Symmetry and symmetry breaking in PDEs

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December 13, 2018

Nonlinear PDEs at Valparaíso
a conference in honor of Patricio Felmer's 60th anniversary

(UFSM, December 10-14, 2018)

Outline

- Introduction

▷ Symmetry, symmetry breaking: the principle of Pierre Curie
▷ The issue of the symmetry for the minimizers
▷ A competition mechanism: potential or weight versus nonlinearity
▷ The flow point of view: connecting the issue of the symmetry with asymptotic states

- Symmetry in some interpolation inequalities

▷ Gagliardo-Nirenberg-Sobolev inequalities on the sphere
▷ Caffarelli-Kohn-Nirenberg inequalities

- Symmetry in interpolation inequalities involving Aharonov-Bohm magnetic fields

▷ Aharonov-Bohm effect
▷ Subquadratic magnetic interpolation inequalities
▷ Aharonov-Bohm magnetic interpolation inequalities in $\mathbb{R}^2$

- The Cucker-Smale model: symmetry, phase transition, dynamics

Symmetry in some interpolation inequalities

- Gagliardo-Nirenberg-Sobolev inequalities on the sphere
- Caffarelli-Kohn-Nirenberg inequalities

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1,\, 2$

Theorem

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas & Spruck, 1981], [Bidaut-Véron & Véron, 1991]

Bifurcation point of view

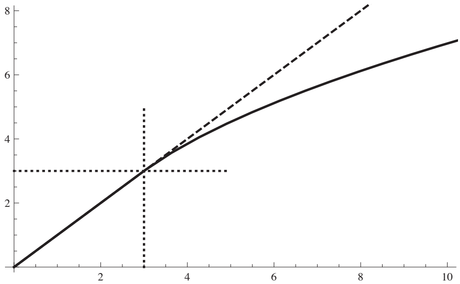


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \ge d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^{\#} := \frac{2\,d^2+1}{(d-1)^2}$

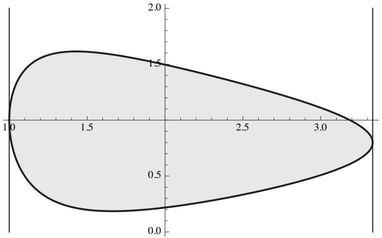
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

▷ The critical Caffarelli-Kohn-Nirenberg inequality
[JD, Esteban, Loss]

[▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities]
[JD. Esteban, Loss, Muratori]

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

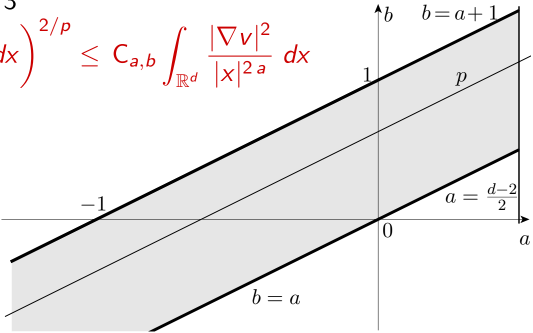
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))

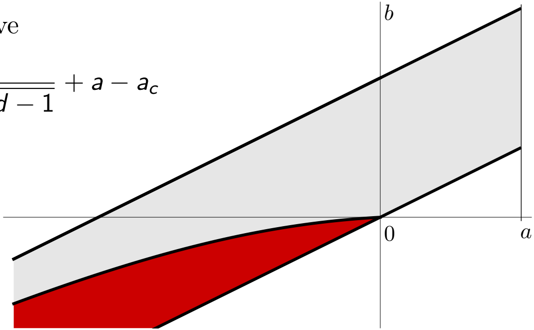
(Caffarelli, Kohn, Nirenberg (1984))

[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

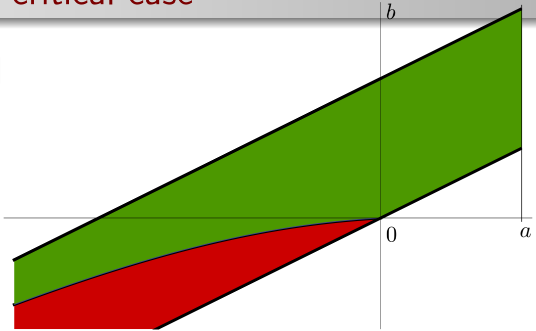
The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r = |x|\,, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} \ge \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*, i.e., for $\Lambda > \Lambda_1$ (explicit)

The variational problem on the cylinder

$$\Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathcal{C})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

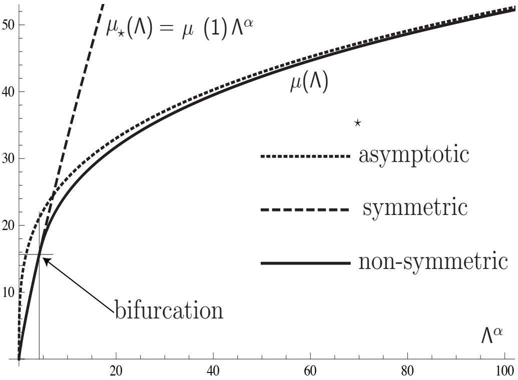
is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathbb{R})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} = \mu_\star(1)\,\Lambda^\alpha$$

Symmetry means $\mu(\Lambda) = \mu_\star(\Lambda)$
Symmetry breaking means $\mu(\Lambda) < \mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

Symmetry in one slide: 3 steps

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $\mathcal{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal{L}_\alpha\,u^m$

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}\\
&\ge(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu
\end{aligned}$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows*... a course on entropy methods (see webpage)

- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: Proc. Int. Cong. of Math., Rio de Janeiro, 3: 2279-2304, 2018.

- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*... the parabolic point of view Journal of elliptic and parabolic equations, 2: 267-295, 2016.

Symmetry in Aharonov-Bohm magnetic fields

- Aharonov-Bohm effect
- Subquadratic magnetic interpolation inequalities
  - ▷ on the circle: magnetic rings
  - ▷ on the torus
- Aharonov-Bohm magnetic interpolation inequalities in $\mathbb{R}^2$

Joint work with D. Bonheure, M.J. Esteban, A. Laptev, & M. Loss

Aharonov-Bohm effect

A major difference between classical mechanics and quantum mechanics is that particles are described by a non-local object, the wave function. Quantum particles can interact with an electromagnetic field even if they are "localized" (from the experimental point of view) in a region where the fields are zero, or if the fields are supported on zero-measure sets

In 1959 Y. Aharonov and D. Bohm proposed a series of experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects*

One of the proposed experiments relies on a long, thin solenoid which produces a magnetic field such that the region in which the magnetic field is non-zero can be approximated by a line in dimension $d = 3$ and by a point in dimension $d = 2$

Notation

The *magnetic Laplacian* is defined via a *magnetic potential* $\mathbf{A}$ by

$$-\Delta_{\mathbf{A}}\,\psi = -\Delta\,\psi - 2\,i\,\mathbf{A}\cdot\nabla\psi + |\mathbf{A}|^2\psi - i\,(\operatorname{div}\mathbf{A})\,\psi$$

The *magnetic field* is $\mathbf{B} = \operatorname{curl}\mathbf{A}$

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d) := \left\{\psi\in\mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\,\psi\in\mathrm{L}^2(\mathbb{R}^d)\right\}$$

The *magnetic gradient* takes the form

$$\nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

- Dimension $d=2$: *polar coordinates* (r,θ)

$$r=|x|=\sqrt{x_1^2+x_2^2}\quad\text{and}\quad r\,e^{i\theta}=x_1+i\,x_2$$

- Dimension $d=3$: cylindrical coordinates (ρ,θ,z)

$$\rho=\sqrt{x_1^2+x_2^2}\,,\quad \rho\,e^{i\theta}=x_1+i\,x_2\quad\text{and}\quad z=x_3$$

Aharonov-Bohm magnetic fields

- Dimension $d = 1$:

$$\nabla_{\mathbf{A}} = \tfrac{\partial}{\partial\theta} - i\,a\,, \quad -\Delta_{\mathbf{A}} = -\left(\tfrac{\partial}{\partial\theta} - i\,a\right)^2$$

- Dimension $d = 2$: $\mathbf{A} = \dfrac{a}{r^2}\,(-x_2, x_1) = \dfrac{a}{r^2}\,\mathbf{e}_\theta$

$$\nabla_{\mathbf{A}} = \left(\tfrac{\partial}{\partial r}, \tfrac{1}{r}\left(\tfrac{\partial}{\partial\theta} - i\,a\right)\right)\,, \quad -\Delta_{\mathbf{A}} = -\tfrac{\partial^2}{\partial r^2} - \tfrac{1}{r}\,\tfrac{\partial}{\partial r} - \tfrac{1}{r^2}\,\left(\tfrac{\partial}{\partial\theta} - i\,a\right)^2$$

$\{\mathbf{e}_r = \frac{x}{r}, \mathbf{e}_\theta\}$ is the local orthogonal basis

- Dimension $d = 3$: $\mathbf{A} = \dfrac{a}{\rho^2}\,(-x_2, x_1, 0)$

$$\nabla_{\mathbf{A}} = \left(\tfrac{\partial}{\partial\rho}, \tfrac{1}{\rho}\left(\tfrac{\partial}{\partial\theta} - i\,a\right), \tfrac{\partial}{\partial z}\right)\,, \quad -\Delta_{\mathbf{A}} = -\tfrac{\partial^2}{\partial\rho^2} - \tfrac{1}{\rho}\,\tfrac{\partial}{\partial\rho} - \tfrac{1}{\rho^2}\,\left(\tfrac{\partial}{\partial\theta} - i\,a\right)^2 - \tfrac{\partial^2}{\partial z^2}$$

$\mathbf{A}$ is singular at $x_1 = x_2 = 0$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is a measure supported in the set $x_1 = x_2 = 0$

Subquadratic magnetic interpolation inequalities

Inequalities involving L^p norms with $1 < p < 2$ are generically designated as *subquadratic inequalities*

▷ Magnetic rings in the subquadratic range

▷ A result of symmetry and symmetry breaking on the torus

Magnetic rings: interpolation inequalities on $\mathbb{S}^1$

$p \in [1,2)$, a non-magnetic interpolation inequality [Bakry, Emrey, 1984]

$$(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^1)$$

Lemma

Let $a \in \mathbb{R}$ and $p \in [1,2)$. Then there exists a concave monotone increasing function $\mu \mapsto \lambda_{a,p}(\mu)$ on $\mathbb{R}^+$ such that

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\, \psi \in \mathrm{H}^1(\mathbb{S}^1,\mathbb{C})$$

- Diamagnetic inequality and non-magnetic interpolation inequality
- Existence of an optimal function: Sobolev's inequalities, compactness and semi-continuity

Properties

- A non-vanishing property: if $\psi \in \mathrm{H}^1(\mathbb{S}^1)$ is a non-trivial optimal function, then $\psi(s) \neq 0$ for any $s \in \mathbb{S}^1$. Take $v_1(s) + i\, v_2(s) = \psi(s)\, e^{ias}$ and consider the Wronskian $w = (v_1\, v_2' - v_1'\, v_2)$
- Use the Euler-Lagrange equation for the phase

$$\mathcal{Q}_{a,p,\mu}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

- The minimization problem is reduced to the study of the inequality

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \geq \lambda_{a,p}(\mu)\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^1)$$

where u is now a real valued function

A rigidity result

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}+\mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\ge\lambda_{a,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^1)$$

Proposition

Let $p\in(1,2)$, $a\in(0,1/2)$, and $\mu>0$.
(i) If $\mu\,(2-p)+4\,a^2\le1$, then $\lambda_{a,p}(\mu)=a^2+\mu$ and equality is achieved only by the constants
(ii) If $\mu\,(2-p)+4\,a^2>1$, then $\lambda_{a,p}(\mu)<a^2+\mu$ and equality is not achieved by the constants

$$\begin{aligned}\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}&+a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}+\mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\\&=(1-4\,a^2)\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\frac{\mu}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\right)\\&\qquad+4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\frac14\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right)\end{aligned}$$

Aharonov-Bohm magnetic interpolation inequalities on $\mathbb{T}^2$

$\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1 \approx [-\pi,\pi) \times [-\pi,\pi) \ni (x,y)$
$d\sigma$ the uniform probability measure

$$\nabla_{\mathbf{A}}\,\psi = \left(\psi_x, \psi_y - i\,a\,\psi\right), \quad \iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma = \iint_{\mathbb{T}^2} \left(|\psi_x|^2 + |\psi_y - \,i\,a\,\psi|^2\right) d\sigma$$

Lemma (A magnetic ground state estimate)

Assume that $a \in (0,1/2)$. *Then*

$$\iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma \ge a^2 \iint_{\mathbb{T}^2} |\psi|^2\,d\sigma \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2)$$

We make a Fourier decomposition on the basis $\left(e^{i\,\ell\,x}\,e^{i\,k\,y}\right)_{k,\ell\in\mathbb{Z}}$
If $a \in (0,1/2)$, then λ_{00} is the lowest mode

$$k = 0\,,\ \ell = 0\ :\ \lambda_{00} = a^2$$
$$k = 1\,,\ \ell = 0\ :\ \lambda_{10} = (1-a)^2 > a^2$$
$$k = 0\,,\ \ell = 1\ :\ \lambda_{01} = 1 + a^2$$

A magnetic interpolation inequality in the flat torus

By tensorization, for any $p \in [1,2)$

$$(2-p)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2)$$

Lemma

Let $p \in [1,2)$, $a \in (0,1/2)$. Then

$$\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \Lambda_{a,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2)$$

$\mu \mapsto \Lambda_{a,p}(\mu)$ *is concave increasing on* $(0,+\infty)$, $\lim_{\mu\to 0_+} \Lambda_{a,p}(\mu) = a^2$

$$\Lambda_{a,p}(\mu) \ge \mu + \big(1 - \mu\,(2-p)\big)\,a^2 \quad \textit{for any} \quad \mu \le \frac{1}{2-p}$$

A symmetry result in the subquadratic regime

$\lambda_{a,p}(\mu)$: the optimal constant on $\mathbb{S}^1$
$\Lambda_{a,p}(\mu)$: the optimal constant on $\mathbb{T}^2$
We recall that the magnetic energy on $\mathbb{T}^2$ is

$$\iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}} \psi|^2 \, d\sigma = \iint_{\mathbb{T}^2} \left(|\psi_x|^2 + |\psi_y - i\, a\, \psi|^2\right) d\sigma$$

Proposition

Let $p \in [1,2)$, $a \in (0,1/2)$. Then

$$\Lambda_{a,p}(\mu) = \lambda_{a,p}(\mu) \quad \text{if} \quad \mu \leq \frac{1}{p-2}$$

and any optimal function is then constant w.r.t. x

$\Lambda_{a,p}(\mu) = a^2 + \mu$ *if and only if* $\mu\,(2-p) + 4\,a^2 \leq 1$

Proof

Notation $⨍ f\,dx := \frac{1}{2\pi}\int_{-\pi}^{\pi} f\,dx$

$$\|\nabla_{\mathbf{A}}\,u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{T}^2)}$$

$$\geq \|\partial_x\psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{T}^2)} - \mu ⨍ \left(⨍ |\psi|^p\,dy\right)^{\frac{2}{p}} dx$$

Let us define $u := |\psi|$, $v(x) := \left(⨍ |u(x,y)|^p\,dy\right)^{1/p}$. By Hölder ($p \leq 2$)

$$|v_x| = v^{1-p} ⨍ u^{p-1}\,u_x\,dy \leq v^{1-p}\left(⨍ u^p\,dy\right)^{\frac{p-1}{p}}\left(⨍ |u_x|^2\,dy\right)^{\frac{1}{2}}$$

that is, $|v_x|^2 \leq ⨍ |u_x|^2\,dy \leq ⨍ |\partial_x\psi|^2\,dy$. With $\mu \leq 1/(2-p)$,

$$\int_{\mathbb{S}^1} |v_x|^2\,d\sigma + \mu\left(\int_{\mathbb{S}^1} |v|^p\,d\sigma\right)^{2/p} - \mu\int_{\mathbb{S}^1} |v|^2\,d\sigma + \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$$
$$\geq \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$$

Aharonov-Bohm magnetic interpolation inequalities in $\mathbb{R}^2$

▷ On $\mathbb{R}^2$ without weights, there is a loss of compactness
▷ On $\mathbb{R}^2$ with weights, optimal functions exist
▷ there is a range of symmetry breaking and a range of symmetry

Magnetic interpolation inequalities without weights

$$\|\nabla_{\mathbf{A}}\,\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} + \lambda\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} \ge \mu_{a,p}(\lambda)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^2)} \quad \forall\,\psi \in \mathrm{H}^1_a(\mathbb{R}^2)$$

Aharonov-Bohm magnetic potential $\mathbf{A}(x) = a\,|x|^{-2}\,\mathbf{e}_\theta$

Proposition

Let $a \in \mathbb{R}\setminus\mathbb{Z}$ and $p \in (2,\infty)$. The optimal constant is

$$\mu_{a,p}(\lambda) = \mathsf{C}_p\,\lambda^{\frac{p}{2}} \quad \forall\,\lambda > 0$$

and equality is not achieved on $\mathrm{H}^1(\mathbb{R}^2)\cap\mathrm{L}^p(\mathbb{R}^2)$

Magnetic Hardy-Sobolev interpolation inequalities ($d = 2$)

The *Caffarelli-Kohn-Nirenberg inequality* with $\mathsf{b} = \mathsf{c} + 2/p$

$$\int_{\mathbb{R}^2} \frac{|\nabla v|^2}{|x|^{2\mathsf{c}}}\,dx \ge \mathsf{C_c} \left(\int_{\mathbb{R}^2} \frac{|v|^p}{|x|^{\mathsf{b}\,p}}\,dx \right)^{2/p}$$

By considering $v(x) = |x|^{\mathsf{c}}\,u(x)$: the *Hardy-Sobolev inequality*

$$\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \mathsf{c}^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx \ge \mathsf{C_c} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2}\,dx \right)^{2/p}$$

The optimal functions are radially symmetric if and only if

$$\mathsf{b} \ge \mathsf{b}_{\mathrm{FS}}(\mathsf{c}) := \mathsf{c} - \frac{\mathsf{c}}{\sqrt{1+\mathsf{c}^2}}$$

according to [Felli-Schneider (2003)], [D.-Esteban-Loss (2015)]

Theorem (magnetic Hardy-Sobolev type inequality)

Let $a \in [0,1/2)$ and $p > 2$. For any $\lambda > -a^2$

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\, \psi|^2\, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\, dx \ge \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\, dx \right)^{2/p} \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)$$

with optimal function $\psi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-\frac{2}{p-2}}$, $\alpha = \frac{p-2}{2}\,\sqrt{\lambda + a^2}$ *if*

$$\lambda \le \lambda_\star := 4\, \frac{1 - 4\, a^2}{p^2 - 4} - a^2$$

Conversely, there is symmetry breaking if

$$\lambda > \lambda_{\rm FS}(a) := \frac{4}{p^2 - 4} - a^2$$

If $\lambda \le \lambda_\star$, $\mu(\lambda) = \frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,\left(\lambda + a^2\right)^{1+\frac{2}{p}} \left(\frac{2\,\sqrt{\pi}\,\Gamma\left(\frac{p}{p-2}\right)}{(p-2)\,\Gamma\left(\frac{p}{p-2}+\frac{1}{2}\right)} \right)^{1-\frac{2}{p}}$

Corollary 1

Corollary (A magnetic Caffarelli-Kohn-Nirenberg inequality)

Assume that $p \in (2, +\infty)$, $\mathbf{A}(x) = a\,|x|^{-2}\,\mathbf{e}_\theta$ *for some* $a \in [0, 1/2)$ *and* $\mathsf{c} \le 0$. *With* μ *as in Theorem 2, for any* $\gamma < \mathsf{c}^2 + a^2$, *we have that*

$$\int_{\mathbb{R}^2} \frac{|\nabla_{\mathbf{A}}\,\phi|^2}{|x|^{2\mathsf{c}}}\,dx \ge \gamma \int_{\mathbb{R}^2} \frac{|\phi|^2}{|x|^{2\mathsf{c}+2}}\,dx + \mu(\mathsf{c}^2 - \gamma)\left(\int_{\mathbb{R}^2} \frac{|\phi|^p}{|x|^{\mathsf{c}\,p+2}}\,dx\right)^{2/p}$$

and $\mu(\mathsf{c}^2 - \gamma)$ *is the optimal constant*

Take $\phi(x) = |x|^{\mathsf{c}}\,\psi(x)$

$$\int_{\mathbb{R}^2} \frac{|\nabla_{\mathbf{A}}\,\phi|^2}{|x|^{2\mathsf{c}}}\,dx = \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx$$

Corollary 2

Proposition (A magnetic Hardy inequality in $\mathbb{R}^2$)

Assume that $q \in (1,2)$, $\mathbf{A}(x) = a\,|x|^{-2}\,\mathbf{e}_\theta$ *for some* $a \in [0,1/2)$. *Then for any function* $\phi \in \mathrm{L}^q\left(\mathbb{R}^2\,|x|^{-2}\,dx\right)$, *we have*

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \mu(0)\left(\int_{\mathbb{R}^2} \frac{|\phi|^q}{|x|^2}\,dx\right)^{-\frac{1}{q}} \int_{\mathbb{R}^2} \frac{\phi}{|x|^2}\,|\psi|^2\,dx \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)$$

Moreover, $\mu(0)$ *is the optimal constant and*

$$\mu(0) = \frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,a^{2+\frac{4}{p}}\left(\frac{2\sqrt{\pi}\,\Gamma\left(\frac{p}{p-2}\right)}{(p-2)\,\Gamma\left(\frac{p}{p-2}+\frac{1}{2}\right)}\right)^{1-\frac{2}{p}} \quad \textit{if} \quad a^2 < \frac{4}{12+p^2}$$

The Cucker-Smale model

- The homogeneous model
- Phase transition
- Dynamics

Xingyu Li, in preparation

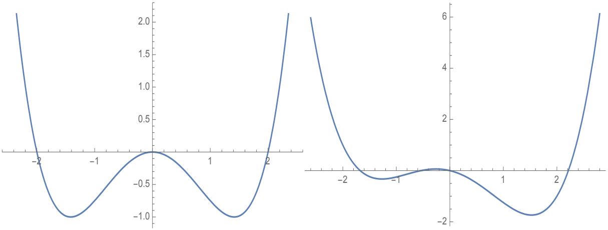
A simple version of the Cucker-Smale model

A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \Phi(v)\, f - \mathcal{U}_f\, f\right)$$

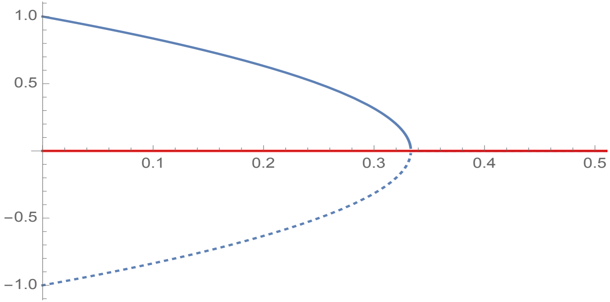
where $\mathcal{U}_f = \int v\, f\, dv$ is the average velocity (f is a probability measure)

$$\Phi(v) = \frac{1}{4}\, v^4 - \frac{1}{2}\, v^2$$



[J. Tugaut, 2014]
[A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016]

Stationary solutions: phase transition



- $d = 1$: there exists a bifurcation point $D = D_*$ such that the only stationary solution corresponds to $\mathcal{U}_f = 0$ if $D > D_*$ and there are three solutions corresponding to $\mathcal{U}_f = 0, \pm u(D)$ if $D < D_*$
- $\mathcal{U}_f = 0$ is linearly unstable if $D < D_*$

Notation: $f_\star^{(0)}$, $f_\star^{(+)}$, $f_\star^{(-)}$

Dynamics

The free energy

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \Phi \, dv - \frac{1}{2} \, |\mathcal{U}_f|^2$$

decays according to

$$\frac{d}{dt} \mathcal{F}[f(t,\cdot)] - \int_{\mathbb{R}^d} \left| D \, \frac{\nabla_v f}{f} + \nabla_v \Phi - \mathcal{U}_f \right|^2 f \, dv$$

- $d = 1$: If $\mathcal{F}[f(t=0,\cdot)] < \mathcal{F}[f_\star^{(0)}]$ and $D < D_*$, then

$$\mathcal{F}[f(t,\cdot)] = -\mathcal{F}\left[f_\star^{(\pm)}\right] \le C \, e^{-\lambda \, t}$$

- λ is the eigenvalue of the linearized problem at $f_\star^{(\pm)}$ in the weighted space $\mathrm{L}^2\left((f_\star^{(\pm)})^{-1}\right)$ with scalar product

$$\langle f, g \rangle_\pm := D \int_{\mathbb{R}^d} f \, g \left(f_\star^{(\pm)}\right)^{-1} dv - \mathcal{U}_f \, \mathcal{U}_g$$

These slides can be found at

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Thank you for your attention !