Entropy methods and stability results in Gagliardo-Nirenberg-Sobolev inequalities

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Joint work on ***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method*** arXiv:2007.03674 with

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Outline

1. A brief introduction to entropy methods
2. A variational point of view on stability
   - Optimality by concentration-compactness
   - Non-constructive stability results
   - Towards constructive stability results
3. Fast diffusion equation and entropy methods
   - Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities
   - The fast diffusion equation in self-similar variables
   - Initial and asymptotic time layers
4. Stability in Gagliardo-Nirenberg-Sobolev inequalities
   - The threshold time and the improved entropy – entropy production inequality (subcritical case)
   - First stability results (subcritical case)
   - Stability in Sobolev's inequality (critical case)

A brief introduction to entropy methods

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \ge 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1, 2$

Theorem

If $\lambda \le d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas, Spruck, 1981], [Bidaut-Véron, Véron, 1991]

Bifurcation point of view

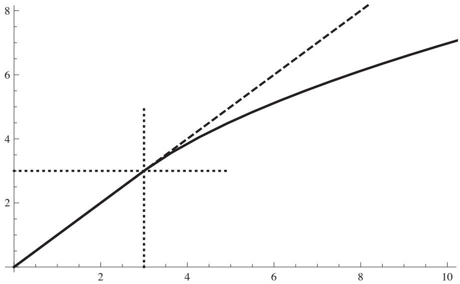


Figure: $(p-2)\lambda \mapsto (p-2)\mu(\lambda)$ with $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

- The Bakry-Emery method (compact manifolds)

▷ The Fokker-Planck equation
▷ The Bakry-Emery method on the sphere: a parabolic method
▷ The Moser-Trudiger-Onofri inequality (on a compact manifold)

- Fast diffusion equations on the Euclidean space (without weights)

▷ Euclidean space: Rényi entropy powers

▷ Euclidean space: self-similar variables and relative entropies
▷ The role of the spectral gap ▷ Second part of the lecture

The Fokker-Planck equation

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathscr{L} v$$

[Bakry, Emery, 1985], [Arnold, Markowich, Toscani, Unterreiter, 2001]
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\, dx} \quad \Longleftrightarrow \quad v_s = 1$$

The Bakry-Emery method

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the *q-entropy* is defined by

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right) d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

$$\text{with} \quad \lambda := \inf_{w \in H^1(\Omega,d\gamma)\setminus\{0\}} \frac{\int_\Omega \left(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi : \nabla w \otimes \nabla w\right) d\gamma}{\int_\Omega |\nabla w|^2\,d\gamma}$$

Proposition

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008] *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ and $\mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

A proof of the interpolation inequality by the *carré du champ* method

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

$p \in [1,2) \cup (2,2^*]$ if $d \ge 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1, 2$

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry, Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2d^2+1}{(d-1)^2}$

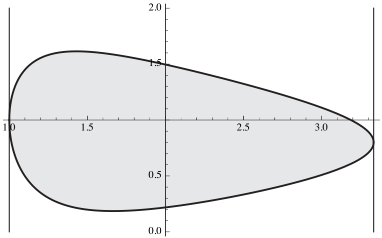
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

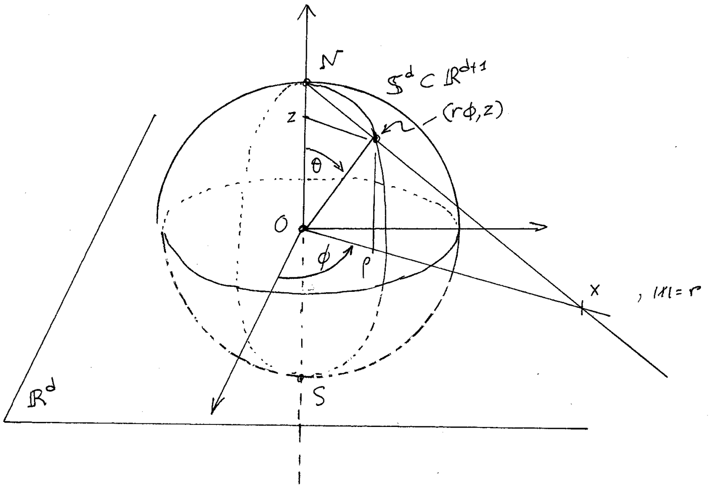
[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region,* $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d := \nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L} f := (1-z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1, \mathcal{L} f_2\rangle = -\int_{-1}^{1} f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. For any $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal{L} f\rangle = \int_{-1}^{1} |f'|^2\,\nu\,d\nu_d \ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathscr{L}\,g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathscr{L}\,f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f,\mathscr{L}\,f\rangle = \langle \mathscr{L}\,f,\mathscr{L}\,f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu,\mathscr{L}\,f\right\rangle$$

$$\frac{d}{dt}\mathscr{I}[g(t,\cdot)] + 2\,d\,\mathscr{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$
$$= -2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^{\#} < \frac{2\,d}{d-2} = 2^*$$

The elliptic point of view (nonlinear flow)

$\frac{\partial u}{\partial t} = u^{2-2\beta}\left(\mathscr{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right),\ \kappa = \beta\,(p-2)+1$

$$-\mathscr{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathscr{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathscr{L}\,u\,u^\kappa\,d\nu_d = -\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

The two terms cancel and we are left only with

$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\,d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathscr{M})\setminus\{0\}} \frac{\displaystyle\int_{\mathscr{M}} \Big[\, \|\,\mathrm{L}_g u - \tfrac{1}{2}\,\mathrm{M}_g u\,\|^2 + \mathfrak{R}(\nabla u, \nabla u)\Big]\, e^{-u/2}\, d\,v_g}{\displaystyle\int_{\mathscr{M}} |\nabla u|^2\, e^{-u/2}\, d\,v_g}$$

Theorem

Assume that $d=2$ and $\lambda_\star>0$. If u is a smooth solution to

$$-\frac{1}{2}\Delta_g u+\lambda=e^u$$

then u is a constant function if $\lambda\in(0,\lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathscr{M}$

$$\frac{1}{4}\|\nabla u\|^2_{\mathrm{L}^2(\mathscr{M})}+\lambda\int_{\mathscr{M}}u\,d\,v_g\ge\lambda\log\left(\int_{\mathscr{M}}e^u\,d\,v_g\right)\quad\forall\,u\in\mathrm{H}^1(\mathscr{M})$$

for some constant $\lambda>0$. Let us denote by λ_1 the first positive e.v. of $-\Delta_g$

Corollary

If $d=2$, then the MTO inequality holds with $\lambda=\Lambda:=\min\{4\pi,\lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g\big(e^{-f/2}\big) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathscr{G}_\lambda[f] := \int_{\mathscr{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathscr{M}} \mathfrak{R}(\nabla f, \nabla f)\,e^{-f/2}\,d\,v_g - \lambda \int_{\mathscr{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathscr{F}_\lambda[f(t,\cdot)] = \int_{\mathscr{M}} \Big(-\tfrac{1}{2}\,\Delta_g f + \lambda\Big)\Big(\Delta_g\big(e^{-f/2}\big) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\Big)\,d\,v_g = -\,\mathscr{G}_\lambda[f(t,\cdot)]$$

Since $\mathscr{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathscr{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathscr{F}_\lambda[u] \ge \int_0^\infty \mathscr{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)-\int_{\mathbb{R}^2}u\,d\mu\right]$$

hold for some $\lambda>0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda<\Lambda_\star$ and the inequality holds with $\lambda=\Lambda_\star$ if equality is achieved among radial functions

Euclidean space: Rényi entropy powers and fast diffusion

- The Euclidean space without weights

▷ Rényi entropy powers, the entropy approach without rescaling: [Savaré, Toscani]: scalings, nonlinearity and a concavity property inspired by information theory

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \ge 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \ge 0$ such that $\int_{\mathbb{R}^d} v_0\,dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 v_0\,dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa\,t^{1/\mu}\right)^d}\,\mathcal{B}_\star\left(\frac{x}{\kappa\,t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m)\,\mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1)\,\mathsf{p}\,\Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m}\left(\frac{1}{d} + m - 1\right) = \frac{2}{d}\,\frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The variation of the Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1 - \frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathsf{p}|^2\,dx = -2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani,Savaré] *Assume that* $m \ge 1-\frac{1}{d}$ *if* $d>1$ *and* $m>0$ *if* $d=1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \le 0$ *and*

$$\lim_{t\to+\infty}\frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma\,\lim_{t\to+\infty}\mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_2^\theta\,\|w\|_{q+1}^{1-\theta} \ge \mathsf{C}_{\rm GN}\,\|w\|_{2q}$$

if $1-\frac{1}{d} \le m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{2q}}$, $q = \frac{1}{2\,m-1}$

A brief introduction to some stability issues in Sobolev and related inequalities

The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal constant S_d),

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \ge 3$?
A question raised in [Brezis, Lieb (1985)]

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \alpha \inf_{\varphi\in\mathfrak{M}} \|\nabla f - \nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $f \mapsto \lambda(f)$ such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \big(1 + \kappa\,\lambda(f)^\alpha\big)\,\mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

*However, the question of **constructive** estimates is still widely open*

Gagliardo-Nirenberg-Sobolev inequalities

We consider the inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\rm GNS}(p)\,\|f\|_{2p}\tag{GNS}$$

$\theta=\frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}$, $p\in(1,+\infty)$ if $d=1$ or 2, $p\in(1,p^*]$ if $d\ge3$, $p^*=\frac{d}{d-2}$

Theorem (del Pino, JD)

Equality case in (GNS) is achieved if and only if

$$f\in\mathfrak{M}:=\left\{g_{\lambda,\mu,y}\,:\,(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$

Aubin-Talenti functions: $g_{\lambda,\mu,y}(x):=\mu\,g\left((x-y)/\lambda\right)$, $g(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}$

[del Pino, JD, 2002], [Gunson, 1987, 1991]

Related inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\rm GNS}(p)\,\|f\|_{2\,p}\qquad\text{(GNS)}$$

▷ *Sobolev's inequality*: $d\ge3$, $p=p^*=d/(d-2)$

$$\|\nabla f\|_2^2\ge\mathsf{S}_d\,\|f\|_{2\,p^*}^2$$

▷ *Euclidean Onofri inequality*

$$\int_{\mathbb{R}^2}e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2}\le e^{\,\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$d=2$, $p\to+\infty$ with $f_p(x):=\mathsf{g}(x)\left(1+\frac{1}{2\,p}\,(h(x)-\overline{h})\right)$, $\overline{h}=\int_{\mathbb{R}^2}h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

▷ *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right)\ge\int_{\mathbb{R}^d}|f|^2\log|f|^2\,dx$$

$\|f\|_2=1$, or $\int_{\mathbb{R}^d}|\nabla f|^2\,dx\ge\frac{1}{2}\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx+\frac{d}{4}\log\left(2\,\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,dx$

A variational point of view on stability

Optimality by concentration-compactness

Deficit functional, scale invariance, weak stability

Deficit functional

$$\delta[f] := (p-1)^2 \, \|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Lemma

(GNS) *is equivalent to* $\delta[f] \geq 0$ *if and only if*

$$\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$$

where $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

Take $f_\lambda(x) = \lambda^{\frac{d}{2p}}\,f(\lambda\,x)$ and optimize on $\lambda > 0$ to get (*weak stability*)

$$\delta[f] \geq \delta[f] - \inf_{\lambda>0} \delta[f_\lambda] =: \delta_\star[f] \geq 0$$

A simplification: $\delta[f] = \delta[|f|]$ so we shall assume that $f \geq 0$ a.e.

Minimization and concentration-compactness

$$I_M=\inf\left\{(p-1)^2\,\|\nabla f\|_2^2+4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1}\,:\,f\in\mathscr{H}_p(\mathbb{R}^d)\,,\quad\|f\|_{2p}^{2p}=M\right\}$$

$$I_1=\mathscr{K}_{\rm GNS}\text{ and }I_M=I_1\,M^\gamma\text{ for any }M>0$$

Lemma

If $d\ge1$ and p is an admissible exponent with $p<d/(d-2)$, then

$$I_{M_1+M_2}<I_{M_1}+I_{M_2}\quad\forall\,M_1,M_2>0$$

Lemma

Let $d\ge1$ and p be an admissible exponent with $p<d/(d-2)$ if $d\ge3$. If $(f_n)_n$ is minimizing and if $\limsup_{n\to+\infty}\sup_{y\in\mathbb{R}^d}\int_{B(y)}|f_n|^{p+1}\,dx=0$, then

$$\lim_{n\to\infty}\|f_n\|_{2p}=0$$

... *Existence*

Existence of a minimizer, further properties

Proposition

Assume that $d \geq 1$ is an integer and let p be an admissible exponent with $p < d/(d-2)$ if $d \geq 3$. Then there is a radial minimizer of δ

- *Pólya-Szegö principle*: there is a radial minimizer solving

$$-2(p-1)^2 \Delta f + 4\big(d - p(d-2)\big) f^p - C f^{2p-1} = 0$$

If $f = \mathbf{g}$, then $C = 8p$

- *A rigidity result*: if f is a (smooth) minimizer and $\mathrm{P} = -\frac{p+1}{p-1} f^{1-p}$, then

$$\big(d - p(d-2)\big) \int_{\mathbb{R}^d} f^{p+1} \left| \Delta \mathrm{P} + (p+1)^2 \frac{\int_{\mathbb{R}^d} |\nabla f|^2 \, dx}{\int_{\mathbb{R}^d} f^{p+1} \, dx} \right|^2 dx + 2\, d\, p \int_{\mathbb{R}^d} f^{p+1} \left\| \mathrm{D}^2 \mathrm{P} - \frac{1}{d} \Delta \mathrm{P} \, \mathrm{Id} \right\|^2 dx = 0$$

$\triangleright$ $\mathbf{g}(x) = (1 + |x|^2)^{-\frac{1}{p-1}}$ is a minimizer and $\delta[\mathbf{g}] = 0$

Non-constructive stability results

Relative entropy and Fisher information

- *Free energy* or *relative entropy functional*

$$\mathcal{E}[f|g] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - g^{p+1} - \tfrac{1+p}{2p}\, g^{1-p} \left(f^{2p} - g^{2p} \right) \right) dx \ge 0$$

Lemma (Csiszár-Kullback inequality)

Let $d \ge 1$ and $p > 1$. There exists a constant $C_p > 0$ such that

$$\left\| f^{2p} - g^{2p} \right\|^2_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p\, \mathcal{E}[f|g] \quad \textit{if} \quad \|f\|_{2p} = \|g\|_{2p}$$

- *Relative Fisher information*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1} \int_{\mathbb{R}^d} \left| (p-1)\nabla f + f^p \nabla g^{1-p} \right|^2 dx$$

Best matching profile

- Nonlinear extension of the *Heisenberg uncertainty principle*

$$\left(\frac{d}{p+1}\int_{\mathbb{R}^d} f^{p+1}\,dx\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2\,dx \int_{\mathbb{R}^d} |x|^2 f^{2p}\,dx$$

▷ Take $g = \mathbf{g}$ in $\mathscr{J}[f|g]$ and expand the square

- If $\mathrm{g}_f := g \in \mathfrak{M}$ is such that $\int_{\mathbb{R}^d} f^{2p}\,(1,x,|x|^2)\,dx = \int_{\mathbb{R}^d} g^{2p}\,(1,x,|x|^2)\,dx$

$$\text{then}\quad \mathscr{E}[f|g] = \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1}-g^{p+1}\right)dx$$

▷ A smaller space: $\mathscr{W}_p(\mathbb{R}^d) := \left\{ f \in \mathscr{H}_p(\mathbb{R}^d)\,:\,|x|\,|f|^p \in \mathrm{L}^2(\mathbb{R}^d)\right\}$

Lemma

For any $f \in \mathscr{W}_p(\mathbb{R}^d)$, $\mathrm{g}_f \in \mathfrak{M}$ is uniquely defined and

$$\mathscr{E}[f|\mathrm{g}_f] = \inf_{g\in\mathfrak{M}} \mathscr{E}[f|g]$$

A first (weak) stability result

Lemma (A weak stability result)

If $g_f = \mathbf{g}$, *then*

$$\delta[f] \geq \delta_\star[f] \approx \mathcal{E}[f|\mathbf{g}]^2$$

▷ Up to the invariances, $\mathbf{g}$ is the unique minimizer for $f \mapsto \delta[f]$

Lemma (Entropy - entropy production inequality)

If $\|f\|_{2p} = \|g\|_{2p}$ *with* $\delta[g] = 0$, *then*

$$\frac{p+1}{p-1}\,\delta[f] = \mathcal{J}[f|g] - 4\,\mathcal{E}[f|g] \geq 0$$

▷ From now on, we will assume that $g_f = \mathbf{g}$, *i.e.*

$$\int_{\mathbb{R}^d} f^{2p}\,(1, x, |x|^2)\,dx = \int_{\mathbb{R}^d} \mathbf{g}^{2p}\,(1, x, |x|^2)\,dx$$

Stability in (GNS)

- [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\mathsf{S}_d\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\ge\alpha\inf_{\varphi\in\mathfrak{M}}\|\nabla f-\nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

- Various extensions

▷ L^q norm of the gradient by [Chianchi, Fusco, Maggi, Pratelli (2009)], [Figalli, Neumayer (2018)], [Neumayer (2020)], [Figalli, Zhang (2020)]
▷ (GNS) inequalities by [Carlen, Figalli (2013)], [Seuffert (2017)], [Nguyen (2019)]

Theorem

There exists a constant $C>0$ such that

$$\delta[f]\ge C\,\mathcal{E}[f|\mathbf{g}]$$

for any $f\in\mathcal{W}_p(\mathbb{R}^d)$ satisfying

$$\int_{\mathbb{R}^d}f^{2p}(1,x,|x|^2)\,dx=\int_{\mathbb{R}^d}\mathbf{g}^{2p}(1,x,|x|^2)\,dx$$

Towards constructive stability results

A strategy based on a spectral gap

- The spectral gap inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, \mathbf{g}^{2p} \, dx \geq \frac{4p}{p-1} \int_{\mathbb{R}^d} |u|^2 \, \mathbf{g}^{3p-1} \, dx$$

valid for any function u such that $\int_{\mathbb{R}^d} u \, \mathbf{g}^{3p-1} \, dx = 0$, can be improved with a constant $\Lambda_\star > 4p/(p-1)$ under the constraint that

$$\int_{\mathbb{R}^d} \left(1, x, |x|^2\right) u \, \mathbf{g}^{3p-1} \, dx = 0$$

- A Taylor expansion with $f = \mathbf{g} + \eta \, h$ gives

$$\lim_{\eta \to 0} \frac{\delta[f_\eta]}{\mathcal{E}[f_\eta | \mathrm{g}]} \geq \frac{(p-1)^2}{p\,(p+1)} \left[\Lambda_\star - \frac{4\,p}{p-1}\right]$$

▷ Analysis along a minimizing sequence...

How can we make this strategy constructive ?

From the carré du champ method to stability results

Carré du champ method (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m \,, \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I} \,, \quad \frac{d\mathscr{I}}{dt} \le -\Lambda\,\mathscr{I}$$

deduce that $\mathscr{I} - \Lambda\mathscr{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathscr{I}[u] \ge \Lambda\,\mathscr{F}[u]}$$

▷ An ***improved entropy – entropy production inequality*** (weak form)

$$\mathscr{I} \ge \Lambda\,\psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda\mathscr{F} \ge \Lambda\big(\psi(\mathscr{F}) - \mathscr{F}\big) \ge 0$$

▷ An ***improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \ge \Lambda$ such that

$$\mathscr{I} - \Lambda\mathscr{F} \ge (\Lambda_\star - \Lambda)\,\mathscr{F}$$

Fast diffusion equation and entropy methods

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$ and $\mathscr{B}$ is the Barenblatt profile with $\int_{\mathbb{R}^d} \mathscr{B} \, dx = \mathcal{M}$

$$\mathscr{B}(x) := \left(1 + |x|^2\right)^{-\frac{1}{1-m}}$$

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation*. With $m \ge m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment*. With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate*. With $m \ge m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GNS}(p) \, \|f\|_{2p} \qquad \text{(GNS)}$$

$$p = \frac{1}{2m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2p} \in [m_1, 1)$$

$u = f^{2p}$ so that $u^m = f^{p+1}$ and $u\,|\nabla u^{m-1}|^2 = (p-1)^2\,|\nabla f|^2$

$$\mathscr{M} = \|f\|_{2p}^{2p}\,, \quad \mathsf{E}[u] = \|f\|_{p+1}^{p+1}\,, \quad \mathsf{I}[u] = (p+1)^2\,\|\nabla f\|_2^2$$

If u solves (FDE) $\frac{\partial u}{\partial t} = \Delta u^m$

$$\mathsf{E}' \ge \frac{p-1}{2p}\,(p+1)^2 \left(\mathscr{C}_{\rm GNS(p)}\right)^{\frac{2}{\theta}} \|f\|_{2p}^{\frac{2}{\theta}} \, \|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}} = C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \ge \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\,t \ge 0$$

Equality case: $u(t,x) = \frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)$, $\mathscr{B}(x) := \left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\Big(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta \mathsf{P})^2\Big)\,dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares

$$-\frac{\mathsf{I}}{2\theta}\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$

$$=\int_{\mathbb{R}^d}u^m\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2dx+(m-m_1)\int_{\mathbb{R}^d}u^m\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^\theta\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathscr{M}^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathscr{B}]^\theta\,\mathsf{E}[\mathscr{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathscr{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^\theta\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\theta}\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{2\theta}\mathscr{M}^{\frac{2\theta}{p}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\Big[v\left(\nabla v^{m-1}-2\,x\right)\Big]=0 \tag{r FDE}$$

Generalized entropy (free energy) and *Fisher information*

$$\mathscr F[v]:=-\frac1m\int_{\mathbb R^d}\Big(v^m-\mathscr B^m-m\,\mathscr B^{m-1}\,(v-\mathscr B)\Big)\,dx$$
$$\mathscr I[v]:=\int_{\mathbb R^d}v\left|\nabla v^{m-1}+2\,x\right|^2\,dx$$

are such that $\mathscr I[v]\ge4\,\mathscr F[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr F[v(t,\cdot)]\le\mathscr F[v_0]\,e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

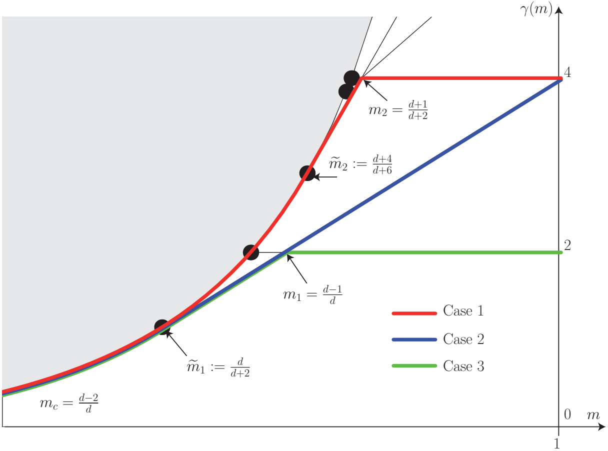
Lemma

Under assumption (H),

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,,\quad \gamma(m) := (1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m) := 2$ *if* $1-1/d \le m < 1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad\text{and}\quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

Hardy-Poincaré inequality. Let $d\ge 1$, $m\in(m_1,1)$ and $g\in \mathrm{L}^2(\mathbb{R}^d,\mathscr{B}^{2-m}\,dx)$ such that $\nabla g\in \mathrm{L}^2(\mathbb{R}^d,\mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx=0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx=0$

$$\mathsf{I}[g]\ge 4\,\alpha\,\mathsf{F}[g] \quad\text{where}\quad \alpha=2-d\,(1-m)$$

Proposition

Let $m\in(m_1,1)$ if $d\ge 2$, $m\in(1/3,1)$ if $d=1$, $\eta=2\,(d\,m-d+1)$ and $\chi=m/(266+56\,m)$. If $\int_{\mathbb{R}^d} v\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx=0$ and

$$(1-\varepsilon)\,\mathscr{B}\le v\le(1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon\in(0,\chi\,\eta)$, then

$$\mathscr{I}[v]\ge(4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0)=\psi'(0)=0$, we have

$$\mathcal{I}-4\mathcal{F}\geq 4\big(\psi(\mathcal{F})-\mathcal{F}\big)\geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathcal{Q}[v]:=\frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt}\leq\mathcal{Q}\,(\mathcal{Q}-4)$$

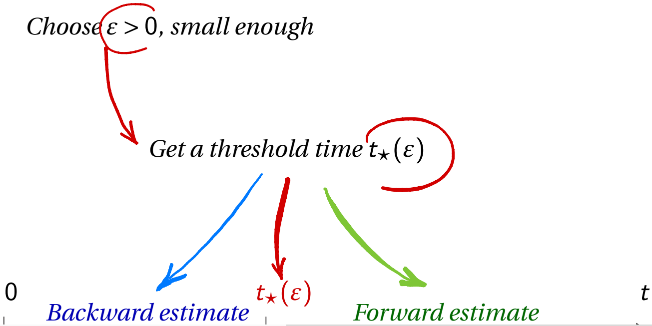
Lemma

Assume that $m>m_1$ and v is a solution to (r FDE) with nonnegative initial datum v_0. If for some $\eta>0$ and $t_\star>0$, we have $\mathcal{Q}[v(t_\star,\cdot)]\geq 4+\eta$, then

$$\mathcal{Q}[v(t,\cdot)]\geq 4+\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}\quad\forall\,t\in[0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

If v is a solves $(r\,\mathrm{FDE})$ for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \tag{H_A}$$

then

$$(1-\varepsilon)\,\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathscr{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Large time asymptotics and Barenblatt solutions

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

admits the self-similar *Barenblatt* solution

$$B(t,x) = \frac{t^{1/(1-m)}}{\left[b_0 \, \frac{t^{2/\mu}}{\mathcal{M}^{2/\mu(1-m)}} + b_1 \, |x|^2\right]^{1/(1-m)}}$$

where $\mu = 2 - d\,(1-m) > 0$, such that

$$\lim_{t\to+\infty} \|u(t) - B(t)\|_{\mathrm{L}^1(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t\to+\infty} t^{d/\mu}\, \|u(t) - B(t)\|_{\mathrm{L}^\infty(\mathbb{R}^d)} = 0$$

The uniform convergence in relative error is a matter of tails

We are interested in the convergence in *relative error*, *i.e.*, the convergence of

$$\left|\frac{u(t,x)-B(t,x)}{B(t,x)}\right|$$

with $\mathcal{M}=\int_{\mathbb{R}^d} u_0\,dx$. If the initial data is $u_0(x)=\left(1+|x|^2\right)^{-m/(1-m)}$, then the solution of (FDE) satisfies

$$\frac{1}{\left[(c\,t+1)^{1/(1-m)}+|x|^2\right]^{\frac{m}{1-m}}}\le u(t,x)\le\frac{(1+t)^{\frac{m}{1-m}}}{\left(1+t+|x|^2\right)^{\frac{m}{1-m}}}$$

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t>0$ large enough

$$\mathscr{B}_{M_1}(t-\tau_1,x)\le u(t,x)\le \mathscr{B}_{M_2}(t+\tau_2,x) \qquad \text{(GHP)}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0]:=\sup_{R>0} R^{\frac{2}{1-m}-d}\int_{\mathbb{R}^d\setminus B_R(0)}|u_0|\,dx<\infty$$

Theorem

[Bonforte, Simonov, 2020] If $M+\mathsf{A}[u_0]<\infty$, then

$$\lim_{t\to\infty}\left\|\frac{u(t)-B(t)}{B(t)}\right\|_\infty=0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m\in(m_1,1)$ if $d\ge2$, $m\in(1/3,1)$ if $d=1$ and let $\varepsilon\in(0,1/2)$, small enough, $A>0$, and $G>0$ be given. There exists an explicit threshold time $T\ge0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t}=\Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0\in\mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0]=\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}}\int_{|x|>r}u_0\,dx\le A<\infty \tag{H_A}$$

$\int_{\mathbb{R}^d}u_0\,dx=\int_{\mathbb{R}^d}B\,dx=\mathcal{M}$ *and* $\mathcal{F}[u_0]\le G$, *then*

$$\sup_{x\in\mathbb{R}^d}\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\varepsilon\quad\forall\,t\ge T$$

The threshold time

Proposition

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$T=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$, $\alpha=d\,(m-m_c)$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,dx=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\,\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad t_\star=t_\star(\varepsilon)=\frac12\log R(T)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,t_\star]\quad\text{where}\quad\zeta=\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}$$

Two consequences

$$\zeta = \mathsf{Z}\big(A, \mathscr{F}[u_0]\big)\,, \quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1,1)$ if $d \ge 2$, $m \in (1/2,1)$ if $d=1$, $A>0$ and $G>0$. If v is a solution of $(r\,\mathrm{FDE})$ with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\,t \ge 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1} \, \frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}, \quad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}} \, \|f\|_{2p}^{2p}} \sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}} f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p} \left(\frac{\kappa[f]^{2p}}{\lambda[f]^2} f^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1} \, \frac{1}{C(p,d)} \, \mathsf{Z}(\mathsf{A}[f], \mathsf{E}[f])$$

Theorem

Let $d \ge 1$, $p \in (1, p^*)$

If $f \in \mathscr{W}_p(\mathbb{R}^d) := \{ f \in \mathrm{L}^{2p}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^p \in \mathrm{L}^2(\mathbb{R}^d) \}$,

$$\left(\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \right)^{2p\gamma} \ge \mathfrak{S}[f] \, \|f\|_{2p}^{2p\gamma} \, \mathsf{E}[f]$$

With $\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d\ge1$ and $p\in(1,p^*)$. There is an explicit $\mathscr{C}=\mathscr{C}[f]$ such that, for any $f\in\mathrm{L}^{2p}(\mathbb{R}^d,(1+|x|^2)\,dx)$ such that $\nabla f\in\mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}[f^{2p}]<\infty$,

$$\delta[f]\ge\mathscr{C}[f]\,\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\left|(p-1)\nabla f+f^p\,\nabla\varphi^{1-p}\right|^2dx$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathsf{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f\in\mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}[f^{2p}]<\infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \ge \frac{\mathscr{C}_\star(A)}{4 + \mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

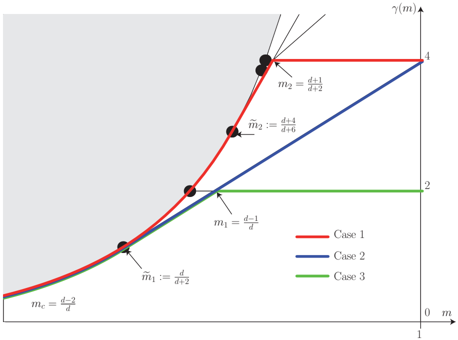
$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

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Thank you for your attention !