - \xi_0|^{1/(q-1)}} > 0 \, .$$

Then, for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$,

$$\limsup_{\tau \to +\infty} \tau^{\alpha - \epsilon} \int_{\mathbb{R}} |U(\tau, \xi) - U_\infty(\tau, \xi - \xi_0)| \, \frac{d\xi}{|\xi - \xi_0|^\alpha} = 0.$$

$\frac{1}{2} < \alpha(1 - 1/q) \Longleftrightarrow q/(2(q-1)) < \alpha$

Corollary 18 [L^1 ESTIMATE] *For any $\beta < 1$, there exists a constant C_β such that*

$$\|U(\tau,\cdot) - U_\infty(\tau,\cdot - \xi_0)\|_1 \le C_\beta\, \tau^{-\beta}$$

Theorem 19 [UNIFORM ESTIMATE]

$$\lim_{\tau\to+\infty} \sup_{\xi\in\mathrm{supp}(U(\tau,\cdot))} \tau^{1/q} \left| U(\tau,\xi) - \left|\frac{\xi-\xi_0}{\tau\, q}\right|^{1/(q-1)} \right| = 0$$

$$\lim_{\tau \to +\infty} (1 + q\,\tau)^{-1/q} \max\left[\mathrm{supp}(U(\tau,\cdot))\right] = \left(\frac{q \int_{\mathbf{R}} U_0\, d\xi}{q-1}\right)^{(q-1)/q} =: c_M$$

$$\max\left[\mathrm{supp}(U(\tau,\cdot))\right] \geq (1 + q\,\tau)^{1/q} c_M \,(1 + O(\tau^{-1}))$$

Two main tools:

1. A *time-dependent rescaling* which preserves the initial data and replaces the characterization of the intermediate asymptotics by the convergence to a stationary solution.

2. A time-decreasing functional which plays the role of an *entropy,* in the sense that it captures global informations on the evolution of the solution and controls its large time asymptotics.

Proposition 20 *Let $U \geq 0$ be a piecewise C^1 entropy solution of (11), whose points of discontinuity are $\xi_1(\tau) < \xi_2(\tau) < \cdots < \xi_n(\tau)$. The rescaled function*

$$u(t,x) = e^t\, U\left(\frac{1}{q}(e^{qt}-1), e^t x\right)$$

is a piecewise C^1 function, whose points of discontinuity are given by $s_i(t) \equiv e^{-t}\xi_i((e^{qt}-1)/q)$ such that

$$s_i'(t) = \frac{(u_i^+)^q - s_i(t)\, u_i^+ - (u_i^-)^q + s_i(t)\, u_i^-}{u_i^+ - u_i^-}\,.$$

Out of the curves $x = s_i(t)$ the function u is a classical solution of

$$u_t = (x\,u - u^q)_x \tag{12}$$

and across these curves it satisfies

$$u_i^- := \lim_{\substack{x\to s_i(t)\\ x<s_i(t)}} u(t,x) > \lim_{\substack{x\to s_i(t)\\ x>s_i(t)}} u(t,x) := u_i^+.$$

$U_0 := U(0,\cdot) = u(0,\cdot) =: u_0$. *If* $U_0 \in L^1(\mathbb{R})$*:* $||u(t)||_1 = ||U_0||_1$ $\forall\, t > 0$.

DEFINITION: u is a solution of (12) iff it is a piecewise C^1 function whose discontinuity points are given by $\{s_i(t)\}_{i=1}^n$ satisfying (20), which solves (12) out of these curves and such that () holds across them. With this definition, if v is a solution of (12) then U is an entropy solution of (11).

For every $c > 0$, let u_∞^c be the stationary solution of (12) defined by

$$u_\infty^c(x) = \begin{cases} x^{1/(q-1)} & 0 \le x \le c \\ 0 & \text{if } x < 0 \text{ or } x > c \end{cases} \tag{13}$$

If $c = c_M := (q\,M/(q-1))^{(q-1)/q}$, $u_\infty := u_\infty^{c_M}$. $\|u_\infty\|_1 = M$.

COMPARISON RESULTS Consider two solutions U, V of (11) with initial data U_0, V_0 such that $0 \le U_0 \le A_0\,V_0$ a.e. for some $A_0 > 0$. Then

$$U(\tau,\cdot) \le A_0\,V(A_0^{q-1}\tau,\cdot)\ a.e. \quad \forall\,\tau \in \mathbb{R}^+ \,.$$

Scaling argument and comparison principle for entropy solutions.

$$u_0 \le A_0\,u_\infty^c\ a.e. \Longrightarrow u(t,x) \le A(t)\,u_\infty^{c(t)}(x)\ a.e. \quad \forall\,t \in \mathbb{R}^+$$

with $A(t) = \dfrac{A_0\,e^{qt/(q-1)}}{\left[1+A_0^{q-1}(e^{qt}-1)\right]^{1/(q-1)}}$ and $c(t) = c\left(\frac{A_0}{A(t)}\right)^{(q-1)/q}$.

A GRAPH CONVERGENCE RESULT

Proposition 21 *Let* $\epsilon > 0$, $M = \int u_0 \, dx$ *and* $c_M = (\frac{qM}{q-1})^{(q-1)/q}$. *Assume that* $u_0 \geq 0$ *is a piecewise* C^1 *initial data with support in* $[0, +\infty)$*, such that* $\liminf_{\substack{x \to 0 \\ x > 0}} x^{1/(1-q)} u_0(x) > 0$.

Then there exists $T > 0$ *such that, for any* $t > T$*:*
(i) the support of $u(\cdot, t)$ *is an interval* $[0, s(t)]$.
(ii) $\inf_{x \in [0, s(t))} x^{1/(1-q)} u(x, t) > 0$.
(iii) there exists a constant $A_0 > 0$ *such that* $u \leq A(t)\, u_\infty^{s(t)}$
(iv) for an arbitrarily small $\epsilon > 0$*, there exists a constant* κ *such that* $u \geq (1 - \kappa\, e^{-qt})\, u_\infty^{c_M - \epsilon}$.

FIGURES

Theorem 22 *Let* $u_0 \geq 0$ *be a piecewise* C^1 *initial data with compact support in* $[0, +\infty)$ *such that* $\liminf_{\substack{x \to 0 \\ x > 0}} x^{1/(1-q)} u_0(x) > 0$.

Then

$$\lim_{t \to \infty} \sup_{x \in (0, s(t))} |u(t,x) - u_\infty^{s(t)}| = 0 \ ,$$

where $[0, s(t)] := \mathrm{Supp}(u(\cdot, t))$ *for* $t > 0$ *large enough. Moreover*

$$\lim_{t \to +\infty} s(t) = c_M =: (\frac{qM}{q-1})^{(q-1)/q} \quad \textit{with } M = \int_{\mathbf{R}} u_0 \, dx$$

and $s(t) \geq c_M - O(e^{-qt})$.

1st step: A CONSEQUENCE OF THE RANKINE-HUGONIOT RELATION

Let $h(t) := \lim_{x\to s(t),\, x<s(t)} u(x,t)$. Then

$$\begin{cases} \dfrac{ds}{dt} = h^{q-1} - s \\ \\ \dfrac{dh}{dt} = h\,(1 - \lim\limits_{\substack{x\to s(t)\\ x<s(t)}} (u^{q-1})_x(x,t)) \end{cases} \tag{14}$$

2nd step: ENTROPY SOLUTION If U is an entropy solution of (11) then it satisfies the *entropy inequality* $(U^{q-1})_\xi \leq \frac{1}{q\,\tau}$ and

$$(u^{q-1})_x \leq (1 - e^{-qt})^{-1}$$

holds in the distribution sense.

3rd step: BY CONTRADICTION Let $u_\infty(x) := x^{1/(q-1)}$. Then
(i) For any $\epsilon > 0$, there exists $t_1 > 0$ such that

$$s(t) \geq C_M - \epsilon \quad \forall\, t > t_1 \,.$$

(ii) For any $\epsilon > 0$, $\delta \in (0,1)$, $t_0 > 0$, there exists $t_1 > t_0$ such that

$$h(t_1) \geq (1-\delta)\, u_\infty(s(t_1)) \,.$$

4th step: COMBINE STEP 1 AND 2

$$\frac{dh}{dt} \geq -\frac{h}{e^{qt}-1} \,.$$

Assume that $h(t_0) = h_0 > 0$ for some $t_0 > 0$. Then

$$h(t) \geq h_0\,(1 - e^{-qt_0})^{1/q} \quad \forall\, t > t_0 \,.$$

5th step: AN ESTIMATE BASED ON THE MASS *As $t \to +\infty$, $s(t)$ converges to C_M and for any $\eta \in (0,1)$, there exists a $t_0 \geq 0$ such that*

$$u(\cdot,t) \geq (1-\eta)\, u_\infty^{s(t)} \quad \forall\, t \geq t_0\,.$$

Proof. This follows from

$$u(x,t) \geq \left((h(t))^{q-1} - \frac{s(t)-x}{1-e^{-qt}} \right)_+^{1/(q-1)} =: v(x,t) \quad \text{on } (C_M - \epsilon, s(t))\,. \tag{15}$$

and for $s(t) > c_M$ from

$$M \geq (1-\eta) \int_0^{C_M-\epsilon} u_\infty \, dx + \int_{C_M-\epsilon}^{s(t)} v \, dx\,.$$

□

L^1 INTERMEDIATE ASYMPTOTICS

For any positive constants c and c', define the entropy by

$$\Sigma(t) = \int_0^{c'} \mu(x)\,|u(t,x) - u_\infty^c(x)|\,dx = \int_0^c \mu\,|u - u_\infty^c|\,dx + \int_c^{c'} \mu\,u\,dx \tag{16}$$

for some nonincreasing function μ, which is continuous on $(0,+\infty)$. The special choice $c' > \sup_{t\in\mathbb{R}^+}\max_{x\in\mathbb{R}}\{\mathrm{supp}\,u(t,x)\}$, $\mu(x) := |x|^{-\alpha}$ and $c = c_M$ will provide exponential rates of convergence. Let $f(v) := v - v^q$.

Proposition 23 [DECAY OF THE ENTROPY] *Let u_0 be an initial data with compact support in $[0,+\infty)$, such that $0 \le u_0(x) \le A_0\,x^{1/(q-1)}$ for some $A_0 > 0$. Assume that $\lim_{x\to 0}\mu(x)\,u_\infty^q(x) = 0$. Let $c' > 0$. Suppose that $\mu' u_\infty^q$ and $\mu\,u_\infty$ are integrable on $(0,c')$.*

Then for every fixed $c \in (0, c')$, for $t \geq 0$ a.e.

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx - \int_c^{c'} \mu'(u_\infty^{c'})^q f\left(\frac{u}{u_\infty^{c'}}\right) dx$$

$$- \mu(c)\, c^{q/(q-1)} \left\{ f\left(\frac{u^+(c)}{c^{\frac{1}{q-1}}}\right) + \left| f\left(\frac{u^-(c)}{c^{\frac{1}{q-1}}}\right)\right| \right\} + \mu(c')\,(c')^{q/(q-1)} f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right)$$

where $u^{\pm}(c) := \lim_{\substack{x \to c \\ \pm(x-c)>0}} u(x)$.

If $c = c'$, then $\frac{d\Sigma}{dt} \leq \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx \leq 0$.

Proof.

$$\frac{d\Sigma}{dt} = \int_0^c \mu\, u_t \left[1\!\!1_{u>u_\infty^c} - 1\!\!1_{u<u_\infty^c} \right] dx + \int_c^{c'} \mu\, u_t \, dx + \left[\mu(s) |u - u_\infty^c(s)| \cdot s'(t) \right]_{u^+}^{u^-}$$

Using the equation and an integration by parts, we get

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu' \, (u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right) \right| dx + \mu(s)\, (u_\infty^c(s))^q \Psi(v^-, v^+)$$

$$- \mu(c)\, c^{\frac{q}{q-1}} \left[\left| f\left(c^{-\frac{1}{q-1}} u^-(t,c)\right) \right| + f\left(c^{-\frac{1}{q-1}} u^+(t,c)\right) \right]$$

$$- \int_c^{c'} \mu' \, (u_\infty^{c'})^q \, f\left(\frac{u}{u_\infty^{c'}}\right) dx + \mu(c')\, (c')^{q/(q-1)} \, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right)$$

where

$$\Psi(v^-, v^+) = \left[f(v^+) - f(v^-) \right] \cdot \frac{|v^+ - 1| - |v^- - 1|}{v^+ - v^-} + |f(v^+)| - |f(v^-)|$$

Since $v^+ < v^-$, we have to distinguish three cases:

(i) $1 \le v^+ \le v^-$: $f(v^-) \le f(v^+) \le 0$ and $\Psi(v^-, v^+) = 0$.

(ii) $v^+ < 1 \le v^-$: $f(v^-) \le 0 < f(v^+)$ and by concavity of $f(v) = v - v^q$

$$\begin{aligned} \tfrac{1}{2}\Psi(v^-, v^+) = \; & \tfrac{v^- - 1}{v^- - v^+} f(v^+) + \tfrac{1 - v^+}{v^- - v^+} f(v^-) \\ & \le f\left(\tfrac{v^- - 1}{v^- - v^+} v^+ + \tfrac{1 - v^+}{v^- - v^+} v^- \right) = f(1) = 0 \, . \end{aligned}$$

(iii) $v^+ < v^- \le 1$: $f(v^-) \ge 0$ and $f(v^+) \ge 0$, $\Psi(v^-, v^+) = 0$.

Rates of decay

Proposition 24 *Assume that* $c \leq c_M$, $c \leq c'$ *and* $c = c_M$ *if* $c' > c$. *If* $\mu(x) = x^{-\alpha} \quad \forall\ x > 0$ *for some* $\alpha \in (0, \frac{q}{q-1})$, *then* $\lim_{t\to+\infty} \Sigma_\alpha(t) = 0$ *and*

$$\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) - \alpha \int_c^{c'} x^{-\alpha}\,u\,dx - r(c') = o(\Sigma_\alpha(t)) \quad \text{as } t \to +\infty$$

with $r(c') = \mu(c')\,(c')^{q/(q-1)}\,f\left((c')^{-1/(q-1)}\,u^-(c')\right)$.

Proof. With $c(t) := \min(s(t), c)$,

$$\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) - r(c') \leq C_q \int_0^{c(t)} x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c} - 1\right)^2 dx + q\,\alpha \int_c^{c'} x^{-\alpha}\,u$$

where $\chi(t) \equiv 0$ if $s(t) > c(t)$, $\chi(t) := \int_{s(t)}^{c} \mu\,u_\infty^c\,dx$ if $s(t) \leq c(t)$.