Entropy methods and non-linear diffusion equations: some recent results (mass transport)

Jean Dolbeault

dolbeaul@ceremade.dauphine.fr

CEREMADE

CNRS & Université Paris-Dauphine

http://www.ceremade.dauphine.fr/∼dolbeaul

NEW DEVELOPMENTS IN PARTIAL DIFFERENTIAL EQUATIONS I

US-CHILE WORKSHOPS

May 22, 2007

Contents

- A brief review of some results on entropy methods in PDEs (fast diffusion equation)
- Hardy-Poincaré inequalities and applications

- Generalized Poincaré inequalities (Beckner): two approaches
  - entropy - entropy production
  - spectral approaches
- More Lyapunov functionals for second and fourth order equations
- L^q Poincaré inequalities for general measures

Introduction to entropy methods: a brief review of some results – I

- Generalized entropy methods for fast diffusion and porous media equations: intermediate asymptotics
- Entropy methods and functional inequalities
- Related topics (mass transport)

Porous media / fast diffusion equations

Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],...
[del Pino, Sáez], [Daskalopulos, Sesum]...

Various approaches:
1) “entropy – entropy-production method”
2) mass transport techniques
3) hypercontractivity for appropriate semi-groups

4) [J.D., del Pino] relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Intermediate asymptotics

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^d$$

$$u_{|t=0} = u_0 \geq 0$$

$$u_0(1+|x|^2) \in L^1\,, \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0\,dx = 1$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$
As $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$$

$\Longrightarrow$ What about $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^1}$?

Time-dependent rescaling

Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy or Free energy

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\;, \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Entropy and entropy production

Stationary solution: choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m} \, |x|^2 \right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form

$$\Sigma[v] = \int \psi \left(\tfrac{v}{v_\infty} \right) \, v_\infty^m \, dx \quad with \; \psi(t) = \tfrac{t^m - 1 - m\,(t-1)}{m-1}$$

Theorem 1 $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2 \, \Sigma[v]$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$

$$\boxed{\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int |w|^{1+p} dx + K \geq 0}$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and, for some γ, K can be written as

$$K = K_0 \left(\int v\, dx = \int w^{2p}\, dx \right)^\gamma$$

$w = w_\infty = v_\infty^{1/2p}$ is optimal

$m = \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Gagliardo-Nirenberg inequalities

Theorem 2 [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *and* $d \geq 3$

$$\|w\|_{2p} \leq A\,\|\nabla w\|_2^\theta\,\|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)}\,, \quad y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \Longleftrightarrow$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \Longleftrightarrow$ Porous medium case: $m > 1$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities

Theorem 3 [Del Pino, J.D.]
(i) $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \| \, [u - u_\infty] \, u_\infty^{m-1} \, \|_{L^1} < +\infty$$

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\left(x/R(t)\right)$$

Optimal L^p-Euclidean logarithmic Sobolev inequality

[Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 4 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n \, \frac{p-1}{p} + 1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} \left(\frac{\sigma}{p} \right)^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

Application to $u_t = \Delta_p u^{\frac{1}{p-1}}$

Extensions and related results

- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg], etc + [J.D.-Nazaret-Savaré, in progress]
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev]

…Mass transport

- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance
- Displacement convexity holds in the same range of exponents, $m \in ((d-1)/d, 1)$, as the Gagliardo-Nirenberg inequalities

Entropy methods and linearization: intermediate asymptotics, vanishing

A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ Extend the domain of validity of the method to the price of a restriction

of the set of admissible solutions

Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0, \cdot) = u_0 \end{cases}$$

where $m \in (0,1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0,T) \times \mathbb{R}^d$
Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$

Barenblatt solutions

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

with

- $R(\tau) := \left[d\,(m-m_c)\,(\tau+T)\right]^{\frac{1}{d\,(m-m_c)}}$ if $m_c < m < 1$
- (vanishing in finite time) if $0 < m < m_c$

$$R(\tau) := \left[d\,(m_c-m)\,(T-\tau)\right]^{-\frac{1}{d\,(m_c-m)}}$$

Time-dependent rescaling: $t := \log\left(\frac{R(\tau)}{R(0)}\right)$ and $x := \frac{y}{R(\tau)}$. The function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves a nonlinear *Fokker-Planck type equation*

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x)\in(0,+\infty)\times\mathbb{R}^d \\ v(0,x) = v_0(x) = R(0)^d\,u_0(R(0)\,x) & x\in\mathbb{R}^d \end{cases}$$

Assumptions

(H1) u_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,y) \le u_0(y) \le U_{D_1,T}(0,y) \quad \forall\, y \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall\, y \in \mathbb{R}^d$$

(H1') v_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \le v_0(x) \le V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d$$

(H2') If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d$$

Convergence to the asymptotic profile (without rate)

$$m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{2}\,, \quad p(m) := \frac{d\,(1-m)}{2\,(2-m)}$$

Theorem 1 *Let $d \geq 3$, $m \in (0,1)$. Consider a solution v with initial data satisfying (H1')-(H2')*

(i) For any $m > m_*$, there exists a unique D_* such that $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = 0$ for any $t > 0$. Moreover, for any $p \in (p(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,dx = 0$

(ii) For $m \leq m_*$, $v(t) - V_{D_*}$ is integrable, $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = \int_{\mathbb{R}^d} f\,dx$ and $v(t)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$ as $t \to \infty$, for any $p \in (1, \infty]$

(iii) (Convergence in Relative Error) For any $p \in (d/2, \infty]$,

$$\lim_{t\to\infty} \|\, v(t)/\, V_{D_*} - 1\,\|_p = \; 0$$

[Daskalopoulos-Sesum, 06], [Blanchet-Bonforte-Grillo-Vázquez, 06-07]

Convergence with rate

$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m)+d\,(1-m)}$$

Theorem 2 *If $m \neq m_*$, there exist $t_0 \geq 0$ and $\lambda_{m,d} > 0$ such that*

(i) *For any $q \in (q_*, \infty]$, there exists a positive constant C_q such that*

$$\|v(t) - V_{D_*}\|_q \leq C_q\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(ii) *For any $\vartheta \in [0, (2-m)/(1-m))$, there exists a positive constant C_ϑ such that*

$$\left\|\, |x|^\vartheta (v(t) - V_{D_*}) \right\|_2 \leq C_\vartheta\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(iii) *For any $j \in \mathbb{N}$, there exists a positive constant H_j such that*

$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j\, \mathrm{e}^{-\frac{\lambda_{m,d}}{d+2(j+1)}\, t} \quad \forall\, t \geq t_0$$

Intermediate asymptotics

Corollary 3 *Let $d \geq 3$, $m \in (0,1)$, $m \neq m_*$. Consider a solution u with initial data satisfying* (H1)-(H2). *For τ large enough, for any $q \in (q_*, \infty]$, there exists a positive constant C such that*

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C\, R(\tau)^{-\alpha}$$

where $\alpha = \lambda_{m,d} + d(q-1)/q$ and large means $T - \tau > 0$, small, if $m < m_c$, and $\tau \to \infty$ if $m \geq m_c$

For any $p \in (d/2, \infty]$, there exists a positive constant $\mathcal{C}$ and $\gamma > 0$ such that

$$\left\| v(t)/\, V_{D_*} - 1 \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathcal{C}\, \mathrm{e}^{-\gamma\, t} \quad \forall\, t \geq 0$$

Preliminaries

L^1-contraction, Maximum Principle, conservation of relative mass...

Passing to the quotient: the function $w(t,x) := \frac{v(t,x)}{V_{D_*}(x)}$ solves

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

with

$$0 < \inf_{x\in\mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \leq w(t,x) \leq \sup_{x\in\mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

... Harnack Principle

$$\|w(t)\|_{C^k(\mathbb{R}^d)} \leq \overline{H}_k < +\infty \quad \forall\, t \geq t_0$$

$\exists\, t_0 \geq 0$ s.t. (H1) holds if $\exists\, R > 0$, $\sup_{|y|>R} u_0(y)\, |y|^{\frac{2}{1-m}} < \infty$, and $m > m_c$

Relative entropy

Relative entropy

$$\mathcal{F}[w] := \frac{1}{1-m} \int_{\mathbb{R}^d} \left[(w-1) - \frac{1}{m}(w^m - 1) \right] V_{D_*}^m \, dx$$

Relative Fisher information

$$\mathcal{J}[w] := \frac{m}{(m-1)^2} \int_{\mathbb{R}^d} \left| \nabla \left[\left(w^{m-1} - 1 \right) V_{D_*}^{m-1} \right] \right|^2 w \, V_{D_*} \, dx$$

Proposition 4 *Under assumptions (H1)-(H2),*

$$\frac{d}{dt}\mathcal{F}[w(t)] = -\,\mathcal{J}[w(t)]$$

Proposition 5 *Under assumptions (H1)-(H2), there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{F}[w(t)] \le \lambda^{-1} \, \mathcal{J}[w(t)]$$

Heuristics: linearization

Take $w(t,x) = 1 + \varepsilon \, \frac{g(t,x)}{V_{D_*}^{m-1}(x)}$ and formally consider the limit $\varepsilon \to 0$ in

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \, \nabla \cdot \left[w \, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1) \, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

Then g solves

$$g_t = m \, V_{D_*}^{m-2}(x) \, \nabla \cdot \left[V_{D_*}(x) \, \nabla g(t,x) \right]$$

and the entropy and Fisher information functionals

$$\mathsf{F}[g] := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2 \, V_{D_*}^{2-m} \, dx \quad \text{and} \quad \mathsf{I}[g] := m \int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \, dx$$

consistently verify $\dfrac{d}{dt} \, \mathsf{F}[g(t)] = - \, \mathsf{I}[g(t)]$

Comparison of the functionals

Lemma 6 *Let $m \in (0,1)$ and assume that u_0 satisfies* (H1)-(H2)
[Relative entropy]

$$C_1 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \; dx \le \mathcal{F}[w] \le C_2 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \; dx$$

[Fisher information]

$$\mathsf{I}[g] \le \beta_1 \, \mathcal{J}[w] + \beta_2 \; \mathsf{F}[g] \quad \textit{with} \quad g := (w-1)\, V_{D_*}^{m-1}$$

Theorem 7 (Hardy-Poincaré) *There exists a positive constant $\lambda_{m,d}$ such that for any $m \neq m_* = (d-4)/(d-2)$, $m \in (0,1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |g-\overline{g}|^2 \; V_{D_*}^{2-m} \; dx \le \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \; dx$$

with $\overline{g} = \int_{\mathbb{R}^d} g \; V_{D_*}^{2-m} \; dx$ if $m > m_*$, $\overline{g} = 0$ otherwise

Hardy-Poincaré inequalities

With $\alpha = \frac{1}{m-1}$, $\alpha_* = \frac{1}{m_*-1} = 1 - \frac{d}{2}$

Theorem 8 *Assume that $d \geq 3$, $\alpha \in \mathbb{R} \setminus \{\alpha^*\}$, $d\mu_\alpha(x) := h_\alpha(x)\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$. Then*

$$\int_{\mathbb{R}^d} \frac{|v|^2}{1+|x|^2}\,d\mu_\alpha \leq \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

holds for some positive constant $\mathcal{C}_{\alpha,d}$, for any $v \in \mathcal{D}(\mathbb{R}^d)$, under the additional condition $\int_{\mathbb{R}^d} v\,d\mu_{\alpha-1} = 0$ if $\alpha \in (-\infty, \alpha^*)$

Limit cases

Poincaré inequality: take $\alpha = -1/\epsilon^2$ to $v_\epsilon(x) := \epsilon^{-d/2}\, v(x/\epsilon)$ and let $\epsilon \to 0$

$$\int_{\mathbb{R}^d} |v|^2 \, d\nu_\infty \le \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_\infty \quad \text{with} \quad d\nu_\infty(x) := e^{-|x|^2} dx$$

... under the additional condition $\int_{\mathbb{R}^d} v \; e^{-|x|^2} dx = 0$

Hardy's inequality: take $v_{1/\epsilon}(x) := \epsilon^{d/2}\, v(\epsilon\, x)$ and let $\epsilon \to 0$

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, d\nu_{0,\alpha} \le \frac{1}{(\alpha - \alpha_*)^2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_{0,\alpha} \quad \text{with} \quad d\nu_{0,\alpha}(x) := |x|^{2\alpha} \, dx$$

... under the additional condition $\bar{v}_\alpha := \int_{\mathbb{R}^d} v \, d\nu_{0,\alpha} = 0$ if $\alpha < \alpha^*$

Some estimates of $\mathcal{C}_{\alpha,d}$

α	$-\infty < \alpha \leq -d$	$-d < \alpha < \alpha^*$	$\alpha^* < \alpha \leq 1$
$\mathcal{C}_{\alpha,d}$	$\frac{1}{2\,\lvert\alpha\rvert}$	$\mathcal{C}_{\alpha,d} \geq \frac{4}{(d+2\alpha-2)^2}$	$\frac{4}{(d+2\alpha-2)^2}$
Optimality	?	?	yes

α	$1 \leq \alpha \leq \bar{\alpha}(d)$	$\bar{\alpha}(d) \leq \alpha \leq d$	d	$\alpha > d$
$\mathcal{C}_{\alpha,d}$	$\frac{4}{d(d+2\alpha-2)}$	$\frac{1}{\alpha(d+\alpha-2)}$	$\frac{1}{2d(d-1)}$	$\frac{1}{d(d+\alpha-2)}$
Optimality	?	?	yes	?

Hardy's inequality: the "completing the square method"

Let $v \in \mathcal{D}(\mathbb{R}^d)$ with supp$(v) \subset \mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha^*$

$$\begin{aligned} 0 \;\; &\leq \;\; \int_{\mathbb{R}^d} \left| \nabla v + \lambda \, \frac{x}{|x|^2} \, v \right|^2 \, |x|^{2\alpha} \, dx \\ &= \int_{\mathbb{R}^d} |\nabla v|^2 \, |x|^{2\alpha} \, dx + \Big[\lambda^2 - \lambda \, (d + 2\alpha - 2) \Big] \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, |x|^{2\alpha} \, dx \end{aligned}$$

An optimization of the right hand side with respect to λ gives $\lambda = \alpha - \alpha^*$, that is $(d + 2\alpha - 2)^2/4 = \lambda^2$. Such an inequality is optimal, with optimal constant λ^2, as follows by considering the test functions:

1) if $\alpha > \alpha^*$: $v_\epsilon(x) = \min\{\epsilon^{-\lambda}, (|x|^{-\lambda} - \epsilon^{\lambda})_+\}$

2) if $\alpha < \alpha^*$: $v_\epsilon(x) = |x|^{1-\alpha-d/2+\epsilon}$ for $|x| < 1$
$v_\epsilon(x) = (2 - |x|)_+$ for $|x| \geq 1$

and letting $\epsilon \to 0$ in both cases

The optimality case: Davies' method

Proposition 9 *Let $d \geq 3$, $\alpha \in (\alpha^*, \infty)$. Then the Hardy-Poincaré inequality holds for any $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathcal{C}_{\alpha,d} := 4/(d-2+2\,\alpha)^2$ if $\alpha \in (\alpha^*, 1]$ and $\mathcal{C}_{\alpha,d} := 4/[d\,(d-2+2\,\alpha)]$ if $\alpha \geq 1$. The constant $\mathcal{C}_{\alpha,d}$ is optimal for any $\alpha \in (\alpha^*, 1]$.*

Proof: $\nabla h_\alpha = 2\alpha\, x\, h_{\alpha-1}$, $\Delta h_\alpha = 2\alpha\, h_{\alpha-2}[d + 2(\alpha - \alpha^*)\,|x|^2] > 0$.
By Cauchy-Schwarz

$$\begin{aligned}
\left| \int_{\mathbb{R}^d} |v|^2\, \Delta h_\alpha\; dx \right|^2 &\leq 4 \left(\int_{\mathbb{R}^d} |v|\, |\nabla v|\, |\nabla h_\alpha|\; dx \right)^2 \\
&\leq 4 \int_{\mathbb{R}^d} |v|^2\, |\Delta h_\alpha|\; dx \int_{\mathbb{R}^d} |\nabla v|^2\, |\nabla h_\alpha|^2\, |\Delta h_\alpha|^{-1}\; dx
\end{aligned}$$

$|\Delta h_\alpha| \geq 2\,|\alpha|\;\min\{d, (d-2+2\,\alpha)\}\, \frac{h_\alpha(x)}{1+|x|^2}$

$\frac{|\nabla h_\alpha|^2}{|\Delta h_\alpha|} \leq \frac{2\,|\alpha|}{d-2+2\,\alpha}\, h_\alpha(x)$

. . . Mass transport ???

Generalized Poincaré inequalities, application to linear diffusion equations (Fokker-Planck)

Coll. A. Arnold, J.-P. Bartier, J.D.

Gaussian measures

[W. Beckner, 1989]: a family of generalized Poincaré inequalities (GPI)

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu\right)^{2/p}\right] \le \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall f \in H^1(d\mu) \qquad (1)$$

where $\mu(x) := \frac{e^{-\frac{1}{2}\,|x|^2}}{(2\pi)^{d/2}}$ denotes the normal centered Gaussian distribution on $\mathbb{R}^d$. For $p = 1$: the Poincaré inequality

$$\int_{\mathbb{R}^d} f^2\,d\mu - \left(\int_{\mathbb{R}^d} f\,d\mu\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall f \in H^1(d\mu)$$

In the limit $p \to 2$: the logarithmic Sobolev inequality (LSI) [L. Gross 1975]

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right)\,d\mu \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu \quad \forall\ f \in H^1(d\mu)$$

Sufficient conditions for generalized Poincaré inequalities ?

[AMTU]: for strictly log-concave distribution functions $\nu(x)$

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \leq \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall f \in H^1(d\nu)$$

where κ is the uniform convexity bound of $-\log\nu(x)$...

...the Bakry-Emery criterion

[Latała and Oleszkiewicz]: under the weaker assumption that $\nu(x)$ satisfies a LSI with constant $0 < \mathcal{C} < \infty$

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\nu}\right) d\nu \leq 2\,\mathcal{C}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall\, f \in H^1(d\nu) \qquad (2)$$

for $1 \leq p < 2$, L-O proved that

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \leq \mathcal{C}\min\left\{\frac{2}{p},\frac{1}{2-p}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$

Proof of the result of Latała and Oleszkiewicz (1/2)

The function $q \mapsto \alpha(q) := q \log\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)$ is convex since

$$\alpha''(q) = \frac{4}{q^3} \, \frac{\left(\int_{\mathbb{R}^d} |f|^{2/q} \left(\log |f|\right)^2 \, d\nu\right)\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right) - \left(\int_{\mathbb{R}^d} |f|^{2/q} \log |f|\, d\nu\right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^2}$$

is nonnegative: $q \mapsto e^{\alpha(q)}$ is also convex, $\varphi(q) := \frac{e^{\alpha(1)} - e^{\alpha(q)}}{q-1}$ is $\searrow$

$$\varphi(q) \le \lim_{q_1 \to 1} \varphi(q_1) = \int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}}\right) d\nu$$

$$\frac{1}{q-1}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q\right] \le 2\mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

$$\frac{1}{q-1}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q\right] = \frac{p}{2-p}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2/p}\right]$$

if $p = 2/q$: $C_p \le 2\,\mathcal{C}/p$

Proof of the result of Latała and Oleszkiewicz (2/2)

Linearization $f = 1 + \varepsilon\, g$ with $\int_{\mathbb{R}^d} g\, d\nu = 0$, limit $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Hölder's inequality, $\left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu \right)^q$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu \right)^q \leq \int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu \right)^2 \leq \mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Generalized Poincaré inequalities for the Gaussian measure

The spectrum of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x \cdot \nabla$ is made of all nonnegative integers $k \in \mathbb{N}$, the corresponding eigenfunctions are the Hermite polynomials. Observe that

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu = \int_{\mathbb{R}^d} f \cdot \mathsf{N} f \, d\mu \quad \forall \; f \in H^1(d\mu)$$

Strategy of Beckner (improved): consider the $L^2(d\mu)$-orthogonal decomposition of f on the eigenspaces of N, i.e.

$$f = \sum_{k \in \mathbb{N}} f_k \, ,$$

where $\mathsf{N} \, f_k = k \, f_k$. If we denote by π_k the orthogonal projection on the eigenspace of N associated to the eigenvalue $k \in \mathbb{N}$, then $f_k = \pi_k[f]$.

$$a_k := \|f_k\|^2_{L^2(d\mu)}\,, \quad \|f\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} a_k \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu = \sum_{k\in\mathbb{N}} k\, a_k$$

The solution of the evolution equation associated to N

$$u_t = -\mathsf{N}\, u = \Delta u - x\cdot\nabla u\,, \quad u(t=0) = f$$

is given by $u(x,t) = \left(e^{-t\,\mathsf{N}}\, f\right)(x) = \sum_{k\in\mathbb{N}} e^{-k\,t}\, f_k(x)$

$$\left\|e^{-t\,\mathsf{N}}\, f\right\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} e^{-2\,k\,t} a_k$$

Lemma 1 *Let $f \in H^1(d\mu)$. If $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \geq 1$, then*

$$\int_{\mathbb{R}^d} |f|^2\, d\mu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}} f\right|^2\, d\mu \leq \frac{1-e^{-2k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu$$

Proof

We use the decomposition on the eigenspaces of N

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu = \left(1 - e^{-2\,k\,t} \right) a_k$$

For any fixed $t > 0$, the function $k \mapsto \frac{1-e^{-2\,k\,t}}{k}$ is monotone decreasing: if $k \geq k_0$, then

$$1 - e^{-2\,k\,t} \leq \frac{1 - e^{-2\,k_0\,t}}{k_0}\, k$$

Thus we get

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu \leq \frac{1 - e^{-2\,k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f_k|^2 \, d\mu$$

which proves the result by summation $\square$

Nelson's hypercontractive estimates

Lemma 2 *For any $f \in L^p(d\mu)$, $p \in (1,2)$, it holds*

$$\left\| e^{-t\mathsf{N}} f \right\|_{L^2(d\mu)} \leq \| f \|_{L^p(d\mu)} \quad \forall\, t \geq -\frac{1}{2}\, \log(p-1)$$

Proof. We set $F(t) := \left(\int_{\mathbb{R}^d} |u(t)|^{q(t)}\, d\mu \right)^{1/q(t)}$ with $q(t)$ to be chosen later and $u(x,t) := \left(e^{-t\mathsf{N}} f \right)(x)$. A direct computation gives

$$\frac{F'(t)}{F(t)} = \frac{q'(t)}{q^2(t)} \int_{\mathbb{R}^d} \frac{|u|^q}{F^q} \log \left(\frac{|u|^q}{F^q} \right) d\mu - \frac{4}{F^q}\, \frac{q-1}{q^2} \int_{\mathbb{R}^d} \left| \nabla \left(|u|^{q/2} \right) \right|^2 d\mu$$

We set $v := |u|^{q/2}$, use the logarithmic Sobolev inequality with $\nu = \mu$ and $\mathcal{C} = 1$, and choose q such that $4\,(q-1) = 2q'$, $q(0) = p$ and $q(t) = 2$. This implies $F'(t) \leq 0$ and ends the proof with $2 = q(t) = 1 + (p-1)\, e^{2t}$ $\square$

A generalization of Beckner's estimates

[Arnold, Bartier, J.D.] First result, for the Gaussian distribution $\mu(x)$

Theorem 3 *Let* $f \in H^1(d\mu)$. *If* $f_1 = f_2 = \ldots = f_{k_0 - 1} = 0$ *for some* $k_0 \geq 1$, *then*

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} |f|^2\, d\mu - \left(\int_{\mathbb{R}^d} |f|^p\, d\mu\right)^{2/p}\right] \leq \frac{1-(p-1)^{k_0}}{k_0\,(2-p)} \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu$$

holds for $1 \leq p < 2$

- In the special case $k_0 = 1$ this is exactly the generalized Poincaré inequality due to Beckner, and for $k_0 > 1$ it is a strict improvement for any $p \in [1,2)$
- Easy to generalize to other measures

$$\nu(x) := e^{-V(x)}$$

using the spectrum of $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$

...Mass transport

- The Fokker-Planck equation can be seen as the gradient flow of the entropy (the free energy) with respect to the Wasserstein distance [Jordan, Kinderlehrer, Otto]
- [J.D., Nazaret, Savaré], in progress: There is a family of distances for which the gradient flow of the p-entropies gives... the Fokker-Planck equation. Entropy - entropy production estimates appear as a consequence of the contraction properties of the flow

Equations de Poincaré généralisées

Coll. J. Carrillo, J.D. , I. Gentil, A. Jüngel

Higher order diffusion equations

The one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

The thin film equation

$$u_t = -(u^m \, u_{xxx})_x \,, \quad x \in S^1 \,, \quad t > 0$$

The Derrida-Lebowitz-Speer-Spohn (DLSS) equation

$$u_t = -(u \, (\log u)_{xx})_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

... with initial condition $u(\cdot, 0) = u_0 \geq 0$ in $S^1 \equiv [0, 1)$

Entropies and energies

Averages:

$$\mu_p[v] := \left(\int_{S^1} v^{1/p} \; dx \right)^p \quad \text{and} \quad \bar{v} := \int_{S^1} v \; dx$$

Entropies: $p \in (0, +\infty)$, $q \in \mathbb{R}$, $v \in H^1_+(S^1)$, $v \not\equiv 0 \, a.e.$

$$\Sigma_{p,q}[v] := \frac{1}{p\,q\,(p\,q-1)} \left[\int_{S^1} v^q \; dx - (\mu_p[v])^q \right] \quad \text{if } p\,q \neq 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{1/q,q}[v] := \int_{S^1} v^q \; \log\left(\frac{v^q}{\int_{S^1} v^q \; dx} \right) dx \quad \text{if } p\,q = 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{p,0}[v] := -\frac{1}{p} \int_{S^1} \log\left(\frac{v}{\mu_p[v]} \right) dx \quad \text{if } q = 0$$

Convexity

$\Sigma_{p,q}[v]$ is non-negative by convexity of

$$u \mapsto \frac{u^{p\,q} - 1 - p\,q\,(u-1)}{p\,q\,(p\,q-1)} =: \sigma_{p,q}(u)$$

By Jensen's inequality,

$$\begin{aligned}
\Sigma_{p,q}[v] &= \mu_p[v]^q \int_{S^1} \sigma_{p,q}\left(\frac{v^{1/p}}{(\mu_p[v])^{1/p}}\right) dx \\
&\geq \mu_p[v]^q \sigma_{p,q}\left(\int_{S^1} \frac{v^{1/p}}{(\mu_p[v])^{1/p}}\, dx\right) = \mu_p[v]^q \sigma_{p,q}(1) = 0
\end{aligned}$$

Limit cases

$p\,q = 1$:

$$\lim_{p\to 1/q} \Sigma_{p,q}[v] = \Sigma_{1/q,q}[v] \quad \text{for } q > 0$$

$q = 0$:

$$\lim_{q\to 0} \Sigma_{p,q}[v] = \Sigma_{p,0}[v] \quad \text{for } p > 0$$

$p = q = 0$:

$$\Sigma_{0,0}[v] = -\int_{S^1} \log\left(\frac{v}{\|v\|_\infty}\right) dx$$

Some references (>2005):

[M. J. Cáceres, J. A. Carrillo, and G. Toscani]

[M. Gualdani, A. Jüngel, and G. Toscani]

[A. Jüngel and D. Matthes]

[R. Laugesen]

Global functional inequalities

Theorem 1 *For all* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*, there exists a positive constant* $\kappa_{p,q}$ *such that, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}} \, J_1[v] := \frac{1}{\kappa_{p,q}} \int_{S^1} |v'|^2 \, dx$$

Corollary 1 *Let* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*. Then, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{4\pi^2 \, \kappa_{p,q}} \, J_2[v] := \frac{1}{4\pi^2 \, \kappa_{p,q}} \int_{S^1} |v''|^2 \, dx$$

A minimizing sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(S^1)$

$$v_n \rightharpoonup v \quad \text{in } H^1(S^1) \quad \text{and} \quad \Sigma_{p,q}[v_n] \to \Sigma_{p,q}[v] \quad \text{as} \quad n \to \infty$$

If $\Sigma_{p,q}[v] = 0$, $\lim_{n\to\infty} J_1[v_n] = 0$. Let $\varepsilon_n := J_1[v_n]$, $w_n := \frac{v_n - 1}{\sqrt{\varepsilon_n}}$ and make a Taylor expansion

$$\left| (1+\sqrt{\varepsilon}\, x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p}\, x \right| \le \frac{1}{p}\, r(\varepsilon_0, p)\, \varepsilon \quad \forall\, (x,\varepsilon) \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \times (0, \varepsilon_0)$$

$$\varepsilon_n := J_1[v_n]\ , \quad \Sigma_{p,q}[v_n] \le c(\varepsilon_0, p, q)\, \varepsilon_n$$

Hence, since $q < 2$,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} = \frac{\varepsilon_n\, J_1[w_n]}{\Sigma_{p,q}[v_n]^{2/q}} \ge [c(\varepsilon_0, p, q)]^{-2/q}\, \varepsilon_n^{1-2/q} \to \infty$$

gives a contradiction

Asymptotic functional inequalities

The regime of small entropies:

$$\mathcal{X}_\varepsilon^{p,q} := \left\{ v \in H^1_+(S^1) \ : \ \Sigma_{p,q}[v] \le \varepsilon \text{ and } \mu_p[v] = 1 \right\}$$

Theorem 2 *For any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\,J_1[v] \quad \forall\, v \in \mathcal{X}_\varepsilon^{p,q}$$

Without the condition $\mu_p[v] = 1$:

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\,(\mu_p[v])^{q-2}\,J_1[v]$$

If $J_1[v] \le 8\,p^2\,\pi^2\,\varepsilon$, define $w := (v-1)/(\kappa_p^\infty\,\sqrt{\varepsilon})$: $J_1[w] \le 1$.

$$
\begin{aligned}
\Sigma_{p,q}[v] &= \frac{1}{pq(pq-1)}\left[\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^q dx - \left(\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^{1/p} dx\right)^{pq}\right] \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\left[\int_{S^1} w^2\;dx - \left(\int_{S^1} w\;dx\right)^2\right] + O(\varepsilon^{3/2}) \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\int_{S^1}(w-\bar{w})^2\;dx + O(\varepsilon^{3/2}) \\
&\le \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\,\frac{J_1[w]}{(2\pi)^2} + O(\varepsilon^{3/2}) = \frac{J_1[v]}{8\,p^2\,\pi^2} + O(\varepsilon^{3/2})
\end{aligned}
$$

using Poincaré's inequality

1^{st} application: Porous media

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1,\ t > 0$$

A one parameter family of entropies :

$$\Sigma_k[u] := \begin{cases} \dfrac{1}{k\,(k+1)} \displaystyle\int_{S^1} \left(u^{k+1} - \bar{u}^{k+1}\right)\, dx & \text{if} \quad k \in \mathbb{R} \setminus \{-1, 0\} \\ \displaystyle\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = 0 \\ -\displaystyle\int_{S^1} \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = -1 \end{cases}$$

With $v := u^p$, $p := \frac{m+k}{2}$, $q := \frac{k+1}{p} = 2\,\frac{k+1}{m+k}$, $\Sigma_k[u] = \Sigma_{p,q}[v]$

Lemma 1 *Let $k \in \mathbb{R}$. If u is a smooth positive solution*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} \left|(u^{(k+m)/2})_x\right|^2 dx = 0$$

with $\lambda := 4\,m/(m+k)^2$ whenever $k+m \neq 0$, and

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} |(\log u)_x|^2 \, dx = 0$$

with $\lambda := m$ for $k+m=0$.

Decay rates

Proposition 1 *Let* $m \in (0, +\infty)$, $k \in \mathbb{R} \setminus \{-m\}$, $q = 2\,(k+1)/(m+k)$, $p = (m+k)/2$ *and* u *be a smooth positive solution*

i) Short-time Algebraic Decay: If $m > 1$ *and* $k > -1$*, then*

$$\Sigma_k[u(\cdot,t)] \le \left[\Sigma_k[u_0]^{-(2-q)/q} + \frac{2-q}{q}\,\lambda\,\kappa_{p,q}\,t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m > 0$ *and* $m + k > 0$*, there exists* $C > 0$ *and* $t_1 > 0$ *such that for* $t \ge t_1$*,*

$$\Sigma_k[u(\cdot,t)] \le \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{8\,p^2\,\pi^2\,\lambda\,\bar{u}^{p(2-q)}\,(t-t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

2^{nd} Application: fourth order equations

$$u_t = -\Big(u^m \; \big(u_{xxx} + a\,u^{-1}\,u_x\,u_{xx} + b\,u^{-2}\,u_x^3\big)\Big)_x, \qquad x \in S^1,\; t > 0$$

Example 1. The thin film equation: $a = b = 0$

$$u_t = -(u^m\,u_{xxx})_x,$$

Example 2. The DLSS equation: $m = 0$, $a = -2$, and $b = 1$

$$u_t = -\Big(u\,(\log u)_{xx}\Big)_{xx},$$

$$L_\pm := \tfrac{1}{4}(3\,a + 5) \pm \tfrac{3}{4}\sqrt{(a-1)^2 - 8\,b}$$

$$A := (k+m+1)^2 - 9\,(k+m-1)^2 + 12\,a\,(k+m-2) - 36\,b$$

Theorem 3 *Assume* $(a-1)^2 \geq 8\,b$

i) Entropy production: If $L_- \leq k+m \leq L_+$

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] \leq 0 \quad \forall\, t>0$$

ii) Entropy production: If $k+m+1 \neq 0$ *and* $L_- < k+m < L_+$*,*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} \left|(u^{(k+m+1)/2})_{xx}\right|^2 dx \leq 0 \quad \forall\, t>0$$

If $k+m+1=0$ *and* $a+b+2-\mu \leq 0$ *for some* $0<\mu<1$*, then*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} |(\log u)_{xx}|^2\, dx \leq 0 \quad \forall\, t>0$$

Decay rates

Theorem 4 *Let $k, m \in \mathbb{R}$ be such that $L_- \leq k + m \leq L_+$*

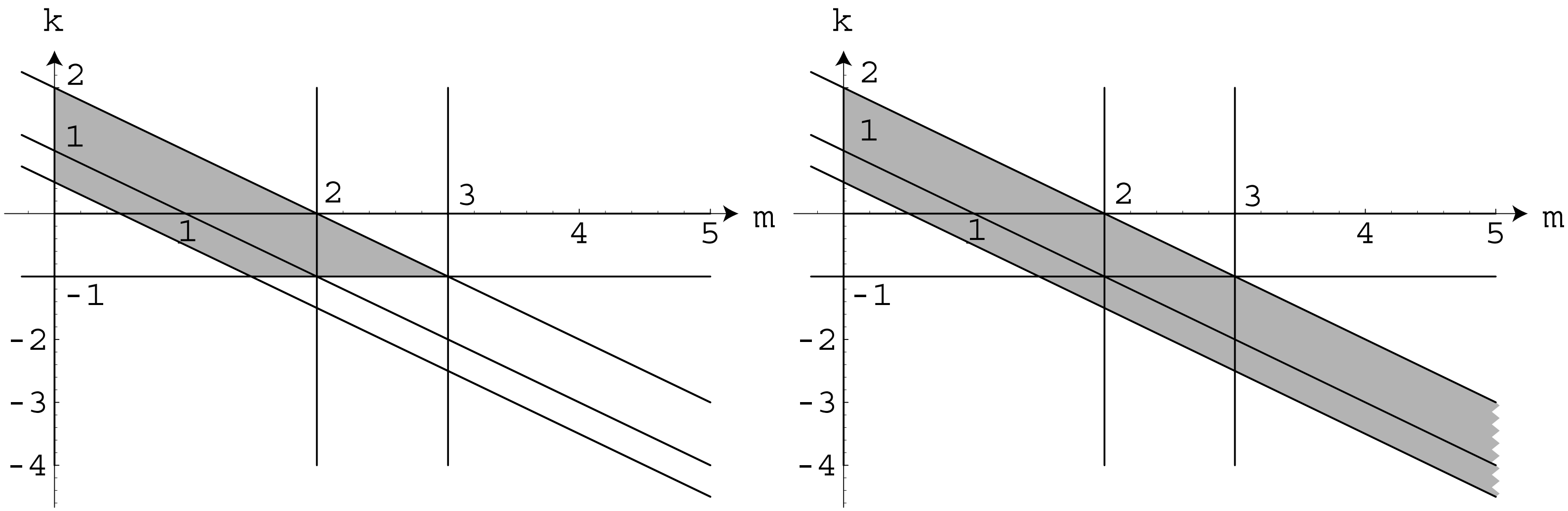
i) Short-time Algebraic Decay: If $k > -1$ and $m > 0$, then

$$\Sigma_k[u(\cdot,t)] \leq \left[\Sigma_k[u_0]^{-(2-q)/q} + 4\pi^2\,\mu\,\kappa_{p,q}\left(\frac{2}{q}-1\right)\,t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m + k + 1 > 0$, then there exists $C > 0$ and $t_1 > 0$ such that

$$\Sigma_k[u(\cdot,t)] \leq \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{32\,p^2\,\pi^4\,\mu\,\bar{u}^{p(2-q)}\,(t-t_1)}{1+C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

Thin film equation: range of the parameters



Left: algebraic decay

Right: asymptotic exponential decay

. . . Mass transport

- [J.D., Nazaret, Savaré], preliminary (formal): what has been done in terms of gradient flows for the linear case (Fokker-Planck equation) seems generalizable to the porous medium case
- Forth higher order equations: not much is understood from the entropy (PDE) point of view, [Jüngel, Matthes], [Laugesen], or from the gradient flow point of view. Gradient flow of the Fisher information: [Gianazza-Savaré-Toscani]

L^q Poincaré inequalities for general measures and consequences for the porous medium equation

J.D., Ivan Gentil, Arnaud Guillin and Feng-Yu Wang

Goal

L^q-Poincaré inequalities, $q \in (1/2, 1]$

$$\left[\mathbf{Var}_\mu(f^q)\right]^{1/q} := \left[\int f^{2q}\,d\mu - \left(\int f^q\,d\mu\right)^2\right]^{1/q} \leqslant \mathrm{C_P}\int |\nabla f|^2\,d\nu$$

Application to the weighted porous media equation, $m \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla\psi\cdot\nabla u^m\,, \quad t \geqslant 0\,, \quad x \in \mathbb{R}^d$$

(Ornstein-Uhlenbeck form). With $d\mu = d\nu = d\mu_\psi = e^{-\psi}\,dx / \int e^{-\psi}\,dx$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = -\,\frac{8}{(m+1)^2}\int |\,\nabla u^{\frac{m+1}{2}}\,|^2\,d\mu_\psi$$

Outline

Equivalence between the following properties:

- L^q-Poincaré inequality
- Capacity-measure criterion
- Weak Poincaré inequality
- BCR (Barthe-Cattiaux-Roberto) criterion

In dimension $d = 1$, there are necessary and sufficient conditions to satisfy the BCR criterion

Motivation: large time asymptotics in connection with functional inequalities

L^q-Poincaré inequality

M Riemanian manifold
Let μ a probability measure, ν a positive measure on M

We shall say that (μ, ν) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$ if for all non-negative functions $f \in \mathcal{C}^1(M)$ one has

$$\left[\mathbf{Var}_\mu(f^q)\right]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2 \, d\nu$$

$q \in (0,1]$ (false for $q > 1$ unless μ is a Dirac measure)
$\mathbf{Var}_\mu(g^2) = \int g^2 \, d\mu - \left(\int g \, d\mu\right)^2 = \mu(g^2) - \mu(g)^2$
$q \mapsto \left[\mathbf{Var}_\mu(f^q)\right]^{1/q}$ increasing wrt $q \in (0,1]$: L^q-Poincaré inequalities form a hierarchy

Capacity-measure criterion

Capacity $\mathrm{Cap}_\nu(A,\Omega)$ of two measurable sets A and Ω such that $A \subset \Omega \subset M$

$$\mathrm{Cap}_\nu(A,\Omega) := \inf\left\{\int |\nabla f|^2\, d\nu \;:\; f \in \mathcal{C}^1(M)\,,\; \mathbb{I}_A \leqslant f \leqslant \mathbb{I}_\Omega\right\}$$

$$\beta_{\mathrm{P}} := \sup\left\{\sum_{k\in\mathbb{Z}} \frac{\left[\mu(\Omega_k)\right]^{1/(1-q)}}{\left[\mathrm{Cap}_\nu(\Omega_k,\Omega_{k+1})\right]^{q/(1-q)}}\right\}^{(1-q)/q}$$

over all $\Omega \subset M$ with $\mu(\Omega) \leq 1/2$ and all sequences $(\Omega_k)_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\Omega_k \subset \Omega_{k+1} \subset \Omega$

Theorem 1 (i) *If $q \in [1/2, 1)$, then $\beta_{\mathrm{P}} \leqslant 2^{1/q}\, \mathrm{C}_{\mathrm{P}}$*

(ii) *If $q \in (0,1)$ and $\beta_{\mathrm{P}} < +\infty$, then $\mathrm{C}_{\mathrm{P}} \leqslant \kappa_{\mathrm{P}}\, \beta_{\mathrm{P}}$*

Weak Poincaré inequalities

Definition 2 [Röckner and Wang] (μ, ν) *satisfies a weak Poincaré inequality if there exists a non-negative non increasing function* $\beta_{\mathrm{WP}}(s)$ *on* $(0, 1/4)$ *such that, for any bounded function* $f \in \mathcal{C}^1(M)$,

$$\forall\, s > 0\,, \quad \mathbf{Var}_\mu(f) \leqslant \beta_{\mathrm{WP}}(s) \int |\nabla f|^2 \, d\nu + s \left[\mathbf{Osc}_\mu(f)\right]^2$$

$\mathbf{Var}_\mu(f) \leqslant \mu((f-a)^2)\ \forall\, a \in \mathbb{R}$

For $a = (\mathrm{supess}_\mu f + \mathrm{infess}_\mu f)/2$, $\mathbf{Var}_\mu(f) \leqslant \left[\mathbf{Osc}_\mu(f)\right]^2/4$: $s \leqslant 1/4$.

Proposition 3 *Let* $q \in [1/2, 1)$. *If* (μ, ν) *satisfies the* L^q*-Poincaré inequality, then it also satisfies a weak Poincaré inequality with* $\beta_{\mathrm{WP}}(s) = (11 + 5\sqrt{5})\, \beta_{\mathrm{P}}\, s^{1-1/q}/2$, $K := (11 + 5\sqrt{5})/2$.

L^q-Poincaré $\Longrightarrow$ BCR criterion $\Longrightarrow$ weak Poincaré

Theorem 4 [Maz'ja] *Let $q \in [1/2, 1)$. For all bounded open set $\Omega \subset M$, if $(\Omega_k)_{k\in\mathbb{Z}}$ is a sequence of open sets such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$, then*

$$\sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{[\mathrm{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \leqslant \frac{1}{1-q} \int_0^{\mu(\Omega)} \left(\frac{t}{\Phi(t)}\right)^{q/(1-q)} dt$$

where $\Phi(t) := \inf\{\mathrm{Cap}_\nu(A, \Omega) \,:\, A \subset \Omega\,,\ \mu(A) \geqslant t\}$

As a consequence: $\beta_{\mathrm{P}} \leqslant (1-q)^{-(1-q)/q} \, \|t/\Phi(t)\|_{L^{q/(1-q)}(0,\mu(\Omega))}$

Corollary 5 *Let $q \in [1/2, 1)$. If (μ, ν) satisfies a weak Poincaré inequality with function β_{WP}, then it satisfies a L^q-Poincaré inequality with*

$$\beta_{\mathrm{P}} \leqslant \frac{11+5\sqrt{5}}{2} \left(\frac{4}{1-q}\right)^{\frac{1-q}{q}} \|\beta_{\mathrm{WP}}(\cdot/4)\|_{L^{\frac{q}{1-q}}(0,1/2)}$$

L^q-Poincaré $\Longrightarrow$ Weak Poincaré with $\beta_{\mathrm{WP}}(s) = C\, s^{\frac{q-1}{q}}$ $\Longrightarrow$ $L^{q'}$-Poincaré $\forall\, q' \in (0, q)$

BCR criterion (1/2)

A variant of two results of [Barthe, Cattiaux, Roberto, 2005] (no absolute continuity of the measure μ with respect to the volume measure)

Theorem 6 [BCR] *Let μ be a probability measure and ν a positive measure on M such that (μ,ν) satisfies a weak Poincaré inequality with function $\beta_{\mathrm{WP}}(s)$. Then for every measurable subsets A, B of M such that $A \subset B$ and $\mu(B) \leqslant 1/2$,*

$$\mathrm{Cap}_\nu(A,B) \geq \frac{\mu(A)}{\gamma(\mu(A))} \quad \textit{with} \quad \gamma(s) := 4\,\beta_{\mathrm{WP}}(s/4)$$

Proof $\triangleleft$ Take f such that $\mathbb{I}_A \leqslant f \leqslant \mathbb{I}_B$: $\mathbf{Osc}_\mu(f) \leqslant 1$

By Cauchy-Schwarz, $\left(\int f\,d\mu\right)^2 \leqslant \mu(B) \int f^2\,d\mu \leqslant \frac{1}{2} \int f^2\,d\mu$

$$\beta_{\mathrm{WP}}(s) \int |\nabla f|^2\,d\nu + s \geqslant \mathbf{Var}_\mu(f) \geq \frac{1}{2} \int f^2\,d\mu \geq \frac{\mu(A)}{2}$$

$\frac{a}{\gamma(a)} = \frac{a}{4\,\beta_{\mathrm{WP}}(a/4)} \leqslant \sup_{s\in(0,1/4)} \frac{a/2-s}{\beta_{\mathrm{WP}}(s)}$ with $a/2 = \mu(A)/2 \leqslant 1/4$ $\triangleright$

BCR criterion (2/2)

Lemma 7 *Take μ and ν as before, $\theta \in (0,1)$, γ a positive non increasing function on $(0,\theta)$. If $\forall\ A, B \subset M$ such that $A \subset B$ are measurable and $\mu(B) \leqslant \theta$,*

$$\mathrm{Cap}_\nu(A,B) \geq \frac{\mu(A)}{\gamma(\mu(A))}$$

then for every function $f \in \mathcal{C}^1(M)$ such that $\mu(\Omega_+) \leqslant \theta$, $\Omega_+ := \{f > 0\}$

$$\int f_+^2 \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int_{\Omega_+} |\nabla f|^2\,d\nu + s\left[\text{supess}_\mu\, f\right]^2 \quad \forall\ s \in (0,1)$$

Theorem 8 *Same assumptions, $\theta = 1/2$. Then $\forall f \in \mathcal{C}^1(M)$*

$$\mathbf{Var}_\mu(f) \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int |\nabla f|^2\,d\nu + s\left[\mathbf{Osc}_\mu(f)\right] \quad \forall\ s \in (0,1/4)$$

$\theta = 1/2$: use the median $m_\mu(f)$, $\mu(f \geqslant m_\mu(f)) \geqslant 1/2$, $\mu(f \leqslant m_\mu(f)) \geqslant 1/2$

Using the BCR criterion: a “Hardy condition”

[Muckenhoupt, 1972] [Bobkov-Götze, 1999] [Barthe-Roberto, 2003] [Barthe-Cattiaux-Roberto, 2005]

$M = \mathbb{R}$, $d\mu = \rho_\nu \, dx$ with median m_μ, $d\nu = \rho_\nu \, dx$

$$R(x) := \mu([x, +\infty)) \ , \quad L(x) := \mu((-\infty, x])$$
$$r(x) := \int_{m_\mu}^{x} \frac{1}{\rho_\nu} \, dx \quad \text{and} \quad \ell(x) := \int_{x}^{m_\mu} \frac{1}{\rho_\nu} \, dx$$

Proposition 9 *Let $q \in [1/2, 1]$. (μ, ν) satisfies a L^q-Poincaré inequality if*

$$\int_{m_\mu}^{\infty} |\, r \, R \,|^{q/(1-q)} \, d\mu < \infty \quad \textit{and} \quad \int_{-\infty}^{m_\mu} |\, \ell \, L \,|^{q/(1-q)} \, d\mu < \infty$$

Proof

Proof $\lhd$ Method: $\mathbf{Var}_\mu(f) \leqslant \mu(|F_-|^2) + \mu(|F_+|^2))$ with $g = (f - f(m_\mu))_\pm$ and prove that

$$\mu(|g|^2) \leqslant \frac{11+5\sqrt{5}}{2}\,\gamma(s) \int |\nabla g|^2\, d\nu + s\left[\text{supess}_\mu g\right]^2 \quad \forall\, s \in (0, 1/2)$$

Let $A \subset B \subset M = (m_\mu, \infty)$ such that $A \subset B$ and $\mu(B) \leqslant 1/2$

$$\mathrm{Cap}_\nu(A, B) \geqslant \mathrm{Cap}_\nu\big(A, (m_\mu, \infty)\big) = \mathrm{Cap}_\nu\big((a, \infty), (m_\mu, \infty)\big) = \frac{1}{r(a)}$$

where $a = \inf A$. Change variables: $t = R(a)$ and choose $\gamma(t) := t\,(r \circ R)^{-1}(t)$ for any $t \in (0, 1/2)$ $\rhd$

Porous media equation

With $\psi \in \mathcal{C}^2(\mathbb{R}^d)$, $d\mu_\psi := \frac{e^{-\psi}\,dx}{Z_\psi}$, define $\mathcal{L}$ on $\mathcal{C}^2(\mathbb{R}^d)$ by

$$\forall\ f \in \mathcal{C}^2(\mathbb{R}^d) \quad \mathcal{L}f := \Delta f - \nabla\psi \cdot \nabla f$$

Such a generator $\mathcal{L}$ is symmetric in $L^2_{\mu_\psi}(\mathbb{R}^d)$,

$$\forall\ f,\ g \in \mathcal{C}^1(\mathbb{R}^d) \quad \int f\,\mathcal{L}g\,d\mu_\psi = -\int \nabla f \cdot \nabla g\,d\mu_\psi$$

Consider for $m > 1$ the weighted porous media equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}\,u^m & \text{in} \quad \mathrm{Q} \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega \\ n \cdot \nabla u = 0 & \text{on} \quad \Sigma \end{cases}$$

$\Omega \subset \mathbb{R}^d$, $Q = \Omega \times [0, +\infty)$, $\Sigma = \partial\Omega \times [0, +\infty)$

$u \in \mathcal{C}^2$, L^1-contraction, existence and uniqueness

Asymptotic behavior

Theorem 10 *Let $m \geqslant 1$ and assume that (μ_ψ, μ_ψ) satisfies a L^q-Poincaré inequality, $q = 2/(m+1)$*

$$\mathbf{Var}_{\mu_\psi}(u(\cdot,t)) \leqslant \left(\left[\mathbf{Var}_{\mu_\psi}(u_0) \right]^{-(m-1)/2} + \frac{4\,m\,(m-1)}{(m+1)^2}\,\mathrm{C_P}\,t \right)^{-2/(m-1)}$$

Reciprocally, if the above inequality is satisfied for any u_0, then (μ_ψ, μ_ψ) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$

Proof $\lhd$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = 2\int u_t\,u\,d\mu_\psi = 2\int u\,\mathcal{L}u^m\,d\mu_\psi = -\frac{8m}{(m+1)^2}\int |\nabla u^{\frac{m+1}{2}}|^2\,d\mu_\psi$$

Apply the L^q-Poincaré inequality with $u = f^{2/(m+1)}$, $q = 2/(m+1)$

Reciprocally, a derivation at $t = 0$ gives the L^q-Poincaré inequality $\rhd$

A conclusion on L^q-Poincaré inequalities

- Observe that we have only algebraic rates
- Weak logarithmic Sobolev inequalities [Cattiaux-Gentil-Guillin, 2006], L^q-logarithmic Sobolev inequalities [D.-Gentil-Guillin-Wang, 2006]

$$\left(\int f^{2q} \, \frac{\log f^{2q}}{\int f^{2q}\, d\mu} \, d\mu\right) =: \mathbf{Ent}_\mu\left(f^{2q}\right)^{1/q} \leq \mathrm{C}_{\mathrm{LS}} \int |\nabla f|^2 \, d\mu$$

- Orlicz spaces, duality, connections with mass transport theory [Bobkov-Götze, 1999] [Cattiaux-Gentil-Guillin, 2006] [Wang, 2006] [Roberto-Zegarlinski, 2003] [Barthe-Cattiaux-Roberto, 2005]

. . . Mass transport

- The Hardy criterion makes the link with mass transport in dimension 1. As already quoted this has been at least partially investigated
- By changing the distance, could one get more generalized entropy functionals ?