φ-hypocoercivity

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A short historical introduction to entropy methods in PDEs

On the notion of entropy

- In physics... the notion of entropy goes back to the 19^{th} century: Gibbs, Clausius, Maxwell, Boltzmann
  ▷ Thermodynamics: the steam engine
  ▷ Boltzmann: how irreversibility arises in large systems
- Fundamental problems in Mathematics
  ▷ Linked to the 6^{th} Hilbert problem
- Information theory
  ▷ Shannon, Rényi,...
  ▷ What von Neumann said to Shannon: *When Shannon first derived his famous formula for information, he asked von Neumann what he should call it and von Neumann replied: "You should call it entropy for two reasons: first because that is what the formula is in statistical mechanics but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage!"*

(to be continued)

On Boltzmann's entropy

Boltzmann's equation describes the evolution of a gas of particles by

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = Q(f, f)$$

t is the time, x the position and v the velocity
$f(t, x, v)$ is the *distribution function* (a probability density on the phase space). In a collision, if v and v_* are the *incoming* velocities and v' and v'_* the *outgoing* velocities, $f_* = f(t, x, v_*)$, etc., then

$$Q(f, f) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \sigma(v - v_*, \omega) \left(f' \, f'_* - f \, f_*\right) dv_* \, d\omega$$

is the collision kernel

The cross-section σ is nonnegative and has symmetry properties

Boltzmann's H theorem

Boltzmann's entropy

$$H=\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,\log f\,dx\,dv$$

Boltzmann's H theorem

$$\frac{dH}{dt}=-\frac{1}{4}\int_{(\mathbb{R}^3)^3\times\mathbb{S}^2}\sigma(v-v_*,\omega)\cdot(f'\,f'_*-f\,f_*)\,\log\left(\frac{f'\,f'_*}{f\,f_*}\right)\,dv_*\,dv\,dx\,d\omega$$

▷ Carleman ($<$ 1949): first mathematical theory

▷ DiPerna and Lions (1989): renormalized solutions

▷ Cercignani, Illner, Pulvirenti (1994): derivation of the equation

Can we compute a rate of convergence to an equilibrium using H ?

On the notion of entropy (continued)

Many other domains of application:

- linear diffusions, Markov processes, semi-group theory... a long story ! Bakry & Emery (1984): the carré du champ method
- PDEs and “entropy methods”: around 1998, Toscani et al., del Pino & JD (1999): the entropy for fast diffusion equations (a nonlinear case)

and also (not discussed here):

- Hyperbolic conservation laws
- Sinai’s entropy for measure-preserving dynamical system
- topological entropy, Perelman’s entropy in differential geometry
- *etc.*

Outline of the lecture

- Entropy methods and diffusion equations
  - ▷ φ-entropies and the *carré du champ* method of Bakry & Emery
  - ▷ the gradient flow point of view
  - ▷ rigidity results and entropy methods on compact manifolds
  - ▷ Rényi entropy powers on the Euclidean space
  - ▷ weighted inequalities and results of symmetry
  - ▷ other applications: the (Patlak)-Keller-Segel in mathematical biology and the Oseen attractor in 2D Euler equations
- Hypocoercivity in kinetic equations
  - ▷ H^1 methods and φ-hypocoercivity
  - ▷ L^2-hypocoercivity

Entropy methods and diffusion equations

φ-entropies: definition

The φ-*entropy* of a nonnegative function $w \in \mathrm{L}^1(\mathbb{R}^d, d\gamma)$ is

$$\mathcal{E}[w] := \int_{\mathbb{R}^d} \varphi(w)\, d\gamma$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is such that

$$\varphi'' \ge 0\,, \quad \varphi \ge \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \le 0$$

A classical example of a such a function φ is given by

$$\varphi_p(w) := \tfrac{1}{p-1}\left(w^p - 1 - p\,(w-1)\right) \quad p \in (1,2]$$

▷ Case $p = 2$: $\varphi_2(w) = (w-1)^2$
▷ Limit case as $p \to 1_+$: $\varphi_1(w) := w \log w - (w-1)$

$d\gamma$ is a probability measure, which is absolutely continuous with respect to Lebesgue's measure

$$d\gamma = e^{-\psi}\, dx$$

φ-entropies and diffusions

If u solves the *Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \Delta u + \nabla_x \cdot (u\,\nabla_x \psi)\,.$$

then $w = u\,e^{\psi}$ solves the *Ornstein-Uhlenbeck* or *backward Kolmogorov* equation

$$\frac{\partial w}{\partial t} = \mathsf{L}\,w := \Delta w - \nabla\psi \cdot \nabla w$$

The Ornstein-Uhlenbeck operator L on $\mathrm{L}^2(\mathbb{R}^d, d\gamma)$ is such that

$$-\int_{\mathbb{R}^d} (\mathsf{L}\,w_1)\,w_2\,d\gamma = \int_{\mathbb{R}^d} \nabla w_1 \cdot \nabla w_2\,d\gamma \quad \forall\,w_1,\,w_2 \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

▷ the mass is conserved: $\int_{\mathbb{R}^d} w(t,\cdot)\,d\gamma = 1$, $\lim_{t\to+\infty} w(t,\cdot) = 1$
▷ the φ-entropy decays

$$\frac{d}{dt}\mathcal{E}[w] = -\int_{\mathbb{R}^d} \varphi''(w)\,|\nabla_x w|^2\,d\gamma =: -\,\mathcal{I}[w]$$

where $\mathcal{I}[w]$ denotes the *φ-Fisher information* functional

φ-entropies: entropy – entropy production inequalities

If for some $\Lambda > 0$ the *entropy – entropy production* inequality

$$\mathcal{I}[w] \geq \Lambda\, \mathcal{E}[w] \quad \forall\, w \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

holds, then

$$\mathcal{E}[w(t,\cdot)] \leq \mathcal{E}[w_0]\, e^{-\Lambda\, t} \quad \forall\, t \geq 0$$

Example: with $e^{-\psi} = (2\pi)^{-d/2}\, e^{-|x|^2/2}$ asn $\varphi = \varphi_p$

▷ $p = 2$, Gaussian Poincaré inequality : $\Lambda = 1$

$$\|f - \bar{f}\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\gamma \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)\,, \quad \bar{f} = \int_{\mathbb{R}^d} f\, d\gamma$$

▷ $p = 1$, Logarithmic Sobolev inequality : $\Lambda = 2$

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)}}\right) d\gamma \leq 2 \int_{\mathbb{R}^d} |\nabla f|^2\, d\gamma \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

φ-entropies: key properties

- Generalized Csiszár-Kullback-Pinsker inequality:
  if $A := \inf_{s\in(0,\infty)} s^{2-p}\,\varphi''(s) > 0$, then

$$\mathcal{E}[w] \ge 2^{-\frac{2}{p}}\,A\,\min\left\{1, \|w\|^{p-2}_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}\right\}\,\|w-1\|^2_{\mathrm{L}^p(\mathbb{R}^d,d\gamma)}$$

- Sub-additivity

$$\mathcal{E}_{\gamma_1\otimes\gamma_2}[w] \le \int_{\mathbb{R}^{d_2}} \mathcal{E}_{\gamma_1}[w]\,d\gamma_2 + \int_{\mathbb{R}^{d_1}} \mathcal{E}_{\gamma_2}[w]\,d\gamma_1$$

- Tensorization

$$\mathcal{I}_{\gamma_1\otimes\gamma_2}[w] = \int_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(w)\,|\nabla w|^2\,d\gamma_1\,d\gamma_2 \ge \min\{\Lambda_1,\Lambda_2\}\,\mathcal{E}_{\gamma_1\otimes\gamma_2}[w]$$

- Generalized Holley-Stroock perturbation lemma:
  if $e^{-b}\,d\gamma \le d\mu \le e^{-a}\,d\gamma$ and $\widetilde{w} := \int_{\mathbb{R}^d} w\,d\mu \,/\, \int_{\mathbb{R}^d} d\mu$, then

$$e^{a-b}\,\Lambda \int_{\mathbb{R}^d} \left[\varphi(w)-\varphi(\widetilde{w})-\varphi'(\widetilde{w})(w-\widetilde{w})\right] d\mu \le \int_{\mathbb{R}^d} \varphi''(w)\,|\nabla w|^2\,d\mu$$

φ-entropies: the *carré du champ* method

(Bakry, Emery, 1984): compute the t-derivative of Fisher

On a convex domain Ω, with $w = z^{2/p}$ so that $\mathcal{I}[w] = \int_\Omega |\nabla z|^2\, d\gamma$

$$\begin{aligned}
\frac{1}{2}\,\frac{d}{dt}\mathcal{I}[w] &= -\frac{2}{p}\,(p-1)\int_\Omega \left\|\mathrm{Hess}\, z\right\|^2 d\gamma - \int_\Omega \mathrm{Hess}\,\psi : \nabla z \otimes \nabla z\, d\gamma \\
&\quad - \frac{2-p}{p}\int_\Omega \left\|\mathrm{Hess}\, z - \frac{\nabla z \otimes \nabla z}{z}\right\|^2 d\gamma \\
&\quad + \int_{\partial\Omega} \mathrm{Hess}\, z : \nabla z \otimes \nu\, e^{-\psi}\, d\sigma \\
&\le -\,\mathcal{I}[w]
\end{aligned}$$

Key observations: $[\nabla, \mathsf{L}] = -\,\mathrm{Hess}\,\psi$... if $\psi(x) = |x|^2/2$

$$\int_\Omega \mathrm{Hess}\,\psi : \nabla z \otimes \nabla z\, d\gamma = \int_\Omega |\nabla z|^2\, d\gamma = \mathcal{I}[w]$$

φ-entropies: a statement

Let $p \in [1,2]$ and assume that for any $X \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)^d$

$$\frac{2}{p}\,(p-1)\int_{\mathbb{R}^d} |\nabla X|^2\,d\gamma + \int_{\mathbb{R}^d} \mathrm{Hess}\,\psi : X \otimes X\,d\gamma \ge \Lambda(p)\int_{\mathbb{R}^d} |X|^2\,d\gamma$$

Theorem

Assume that $q \in [1,2)$*. If* $\Lambda = \Lambda(2/q) > 0$*, then*

$$\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \|f\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}}{2-q} \le \frac{1}{\Lambda}\int_{\mathbb{R}^d} |\nabla f|^2\,d\gamma \quad \forall\, f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$$

φ-entropies: improved inequalities

Remainder terms: with $\kappa_p = (p-1)\,(2-p)/p$

$$\frac{d}{dt}\mathcal{I}[w] + 2\,\mathcal{I}[w] \le -\,\kappa_p\,\frac{\mathcal{I}[w]^2}{1+(p-1)\,\mathcal{E}[w]}$$

Let $\mathsf{e}(t) := \frac{1}{p-1}\left(\int_{\mathbb{R}^d} f^2\,d\gamma - 1\right)$ where $f = w^{p/2}$

$$\mathsf{e}'' + 2\,\mathsf{e}' \ge \frac{\kappa_p\,|\mathsf{e}'|^2}{1+(p-1)\,\mathsf{e}} \ge \frac{\kappa_p\,|\mathsf{e}'|^2}{1+\mathsf{e}}$$

Proposition

Assume that $q \in (1,2)$ *and* $d\gamma = (2\pi)^{-d/2}\,e^{-|x|^2/2}\,dx$
With $F(s) := \frac{1}{1-\kappa_p}\left[1+s-(1+s)^{\kappa_p}\right]$, *for any* $f \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|f\|_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)} = 1$

$$\frac{1}{q}\,F\left(q\,\frac{\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-1}{2-q}\right) \le \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}$$

Bakry-Emery

Photo: Nassif Ghoussoub

φ-entropies: a summary

▷ To prove the decay of the entropy, we need an entropy – entropy production inequality $\Lambda\,\mathcal{E} \le \mathcal{I}$

▷ The best constant in the entropy – entropy production inequality determines the (exponential) rate of decay: $\mathcal{E}(t) \le \mathcal{E}(O)\,e^{-\Lambda t}$

▷ By differentiating this estimate at $t = 0$, we see that an exponential rate of decay is equivalent to an entropy – entropy production inequality: $-\mathcal{I}(0) = \frac{d}{dt}\mathcal{E}(0) \le -\Lambda\,\mathcal{E}(0)$

▷ The *carré du champ* method: prove that $\frac{d}{dt}\left(\mathcal{I}(t) - \Lambda\,\mathcal{E}(t)\right) \le 0$

▷ With an improved inequality $\Lambda\,F(\mathcal{E}) \le \mathcal{I}$ where $F'' > 0$, $F(0) = 0$ and $F'(0) = 0$, optimality in the entropy – entropy production inequality can be achieved only in the asymptotic regime and Λ is given by a spectral gap of a linearized problem

▷ The Fokker-Planck equation can be seen as a gradient flow of the φ-entropy under an appropriate notion of distance

A (short) review of applications to nonlinear equations

▷ Nonlinear interpolation inequalities

▷ Rigidity results for nonlinear elliptic equations

▷ Monotonicity along nonlinear flows

▷ Symmetry results in weighted inequalities

▷ Other applications

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: (B. Gidas, J. Spruck, 1981), (M.-F. Bidaut-Véron, L. Véron, 1991)
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): (D. Bakry, M. Emery, 1984), (Bentaleb), (Bakry, Ledoux, 1996), (Demange, 2008), (JD, Esteban, Kolwalczyk, Loss, 2014 & 2015) → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs
  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., (del Pino, JD, 2001), (Blanchet, Bonforte, JD, Grillo, Vázquez) → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*
  ▷ Mass transportation: (Otto) → *C. Villani, Optimal transport. Old and new (2009)*
  ▷ Rényi entropy powers (information theory) (Savaré, Toscani, 2014), (Dolbeault, Toscani)

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...

Rigidity: the Bakry-Emery method on $\mathbb{S}^d$

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

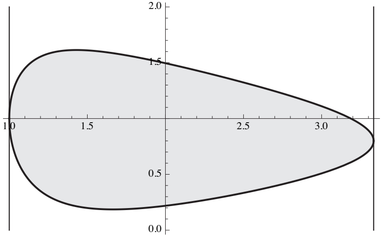
Rigidity: the fast diffusion flow on $\mathbb{S}^d$

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Rigidity: the functional inequality

The entropy – entropy production method establishes the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where $d\mu$ is the uniform probability measure on $\mathbb{S}^d$ and $p \ge 1$, $p \neq 2$ and $p \le 2^* := \frac{2\,d}{d-2}$ if $d \ge 3$
The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$$

(Beckner, 1993)
(Bidaut-Véron, Véron, 1991)
(JD, Esteban, Kowalczyk, Loss)

Rényi entropy powers: the fast diffusion equation on $\mathbb{R}^d$

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

Rényi entropy power and Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx = -\,2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

Critical case: if $m = 1 - \frac{1}{d}$: $\mathsf{F}' = \mathsf{I}$ and the inequality $\mathsf{I} \ge \mathsf{I}_\star =: \mathsf{I}[\mathcal{B}_\star]$ is Sobolev's inequality

Rényi entropy power: the subcritical case

Theorem

(Toscani-Savaré) *Assume that* $m \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $(1 - m)\,\mathsf{F}''(t) \leq 0$

(Dolbeault-Toscani) The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$

Weighted inequalities and results of symmetry

Symmetry: critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b:
$a < a_c = (d-2)/2$, $a \le b \le a+1$ if $d \ge 3$

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

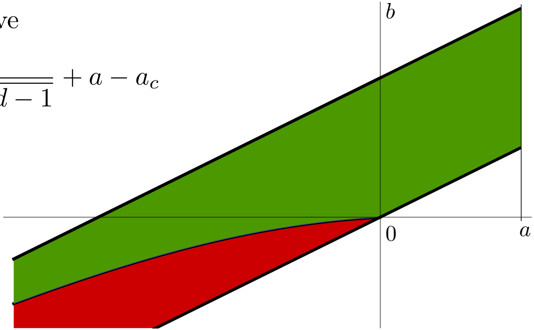
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Symmetry: the sharp result in the critical case

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



(JD, Esteban, Loss, 2016)

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

Symmetry: an approach based on Rényi entropy powers

We compute the derivative of the generalized *Rényi entropy power* functional

$$\frac{d}{dt}\mathsf{F}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathsf{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d}\,dx$ and the pressure variable is

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

Symmetry: the formal computation

With $\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathsf{L}_\alpha u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$-\,\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}$$
$$\geq (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathsf{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right) u^m\,d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result

Other applications

- The subcritical regime: (JD, Esteban, Loss, Muratori, 2017)
- Equations with a mean field coupling

  ▷ the (Patlak)-Keller-Segel in mathematical biology (Campos, JD, 2014)

  ▷ the Oseen attractor in 2D Euler equations (with positive vorticity) (Gallay)

Hypocoercivity in kinetic equations

H^1 methods and φ-hypocoercivity

Some references

- hypoelliptic methods: (Hörmander), (Hérau, Nier), and many others
- H^1-hypocoercive methods: (Gallay), (Villani), (Mouhot-Neumann), (Baudoin), *etc.*
- φ-hypocoercivity: (Arnold, Erb), (Achleitner, Arnold, Stürzer), (Achleitner, Arnold, Carlen), (Monmarché et al.), (Evans), (JD, Li)
- L^2-hypocoercive methods: (JD, Mouhot, Schmeiser), (Bouin, JD, Mouhot, Mischler, Schmeiser), (Arnold et al.)

▷ Motivation: coupling with mean field equations
Partial results: (Hérau, Thomann), (Herda, Rodrigues)

The kinetic Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v \nabla_v f)$$

with $\psi(x) = |x|^2/2$. Under the condition $\|f\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$, it has a unique stationary solution

$$f_\star(x,v) = (2\pi)^{-\frac{d}{2}} e^{-\psi(x)} e^{-\frac{1}{2}|v|^2} = (2\pi)^{-d} e^{-\frac{1}{2}(|x|^2+|v|^2)} \quad \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d$$

The function $g := f/f_\star$ solves

$$\frac{\partial g}{\partial t} + \mathsf{T} g = \mathsf{L}\, g$$

where the transport operator T and the Ornstein-Uhlenbeck operator L are defined respectively by

$$\mathsf{T} g := v \cdot \nabla_x g - x \cdot \nabla_v g \quad \text{and} \quad \mathsf{L}\, g := \Delta_v g - v \cdot \nabla_v g$$

Let $d\mu := f_\star\, dx\, dv$ be the invariant measure on $\mathbb{R}^d \times \mathbb{R}^d$

An optimal decay rate

The function $h := g^{p/2}$ solves

$$\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\,\frac{|\nabla_v h|^2}{h}\,.$$

At the kinetic level, we consider the φ_p-entropy given by

$$\mathcal{E}[g] := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \varphi_p(g)\,d\mu$$

Proposition

(Arnold, Erb) *With the above notations there exists a constant $\mathcal{C} > 0$ for which*

$$\mathcal{E}[g(t,\cdot,\cdot)] \le \mathcal{C}\,e^{-t} \quad \forall\, t \ge 0$$

φ-hypocoercivity: the H^1 approach

The method is based on the *Fisher information*

$$\mathcal{J}[h] = \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla_v h|^2\,d\mu + \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla_x h|^2\,d\mu + \tfrac{1}{2}\int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\,d\mu$$

which involves derivatives in x and v
A *carré du champ* computation shows that

$$\frac{d}{dt}\mathcal{J}[h(t,\cdot)] \le -\mathcal{J}[h(t,\cdot)]$$

The result follows from the entropy – entropy production inequality

$$\Lambda\,\mathcal{E}[g(t,\cdot,\cdot)] = \Lambda\,\mathcal{E}[h^{2/p}] \le \mathcal{J}[h]$$

φ-hypocoercivity: improved rate

Generalized (twisted, time-dependent) *Fisher information* functional

$$\mathcal{J}_\lambda[h] = (1-\lambda)\int_{\mathbb{R}^d} |\nabla_v h|^2\,d\mu + (1-\lambda)\int_{\mathbb{R}^d} |\nabla_x h|^2\,d\mu + \lambda\int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\,d\mu$$

Theorem

(JD, Li) Let $p \in (1,2)$. There exists a function $\lambda : \mathbb{R}^+ \to [1/2, 1)$ and a continuous function ρ on $\mathbb{R}^+$ such that $\rho > 1/2$ a.e., for which we have

$$\frac{d}{dt}\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le -\,2\,\rho(t)\,\mathcal{J}_{\lambda(t)}[h(t,\cdot)]$$

As a consequence, for any $t \ge 0$ we have the global estimate

$$\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le \mathcal{J}_{\lambda(0)}[h_0]\,\exp\left(-\,2\int_0^t \rho(s)\,ds\right)$$

L^2-hypocoercivity

- ▷ Abstract statement
- ▷ A toy model
- ▷ Decay estimates
- ▷ Diffusion limits

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F \tag{2}$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$

$$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

Fokker-Planck kernels with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \,, \quad f(0,x,v) = f_0(x,v) \tag{3}$$

for a distribution function $f(t,x,v)$, with *position* variable $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$ the flat d-dimensional torus

Fokker-Planck collision operator with a general equilibrium M

$$\mathsf{L} f = \nabla_v \cdot \Big[M \, \nabla_v \left(M^{-1} f \right) \Big]$$

An *admissible local equilibrium* M is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v)\, dv = 1 \,, \quad d\gamma = \gamma(v)\, dv := \frac{dv}{M(v)}$$

Typical example: $M(v) = (2\pi)^{-d/2}\, e^{-\frac{1}{2}\,|v|^2}$ + some technical assumptions (H)

Scattering collision operators

Scattering collision operator

$$\mathsf{L}f = \int_{\mathbb{R}^d} \sigma(\cdot, v')\,\big(f(v')\,M(\cdot) - f(\cdot)\,M(v')\big)\,dv'$$

Main assumption on the *scattering rate* σ: for some positive, finite $\overline{\sigma}$

$$1 \le \sigma(v,v') \le \overline{\sigma} \quad \forall\, v,\, v' \in \mathbb{R}^d$$

Example: linear BGK operator

$$\mathsf{L}f = M\rho_f - f\,, \quad \rho_f(t,x) = \int_{\mathbb{R}^d} f(t,x,v)\,dv$$

Local mass conservation

$$\int_{\mathbb{R}^d} \mathsf{L}f\,dv = 0$$

The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \, \|(1-\Pi)F\| \tag{H4}$$

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \leq -\,\lambda_m \, \|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

as in (Dolbeault-Mouhot-Schmeiser) so that $\langle \mathsf{AT}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$\begin{aligned} -\frac{d}{dt}\mathsf{H}[F] &=: \mathsf{D}[F] \\ &= -\,\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \\ &\quad -\,\delta\,\mathrm{Re}\langle \mathsf{TA}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{AL}F, F\rangle \end{aligned}$$

▷ for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \le \mathsf{D}[F]$$

▷ norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Exponential decay of the entropy

$$\lambda=\frac{\lambda_M}{3\,(1+\lambda_M)}\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\},\ \delta=\tfrac12\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta,\lambda):=(\delta\,C_M)^2-4\left(\lambda_m-\delta-\frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M}-\frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathcal{H}$*. Under* (H1)–(H4)*, for any* $t\ge0$

$$\mathsf{H}[F(t,\cdot)]\le\mathsf{H}[F_0]\,e^{-\lambda_\star\,t}$$

where $\lambda_\star$ *is characterized by*

$$\lambda_\star:=\sup\left\{\lambda>0\,:\,\exists\,\delta>0\ s.t.\ h_1(\delta,\lambda)=0\,,\ \lambda_m-\delta-\tfrac14\,(2+\delta)\,\lambda>0\right\}$$

Hypocoercivity

Corollary

For any $\delta\in(0,2)$, if $\lambda(\delta)$ is the largest positive root of $h_1(\delta,\lambda)=0$ for which $\lambda_m-\delta-\frac14\,(2+\delta)\,\lambda>0$, then for any solution F of (2)

$$\|F(t)\|^2\le\frac{2+\delta}{2-\delta}\,e^{-\lambda(\delta)\,t}\,\|F(0)\|^2\quad\forall\,t\ge0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2\le\mathsf{H}[F]\le\frac{2+\delta}{4}\,\|F\|^2$$

We use $\frac{2-\delta}{4}\,\|F_0\|^2\le\mathsf{H}[F_0]$ so that $\lambda_\star\ge\sup_{\delta\in(0,2)}\lambda(\delta)$

A toy problem

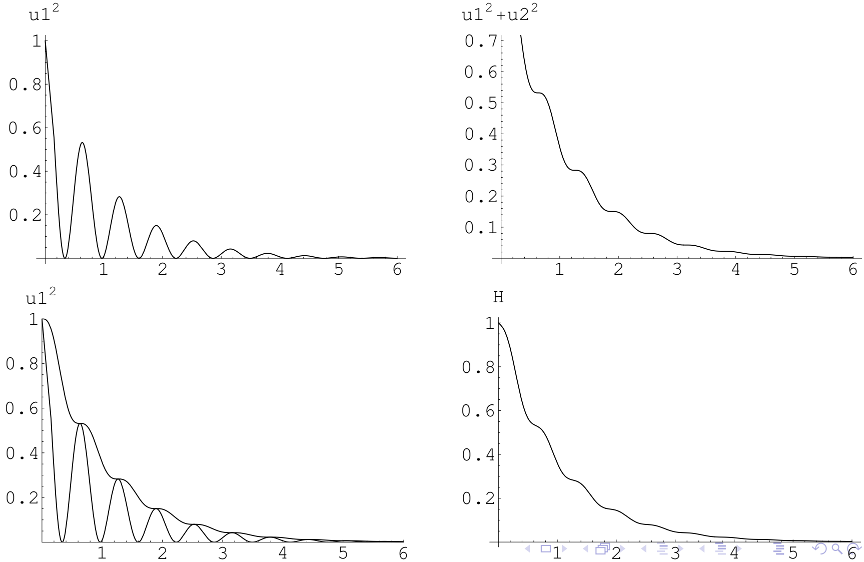
$$\frac{du}{dt} = (L-T)\,u\,, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = -\,2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -\,k^2\,u_1$
- generalized entropy: $H(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{dH}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right)u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\leq -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1u_2 \end{aligned}$$

Plots for the toy problem



Further results

(Bouin, JD, Mouhot, Mischler, Schmeiser)

▷ It is possible to make a mode by mode analysis in Fourier variables

▷ On the whole space without confinement potential ($\psi \equiv 0$), we obtain rates of decay using Nash's inequality

Consider

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f_0(x,v)$$

for a distribution function $f(t,x,v)$, with *position* variable in the whole space, $x \in \mathbb{R}^d$, and with *time* $t \geq 0$

(a) *Fokker-Planck* collision operator:

$$\mathsf{L} f = \nabla_v \cdot \Big[M \, \nabla_v \left(M^{-1} f \right) \Big]$$

(b) *Scattering* collision operator:

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, M(\cdot) - f(\cdot) \, M(v') \right) dv'$$

A statement

A typical example of a *local equilibrium* M is the Gaussian function

$$M(v)=\frac{e^{-\frac{1}{2}\,|v|^2}}{(2\pi)^{d/2}}$$

but our results apply to more general functions M

Theorem

Let us consider an admissible M and a collision operator L satisfying some additional technical assumptions. Assume that $x\in\mathbb{R}^d$, and $\gamma_k(v)=\big(1+|v|^2\big)^{k/2}$ for some $k\in(d,\infty]$. There exists a constant $C>0$ such that the solution f satisfies

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)}\le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)}+\|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

Diffusion limits:
from kinetic equations to diffusions

BGK-type kinetic equation as a motivation for nonlinear diffusions – polytropes and fast diffusion / porous medium

$$\begin{aligned} \varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon - \varepsilon \nabla_x V(x) \cdot \nabla_v f^\varepsilon &= G_{f^\varepsilon} - f^\varepsilon \\ f^\varepsilon(x,v,t=0) &= f_I(x,v)\,, \quad x,v \in \mathbb{R}^3 \end{aligned}$$

with the Gibbs equilibrium $G_f := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right)$

The Fermi energy $\mu_{\rho_f}(x,t)$ is implicitly defined by

$$\int_{\mathbb{R}^3} \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right) dv = \int_{\mathbb{R}^3} f(x,v,t)\,dv =: \rho_f(x,t)$$

$f^\varepsilon(x,v,t)$ ... phase space particle density

$V(x)$ ... potential

ε ... mean free path

$$\Longrightarrow \quad \mu_{\rho_f} = \bar\mu(\rho_f)$$

Diffusion limits

(J.D., P. Markowich, D. Ölz, C. Schmeiser)

Theorem

For any $\varepsilon > 0$, the equation has a unique weak solution $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$. As $\varepsilon \to 0$, f^ε weakly converges to a local Gibbs state f^0 given by

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}\,|v|^2 - \bar{\mu}(\rho(x,t))\right)$$

where ρ is a solution of the nonlinear diffusion equation

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho\, \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$

$$\nu(\rho) = \int_0^\rho s\,\bar{\mu}'(s)\,ds$$

Some slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

The papers can be found at

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Thank you for your attention !