Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations (results, formal expansions, numerical results, conjecture)

Jean Dolbeault

dolbeaul@ceremade.dauphine.fr

CEREMADE
CNRS & Université Paris-Dauphine

http://www.ceremade.dauphine.fr/~dolbeaul

November 29, 2013

Université Libre de Bruxelles
Groupe de contact FNRS
Some slides related to this talk:
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+ Collaborators: M. del Pino, M. Esteban, S. Filippas, M. Loss, G. Tarantello, A. Tertikas
Introduction
A symmetry breaking mechanism

\[ f(u) + V(x) \Rightarrow \]

\[ |x| \]
The energy point of view (ground state)

- $\mu(x)$
- $\lambda = |x|$
- No potential

- $V(x)$
- The potential is pushing the mass away from the "center"

- The solution with broken symmetry is energetically more favourable than the one with radial symmetricity
Caffarelli-Kohn-Nirenberg inequalities (Part I)

Joint work(s) with M. Esteban, M. Loss and G. Tarantello
Caffarelli-Kohn-Nirenberg (CKN) inequalities

\[
\left( \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}} \, dx \quad \forall \, u \in D_{a,b}
\]

with \( a \leq b \leq a + 1 \) if \( d \geq 3 \), \( a < b \leq a + 1 \) if \( d = 2 \), and \( a \neq \frac{d - 2}{2} =: a_c \)

\[
p = \frac{2d}{d - 2 + 2(b - a)}
\]

\[
D_{a,b} := \left\{ |x|^{-b} u \in L^p(\mathbb{R}^d, dx) : |x|^{-a} |\nabla u| \in L^2(\mathbb{R}^d, dx) \right\}
\]
The symmetry issue

\[
\left( \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla u|^2}{|x|^{2a}} \, dx \quad \forall \, u \in D_{a,b}
\]

\( C_{a,b} = \) best constant for general functions \( u \)

\( C_{a,b}^* = \) best constant for radially symmetric functions \( u \)

\[
C_{a,b}^* \leq C_{a,b}
\]

Up to scalar multiplication and dilation, the optimal radial function is

\[
u_{a,b}^*(x) = |x|^{a + \frac{d}{2} \frac{b-a}{b-a+1}} \left( 1 + |x|^2 \right)^{-\frac{d-2+2(b-a)}{2(1+a-b)}}
\]

Questions: is optimality (equality) achieved? do we have \( u_{a,b} = u_{a,b}^* \) ?
Known results

[Aubin, Talenti, Lieb, Chou-Chu, Lions, Catrina-Wang, ...]

- Extremals exist for $a < b < a + 1$ and $0 \leq a \leq \frac{d-2}{2}$,
  for $a \leq b < a + 1$ and $a < 0$ if $d \geq 2$

- Optimal constants are never achieved in the following cases
  - “critical / Sobolev” case: for $b = a < 0$, $d \geq 3$
  - “Hardy” case: $b = a + 1$, $d \geq 2$

- If $d \geq 3$, $0 \leq a < \frac{d-2}{2}$ and $a \leq b < a + 1$, the extremal functions are radially symmetric ... $u(x) = |x|^a v(x) +$ Schwarz symmetrization
More results on symmetry

Radial symmetry has also been established for \( d \geq 3, \ a < 0, \ |a| \) small and \( 0 < b < a + 1 \): [Lin-Wang, Smets-Willem]

Schwarz foliated symmetry [Smets-Willem]

\( d = 3 \): optimality is achieved among solutions which depend only on the “latitude” \( \theta \) and on \( r \). Similar results hold in higher dimensions.
Symmetry breaking

[Catrina-Wang, Felli-Schneider] if \( a < 0, a \leq b < b^{FS}(a) \), the extremal functions ARE NOT radially symmetric!

\[
b^{FS}(a) = \frac{d (d - 2 - 2a)}{2 \sqrt{(d - 2 - 2a)^2 + 4(d - 1)}} - \frac{1}{2} (d - 2 - 2a)
\]

[Catrina-Wang] As \( a \to -\infty \), optimal functions look like some decentered optimal functions for some Gagliardo-Nirenberg interpolation inequalities (after some appropriate transformation)
Approaching Onofri’s inequality \((d = 2)\)

[1] [J.D., M. Esteban, G. Tarantello] A generalized Onofri inequality

On \(\mathbb{R}^2\), consider \(d\mu_\alpha = \frac{\alpha + 1}{\pi} \frac{|x|^{2\alpha} \, dx}{(1 + |x|^{2(\alpha + 1)})^2}\) with \(\alpha > -1\)

\[
\log \left( \int_{\mathbb{R}^2} e^v \, d\mu_\alpha \right) - \int_{\mathbb{R}^2} v \, d\mu_\alpha \leq \frac{1}{16 \pi (\alpha + 1)} \| \nabla v \|^2_{L^2(\mathbb{R}^2, dx)}
\]

For \(d = 2\), radial symmetry holds if \(-\eta < a < 0\) and \(-\varepsilon(\eta) a \leq b < a + 1\)

**Theorem 1.** [J.D.-Esteban-Tarantello] *For all \(\varepsilon > 0 \exists \eta > 0 \text{ s.t. for } a < 0, |a| < \eta\)*

(i) if \(|a| > \frac{2}{p - \varepsilon} (1 + |a|^2)\), then
\[C_{a,b} > C_{a,b}^* \text{ (symmetry breaking)}\]

(ii) if \(|a| < \frac{2}{p + \varepsilon} (1 + |a|^2)\), then
\[C_{a,b} = C_{a,b}^* \text{ and } u_{a,b} = u_{a,b}^*\]
For $d \geq 2$, radial symmetry can be proved when $b$ is close to $a + 1$

Theorem 2. [J.D.-Esteban-Loss-Tarantello] Let $d \geq 2$. For every $A < 0$, there exists $\varepsilon > 0$ such that the extremals are radially symmetric if $a + 1 - \varepsilon < b < a + 1$ and $a \in (A, 0)$. So they are given by $u_{a,b}^*$, up to a scalar multiplication and a dilation.
The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve.

**Theorem 3.** [J.D.-Esteban-Loss-Tarantello] For all \( d \geq 2 \), there exists a continuous function \( \ast : (2, 2^* ) \longrightarrow (-\infty, 0) \) such that \( \lim_{p \to 2^-} \ast (p) = 0 \), \( \lim_{p \to 2^+} \ast (p) = -\infty \) and

(i) If \((a, p) \in (\ast (p), \frac{d-2}{2}) \times (2, 2^*)\), all extremals radially symmetric

(ii) If \((a, p) \in (-\infty, \ast (p)) \times (2, 2^*)\), none of the extremals is radially symmetric

Open question. Do the curves obtained by Felli-Schneider and ours coincide?
Emden-Fowler transformation and the cylinder $C = \mathbb{R} \times S^{d-1}$

t = \log |x|, \quad \omega = \frac{x}{|x|} \in S^{d-1}, \quad w(t, \omega) = |x|^{-a} v(x), \quad \Lambda = \frac{1}{4} (d - 2 - 2a)^2

Caffarelli-Kohn-Nirenberg inequalities rewritten on the cylinder become standard interpolation inequalities of Gagliardo-Nirenberg type

$$\|w\|_{L^p(C)}^2 \leq C_{\Lambda,p} \left[ \|\nabla w\|_{L^2(C)}^2 + \Lambda \|w\|_{L^2(C)}^2 \right]$$

$$\mathcal{E}_\Lambda[w] := \|\nabla w\|_{L^2(C)}^2 + \Lambda \|w\|_{L^2(C)}^2$$

$$C_{\Lambda,p}^{-1} := C_{a,b}^{-1} = \inf \left\{ \mathcal{E}_\Lambda(w) : \|w\|_{L^p(C)} = 1 \right\}$$

$$a < 0 \implies \Lambda > a_c^2 = \frac{1}{4} (d - 2)^2$$

"critical / Sobolev" case: $b - a \to 0 \iff p \to \frac{2d}{d - 2}$

"Hardy" case: $b - (a + 1) \to 0 \iff p \to 2_+$
Scaling and consequences

A scaling property along the axis of the cylinder \((d \geq 2)\)

Let \(w_{\sigma}(t, \omega) := w(\sigma t, \omega)\) for any \(\sigma > 0\)

\[
\mathcal{F}_{\sigma^2 \Lambda, p}(w_{\sigma}) = \sigma^{1+2/p} \mathcal{F}_{\Lambda, p}(w) - \sigma^{-1+2/p} (\sigma^2 - 1) \frac{\int_C |\nabla_\omega w|^2 \, dy}{\left(\int_C |w|^p \, dy\right)^{2/p}}
\]

**Lemma 4.** [JD, Esteban, Loss, Tarantello] If \(d \geq 2, \Lambda > 0\) and \(p \in (2, 2^*)\)

(i) If \(C_{\Lambda,p}^d = C_{\Lambda,p}^{d,*}\), then \(C_{\lambda,p}^d = C_{\lambda,p}^{d,*}\) and \(w_{\lambda,p} = w_{\lambda,p}^*\), for any \(\lambda \in (0, \Lambda)\)

(ii) If there is a non radially symmetric extremal \(w_{\Lambda,p}\), then \(C_{\lambda,p}^d > C_{\lambda,p}^{d,*}\) for all \(\lambda > \Lambda\)
Corollary 5. [JD, Esteban, Loss, Tarantello] Let $d \geq 2$. For all $p \in (2, 2^*)$, \( \Lambda^*(p) \in (0, \Lambda^{FS}(p)) \) and

(i) If \( \lambda \in (0, \Lambda^*(p)) \), then \( w_{\lambda,p} = w^*_{\lambda,p} \) and clearly, \( C^d_{\lambda,p} = C^d_{\lambda,p} \)

(ii) If \( \lambda = \Lambda^*(p) \), then \( C^d_{\lambda,p} = C^d_{\lambda,p} \)

(iii) If \( \lambda > \Lambda^*(p) \), then \( C^d_{\lambda,p} > C^d_{\lambda,p} \)

Upper semicontinuity is easy to prove
For continuity, a delicate spectral analysis is needed
One more result on symmetry

[Betta, Brock, Mercaldo, Posteraro]
Caffarelli-Kohn-Nirenberg inequalities (Part II) and Logarithmic Hardy inequalities

Joint work with M. del Pino, S. Filippas and A. Tertikas
Let $2^* = \infty$ if $d = 1$ or $d = 2$, $2^* = 2d/(d - 2)$ if $d \geq 3$ and define

$$\vartheta(p, d) := \frac{d(p - 2)}{2p}$$

**Theorem 6.** [Caffarelli-Kohn-Nirenberg-84] Let $d \geq 1$. For any $\theta \in [\vartheta(p, d), 1]$, with $p = \frac{2d}{d - 2 + 2(b - a)}$, there exists a positive constant $C_{\text{CKN}}(\theta, p, a)$ such that

$$
\left( \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^b} \, dx \right)^{\frac{2}{p}} \leq C_{\text{CKN}}(\theta, p, a) \left( \int_{\mathbb{R}^d} \frac{\nabla u^2}{|x|^{2a}} \, dx \right)^{\theta} \left( \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \, dx \right)^{1 - \theta}
$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $C^*_{\text{CKN}}(\theta, p, a)$ and

$$C_{\text{CKN}}(\theta, p, a) \geq C^*_{\text{CKN}}(\theta, p, a) = C^*_{\text{CKN}}(\theta, p) \Lambda^{\frac{p-2}{2p} - \theta}$$

$$C^*_{\text{CKN}}(\theta, p) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \left[ \frac{2+(2\theta-1)p}{2p} \right]^{\frac{6-p}{2p}} \left[ \frac{\Gamma\left(\frac{2}{p-2} + \frac{1}{2}\right)}{\sqrt{\pi} \, \Gamma\left(\frac{2}{p-2}\right)} \right]^{\frac{p-2}{p}}$$
A “logarithmic Hardy inequality”

**Theorem 7.** [del Pino, J.D. Filippas, Tertikas] Let $d \geq 3$. There exists a constant $C_{\text{LH}} \in (0, S]$ such that, for all $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \, dx = 1$, we have

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log \left( |x|^{d-2} |u|^2 \right) \, dx \leq \frac{d}{2} \log \left[ C_{\text{LH}} \int_{\mathbb{R}^d} |\nabla u|^2 \, dx \right]$$

A “weighted logarithmic Hardy inequality” (WLH)

**Theorem 8.** [del Pino, J.D. Filippas, Tertikas] Let $d \geq 1$. Suppose that $a < (d - 2)/2$, $\gamma \geq d/4$ and $\gamma > 1/2$ if $d = 2$. Then there exists a positive constant $C_{\text{WLH}}$ such that, for any $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ normalized by $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(\alpha + 1)}} \, dx = 1$, we have

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(\alpha + 1)}} \log \left( |x|^{d-2-2\alpha} |u|^2 \right) \, dx \leq 2\gamma \log \left[ C_{\text{WLH}} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\alpha}} \, dx \right]$$
Theorem 9. [del Pino, J.D. Filippas, Tertikas] Let $d \geq 1$, $a < (d - 2)/2$ and $\gamma \geq 1/4$. If $u = u(|x|) \in D^{1,2}_a(\mathbb{R}^d)$ is radially symmetric, and $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \ dx = 1$, then

$$
\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \log (|x|^{d-2-2a} |u|^2) \ dx \leq 2 \gamma \log \left[ C^*_{WLH} \int_{\mathbb{R}^d} \frac{|
abla u|^2}{|x|^{2a}} \ dx \right]
$$

where

$$
C^*_{WLH} = \frac{1}{\gamma} \frac{[\Gamma(d/2)]^{1/2}}{(8 \pi^{d+1} e)^{1/4} (d-2-2a)^2} \left( \frac{4 \gamma - 1}{(d-2-2a)^2} \right)^{4 \gamma - 1/4 - 1/4} \text{ if } \gamma > \frac{1}{4}
$$

$$
C^*_{WLH} = 4 \frac{[\Gamma(d/2)]^2}{8 \pi^{d+1} e} \text{ if } \gamma = \frac{1}{4}
$$

If $\gamma > \frac{1}{4}$, equality is achieved by the function

$$
u = \frac{\tilde{u}}{\int_{\mathbb{R}^d} \frac{|	ilde{u}|^2}{|x|^2} \ dx} \text{ where } \tilde{u}(x) = |x|^{-\frac{d-2-2a}{2}} \exp \left( -\frac{(d-2-2a)^2}{4(4 \gamma - 1)} \left[ \log |x| \right]^2 \right)$$
Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

Joint work with Maria J. Esteban
First existence result: the sub-critical case

**Theorem 10.** [J.D. Esteban] Let \( d \geq 2 \) and assume that \( a \in (-\infty, a_c) \)

(i) For any \( p \in (2, 2^*) \) and any \( \theta \in (\vartheta(p, d), 1) \), the Caffarelli-Kohn-Nirenberg inequality (CKN)

\[
\left( \int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b_p}} \, dx \right)^{\frac{2}{p}} \leq C(\theta, p, a) \left( \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}} \, dx \right)^\theta \left( \int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \, dx \right)^{1-\theta}
\]

admits an extremal function in \( \mathcal{D}^{1,2}_a(\mathbb{R}^d) \)

**Critical case:** there exists a continuous function \( a^* : (2, 2^*) \to (-\infty, a_c) \) such that the inequality also admits an extremal function in \( \mathcal{D}^{1,2}_a(\mathbb{R}^d) \) if \( \theta = \vartheta(p, d) \) and \( a \in (a^*(p), a_c) \)

(ii) For any \( \gamma > d/4 \), the weighted logarithmic Hardy inequality (WLH)

\[
\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2(a+1)}} \log \left( |x|^{d-2a} |u|^2 \right) \, dx \leq 2 \gamma \log \left[ C_{WLH} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2a}} \, dx \right]
\]

admits an extremal function in \( \mathcal{D}^{1,2}_a(\mathbb{R}^d) \)

**Critical case:** idem if \( \gamma = d/4 \), \( d \geq 3 \) and \( a \in (a^*, a_c) \) for some \( a^* \in (-\infty, a_c) \)
Existence for CKN

$d = 3, \theta = 1$

d = 3, $\theta = 0.8$
Second existence result: the critical case

**Theorem 11** (Critical cases). [J.D. Esteban]

(i) If $\theta = \vartheta(p, d)$ and $C_{GN}(p) < C_{CKN}(\theta, p, a)$, then (CKN) admits an extremal function in $\mathcal{D}^{1,2}_{a,d}(\mathbb{R}^d)$.

(ii) If $\gamma = d/4$, $d \geq 3$, and $C_{LS} < C_{WLH}(\gamma, a)$, then (WLH) admits an extremal function in $\mathcal{D}^{1,2}_{a,d}(\mathbb{R}^d)$.

If $a \in (a_*, a_c)$ then

$$C_{LS} < C_{WLH}(d/4, a)$$

$$a_* := a_c - \sqrt{(d - 1)e (2^{d+1} \pi)^{-1/(d-1)} \Gamma(d/2)^2/(d-1)}$$
Radial symmetry and symmetry breaking

Joint work with
M. del Pino, S. Filippas and A. Tertikas (symmetry breaking)
Maria J. Esteban, Gabriella Tarantello and Achilles Tertikas
Implementing the method of Catrina-Wang / Felli-Schneider

Among functions $w \in H^1(C)$ which depend only on $s$, the minimum of

$$
\mathcal{J}[w] := \int_C \left( |\nabla w|^2 + \frac{1}{4} (d - 2 - 2a)^2 |w|^2 \right) dy - [C^*(\theta, p, a)]^{-\frac{1}{p}} \frac{(\int_C |w|^p \, dy)^{\frac{2}{p} - \theta}}{(\int_C |w|^2 \, dy)^{\frac{1}{1 - \theta}}}
$$

is achieved by $\bar{w}(y) := \left[ \cosh(\lambda s) \right]^{-\frac{2}{p-2}}$, $y = (s, \omega) \in \mathbb{R} \times S^{d-1} = C$ with

$$
\lambda := \frac{1}{4} (d - 2 - 2a) (p - 2) \sqrt{\frac{p+2}{2p \theta - (p-2)}}
$$

as a solution of

$$
\lambda^2 (p-2)^2 w'' - 4w + 2p |w|^{p-2} w = 0
$$

Spectrum of $L := -\Delta + \kappa \bar{w}^{p-2} + \mu$ is given for $\sqrt{1 + 4 \kappa / \lambda^2} \geq 2j + 1$ by

$$
\lambda_{i,j} = \mu + i (d + i - 2) - \frac{\lambda^2}{4} \left( \sqrt{1 + \frac{4 \kappa}{\lambda^2}} - (1 + 2j) \right)^2 \quad \forall \, i, j \in \mathbb{N}
$$

- The eigenspace of $L$ corresponding to $\lambda_{0,0}$ is generated by $\bar{w}$
- The eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_C \bar{w} \phi_{(1,0)} \, dy = 0$ and $\int_C \bar{w}^{p-1} \phi_{(1,0)} \, dy = 0$
- If $\lambda_{1,0} < 0$, optimal functions for (CKN) cannot be radially symmetric and

$$
C(\theta, p, a) > C^*(\theta, p, a)
$$
Schwarz’ symmetrization

With \( u(x) = |x|^a v(x) \), (CKN) is then equivalent to

\[
\| |x|^{a-b} v \|_{L^p(\mathbb{R}^N)}^2 \leq C_{\text{CKN}}(\theta, p, \Lambda) (A - \lambda B)^\theta B^{1-\theta}
\]

with \( A := \| \nabla v \|_{L^2(\mathbb{R}^N)}^2, B := \| |x|^{-1} v \|_{L^2(\mathbb{R}^N)}^2 \) and \( \lambda := a (2 a_c - a) \). We observe that the function \( B \mapsto h(B) := (A - \lambda B)^\theta B^{1-\theta} \) satisfies

\[
\frac{h'(B)}{h(B)} = \frac{1 - \theta}{B} - \frac{\lambda \theta}{A - \lambda B}
\]

By Hardy’s inequality \((d \geq 3)\), we know that

\[
A - \lambda B \geq \inf_{a > 0} (A - a (2 a_c - a) B) = A - a_c^2 B > 0
\]

and so \( h'(B) \leq 0 \) if \((1 - \theta) A < \lambda B \iff \frac{A}{B} < \lambda / (1 - \theta)\)

By interpolation \( \frac{A}{B} \) is small if \( a_c - a > 0 \) is small enough, for \( \theta > \vartheta(p, d) \)
and \( d \geq 3 \)
Regions in which Schwarz’ symmetrization holds

Here $d = 5$, $a_c = 1.5$ and $p = 2.1, 2.2, \ldots 3.2$
Symmetry holds if $a \in [a_0(\theta, p), a_c)$, $\theta \in (\vartheta(p, d), 1)$
Horizontal segments correspond to $\theta = \vartheta(p, d)$
Hardy’s inequality: the above symmetry region is contained in $\theta > (1 - \frac{a}{a_c})^2$

Alternatively, we could prove the symmetry by the moving planes method in the same region
The zones in which existence is known are:

(1) extremals are achieved among radial functions, by the Schwarz symmetrization method

(1)+(2) this follows from the explicit *a priori* estimates; $\Lambda_1 = (a_c - a_1)^2$

(1)+(2)+(3) this follows by comparison of the optimal constant for (CKN) with the optimal constant in the corresponding Gagliardo-Nirenberg-Sobolev inequality
Summary (2/2): Symmetry and symmetry breaking for (CKN)

The zone of symmetry breaking contains:

1. by linearization around radial extremals
2. by comparison with the Gagliardo-Nirenberg-Sobolev inequality

In (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4), that is, for $a_0 \leq a < a_c$, symmetry holds by the Schwarz symmetrization.
One bound state Lieb-Thirring inequalities and symmetry

Joint work with Maria J. Esteban and M. Loss
Symmetry: a new quantitative approach

\[ b_\star(a) := \frac{d (d - 1) + 4 d (a - a_c)^2}{6 (d - 1) + 8 (a - a_c)^2} + a - a_c. \]

**Theorem 12.** Let \( d \geq 2 \). When \( a < 0 \) and \( b_\star(a) \leq b < a + 1 \), the extremals of the Caffarelli-Kohn-Nirenberg inequality with \( \theta = 1 \) are radial and

\[ C_{a,b}^d = \left| \mathbb{S}^{d-1} \right| \frac{p-2}{p} \left[ \frac{(a-a_c)^2 (p-2)^2}{p+2} \right]^{\frac{p-2}{2p}} \left[ \frac{p+2}{2 p (a-a_c)^2} \right]^{\frac{6-p}{2p}} \left[ \frac{4}{p+2} \right]^{\frac{6-p}{2p}} \left[ \frac{\Gamma \left( \frac{2}{p-2} + \frac{1}{2} \right)}{\sqrt{\pi} \, \Gamma \left( \frac{2}{p-2} \right)} \right]^{\frac{p-2}{p}} \]
The symmetry region

\[ b = a + 1 \]

Symmetry region

\[ b = b_{FS}(a) \]

Symmetry breaking region

\[ b = b_*(a) \]
The symmetry result on the cylinder

\[ \Lambda_*(p) := \frac{(d - 1)(6 - p)}{4(p - 2)} \]

\[ d\omega : \text{the uniform probability measure on } S^{d-1} \]

\[ L^2: \text{the Laplace-Beltrami operator on } S^{d-1} \]

**Theorem 13.** Let \( d \geq 2 \) and let \( u \) be a non-negative function on \( C = \mathbb{R} \times S^{d-1} \) that satisfies

\[ -\partial_s^2 u - L^2 u + \Lambda u = u^{p-1} \]

and consider the symmetric solution \( u_* \). Assume that

\[ \int_C |u(s, \omega)|^p \, ds \, d\omega \leq \int_{\mathbb{R}} |u_*(s)|^p \, ds \]

for some \( 2 < p < 6 \) satisfying \( p \leq \frac{2d}{d-2} \). If \( \Lambda \leq \Lambda_*(p) \), then for a.e. \( \omega \in S^{d-1} \) and \( s \in \mathbb{R} \), we have \( u(s, \omega) = u_*(s - s_0) \) for some constant \( s_0 \).
The one-bound state version of the Lieb-Thirring inequality

Let $K(\Lambda, p, d) := C^d_{a,b}$ and

$$\Lambda^d_\gamma(\mu) := \inf \left\{ \Lambda > 0 : \mu^{\frac{2\gamma}{\gamma+1}} = 1/K(\Lambda, p, d) \right\}$$

**Lemma 14.** For any $\gamma \in (2, \infty)$ if $d = 1$, or for any $\gamma \in (1, \infty)$ such that $\gamma \geq \frac{d-1}{2}$ if $d \geq 2$, if $V$ is a non-negative potential in $L^{\gamma+\frac{1}{2}}(C)$, then the operator $-\partial^2 - L^2 - V$ has at least one negative eigenvalue, and its lowest eigenvalue, $-\lambda_1(V)$ satisfies

$$\lambda_1(V) \leq \Lambda^d_\gamma(\mu) \quad \text{with} \quad \mu = \mu(V) := \left( \int_C V^{\gamma+\frac{1}{2}} \, ds \, d\omega \right)^{\frac{1}{\gamma}}$$

Moreover, equality is achieved if and only if the eigenfunction $u$ corresponding to $\lambda_1(V)$ satisfies $u = V^{(2\gamma-1)/4}$ and $u$ is optimal for (CKN)

Symmetry $\iff \Lambda^d_\gamma(\mu) = \Lambda^d_\gamma(1) \mu$
The one-bound state Lieb-Thirring inequality (2)

Let \( V = V(s) \) be a non-negative real valued potential in \( L^{\gamma + \frac{1}{2}}(\mathbb{R}) \) for some \( \gamma > 1/2 \) and let \(-\lambda_1(V)\) be the lowest eigenvalue of the Schrödinger operator \(-\frac{d^2}{ds^2} - V\). Then

\[
\lambda_1(V)^\gamma \leq c_{LT}(\gamma) \int_{\mathbb{R}} V^{\gamma + 1/2}(s) \, ds
\]

with \( c_{LT}(\gamma) = \frac{\pi^{-1/2}}{\gamma - 1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left( \frac{\gamma - 1/2}{\gamma + 1/2} \right)^{\gamma + 1/2} \), with equality if and only if, up to scalings and translations,

\[
V(s) = \frac{\gamma^2 - 1/4}{\cosh^2(s)} =: V_0(s)
\]

Moreover \( \lambda_1(V_0) = (\gamma - 1/2)^2 \) and the corresponding ground state eigenfunction is given by

\[
\psi_\gamma(s) = \pi^{-1/4} \left( \frac{\Gamma(\gamma)}{\Gamma(\gamma - 1/2)} \right)^{1/2} \left[ \cosh(s) \right]^{-\gamma + 1/2}
\]
The generalized Poincaré inequality

**Theorem 15.** [Bidaut-Véron, Véron] \((\mathcal{M}, g)\) is a compact Riemannian manifold of dimension \(d - 1 \geq 2\), without boundary, \(\Delta_g\) is the Laplace-Beltrami operator on \(\mathcal{M}\), the Ricci tensor \(R\) and the metric tensor \(g\) satisfy \(R \geq \frac{d-2}{d-1} (q - 1) \lambda g\) in the sense of quadratic forms, with \(q > 1\), \(\lambda > 0\) and \(q \leq \frac{d+1}{d-3}\). Moreover, one of these two inequalities is strict if \((\mathcal{M}, g)\) is \(S^{d-1}\) with the standard metric.

If \(u\) is a positive solution of

\[
\Delta_g u - \lambda u + u^q = 0
\]

then \(u\) is constant with value \(\lambda^{1/(q-1)}\). Moreover, if \(\text{vol}(\mathcal{M}) = 1\) and

\[
D(\mathcal{M}, q) := \max \{\lambda > 0 : R \geq \frac{N-2}{N-1} (q - 1) \lambda g\}
\]

is positive, then

\[
\frac{1}{D(\mathcal{M}, q)} \int_{\mathcal{M}} |\nabla v|^2 + \int_{\mathcal{M}} |v|^2 \geq \left( \int_{\mathcal{M}} |v|^{q+1} \right)^{\frac{2}{q+1}} \quad \forall \, v \in W^{1,1}(\mathcal{M})
\]

Applied to \(\mathcal{M} = S^{d-1}\):\( D(S^{d-1}, q) = \frac{q-1}{d-1}\)
The case: $\theta < 1$

$$
\mathcal{C}(p, \theta) := \frac{(p + 2)^{\frac{p+2}{2\theta-1} - 1}}{2\theta - 1} p^2 + 2 \left(2 - p(1 - \theta)\right)^{\frac{2 - 2p(1 - \theta)}{(2\theta-1) p^2}} \frac{4(p-2)}{(2\theta-1) p^2} \left(\frac{\Gamma\left(\frac{p}{p-2}\right)}{\Gamma\left(\frac{\theta p}{p-2}\right)}\right) \left(\frac{\Gamma\left(\frac{2\theta p}{p-2}\right)}{\Gamma\left(\frac{2p}{p-2}\right)}\right)^{\frac{2(p-2)}{(2\theta-1) p^2}}
$$

Notice that $\mathcal{C}(p, \theta) \geq 1$ and $\mathcal{C}(p, \theta) = 1$ if and only if $\theta = 1$

**Theorem 16.** With the above notations, for any $d \geq 3$, any $p \in (2, 2^*)$ and any $\theta \in [\vartheta(p, d), 1)$, we have the estimate

$$
C_{CKN}^*(\theta, a, p) \leq \mathcal{C}_{CKN}(\theta, a, p) \leq C_{CKN}^*(\theta, a, p) \mathcal{C}(p, \theta)^{(2\theta - 1) p^2} 2p
$$

under the condition

$$(a - a_c)^2 \leq \frac{(d - 1)(2\theta - 3) p + 6}{4(p - 2)} \frac{1}{\mathcal{C}(p, \theta)}$$
Numerical results, formal expansion

Collaboration with Maria J. Esteban
\[ \Lambda \mapsto \min \left\{ \| \nabla w \|_{L^2(C)}^2 + \Lambda \| w \|_{L^2(C)}^2 : \| w \|_{L^p(C)} = 1 \right\} \]
Non symmetric optimal functions: grid issues

Coarse / refined / self-adaptive grids
A self-adaptive grid
Comparison with the asymptotic regime
Reparametrization

\[ Q^\theta_\Lambda[u] := \frac{\left( \| \nabla u \|_{L^2(C)}^2 + \Lambda \| u \|_{L^2(C)}^2 \right)^\theta}{\| u \|_{L^p(C)}^2} \| u \|_{L^2(C)}^{2(1-\theta)} \]

A minimizer solves the Euler-Lagrange equation

\[-\theta \Delta u + \left[ (1 - \theta) t[u] + \Lambda \right] u = u^{p-1} \quad \text{with} \quad t[u] := \frac{\int_C |\nabla u|^2 \, dy}{\int_C u^2 \, dy} \]

When \( \theta = 1 \), denote the solution by \( u_\mu \). We may parametrize the branch for any \( \theta \leq 1 \) by

\[ \Lambda^\theta(\mu) = \theta \mu - (1 - \theta) \tau(\mu) , \]
\[ J^\theta(\mu) := Q^\theta_\Lambda[u_\mu] = \nu(\mu) \theta^\theta (\mu + \tau(\mu))^{\theta} \]

where

\[ \tau(\mu) := t[u_\mu] \quad \text{and} \quad \nu(\mu) := \frac{\| u_\mu \|_{L^2(C)}^2}{\| u_\mu \|_{L^p(C)}^2} \]
Asymptotic behavior of the branch

With \( \vartheta = \vartheta(p, d) = d \frac{p-2}{2p} \), denote by \( K_{GN} = K_{GN}(p, d) \) the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

\[
\| u \|_{L^p(\mathbb{R}^N)}^2 \leq \frac{K_{GN}}{|\mathbb{S}^{d-1}| \frac{p-2}{p}} \| \nabla u \|_{L^2(\mathbb{R}^N)}^2 \| u \|_{L^2(\mathbb{R}^N)}^{2(1-\vartheta)} \quad \forall u \in H^1(\mathbb{R}^d)
\]

**Theorem 17.** With the previous notations, for all \( \theta > \vartheta = \vartheta(p, d) \), we have

\[
\lim_{\mu \to \infty} \mu^{\vartheta-\theta} J^\theta(\mu) = \frac{\theta^\theta}{\vartheta^\vartheta (1-\vartheta)^{\vartheta-\theta}} \frac{1}{K_{GN}}
\]

Moreover, the parametric curve \( \mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu)) \) is asymptotic to the curve

\[
\Lambda \mapsto \frac{\theta^\theta}{\vartheta(p, d)^{\vartheta(p, d)} (\theta - \vartheta(p, d))^{\vartheta-\vartheta(p, d)}} \frac{\Lambda^{\vartheta-\vartheta(p,d)}}{K_{GN}}
\]

for large values of \( \mu \) or, equivalently, for large values of \( \Lambda = \Lambda^\theta(\mu) \).
Expansion around the bifurcation point

**Theorem 18.** Assume that $\theta = 1$, $d \geq 3$ and $p \in (2, 2^*)$. Under assumption (H), there exist a constant $c_{p,d}$ and

\[ u(\mu) := u_{\mu,*} + \sqrt{c_{p,d} (\mu - \mu_{FS})} \varphi + c_{p,d} (\mu - \mu_{FS}) \psi \]

where $\varphi$ and $\psi$ are two smooth functions with exponential decay as $|s| \to \infty$ such that for $c_{p,d} (\mu - \mu_{FS}) > 0$

\[ Q_\mu[u(\mu)] = Q_\mu[u_{\mu,*}] \left(1 - \frac{p^2 - 4}{8} c_{p,d} (\mu - \mu_{FS})^2 + o \left((\mu - \mu_{FS})^2\right)\right) \]

Moreover, if $c_{p,d}$ is positive, then for $\mu > \mu_{FS}$, $Q_\mu[u(\mu)]$ minimizes $Q_\mu$ in a neighborhood of $u_{\mu,*}$ among smooth functions with exponential decay as $|s| \to \infty$, up to terms of order $o \left((\mu - \mu_{FS})^2\right)$.
Redefine

$$\tau(\mu) := t[u(\mu)] \quad \text{and} \quad \nu(\mu) := \frac{||u(\mu)||^2_{L^2(C)}}{||u(\mu)||^2_{L^p(C)}}$$

and then $$\Lambda^\theta(\mu)$$ and $$J^\theta(\mu)$$ accordingly. Let $$\vartheta_2(p,d) := \frac{\tau'(\mu_{FS})}{1+\tau'(\mu_{FS})}$$

**Theorem 19.** Under assumption (H) and with definition (A), if $$c_{p,d}$$ is positive, if $$\mu \sim (\mu_{FS})_+$$, then

- **Either** $$\vartheta_2(p,d) \leq \vartheta(p,d)$$ **and then for all** $$\theta \in (\vartheta(p,d), 1]$$, **the branch** $$\left(\Lambda^\theta(\mu), J^\theta(\mu)\right)$$ **is concave, nondecreasing in** $$\mu$$ **and it is below the symmetric branch** $$\left(\Lambda^\theta_*(\mu), J^\theta_*(\mu)\right)$$.

- **Or,** on the contrary, $$\vartheta_2(p,d) > \vartheta(p,d)$$ **and then we find two different behaviors:**
  - **if** $$\theta \in (\vartheta_2(p,d), 1]$$, **the branch is concave, nondecreasing in** $$\mu$$ **and below the symmetric branch**
  - **if** $$\theta \in (\vartheta(p,d), \vartheta_2(p,d))$$, **then the branch** $$\left(\Lambda^\theta(\mu), J^\theta(\mu)\right)$$ **is above the symmetric branch** $$\left(\Lambda^\theta_*(\mu), J^\theta_*(\mu)\right)$$ **and** $$\frac{d}{d\mu} \Lambda^\theta(\mu_{FS}) < 0$$
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8, d = 5, \theta = 1$
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8, d = 5, \theta = 0.8$
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8, d = 5, \theta = 0.72$
Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.95$
Enlargement for $p = 2.8, d = 5, \theta = 0.72$
Critical case $\theta = \vartheta(p, d)$

\[ J^\theta(\mu) \]

\[ \Lambda^\theta(\mu) \]

Bifurcation

Symmetric

Non-symmetric

$1/K_{\text{CKN}}(\vartheta(p,d), \Lambda_{FS}(p,\theta), p)$

$1/K_{\text{GN}}$
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 1$
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 0.95$
Case $p = 3.15$, $d = 5$, $\theta = \vartheta(3.15, 5) \approx 0.9127$
$c_{p,d}$ with $d = 5$ as a function of $p$
local and asymptotic criteria
Thank you!