

Interpolation inequalities: rigidity results, nonlinear flows and applications

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Scope (1/3): rigidity results

Rigidity results for semilinear elliptic PDEs on manifolds...

Let $(\mathfrak{M}, g)$ be a smooth compact connected Riemannian manifold of dimension $d \geq 2$, no boundary, Δ_g is the Laplace-Beltrami operator the Ricci tensor $\mathfrak{R}$ has good properties (which ones ?)

Let $p \in (2, 2^*)$, with $2^* = \frac{2d}{d-2}$ if $d \geq 3$, $2^* = \infty$ if $d = 2$

For which values of $\lambda > 0$ the equation

$$-\Delta_g v + \lambda v = v^{p-1}$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv \lambda^{\frac{1}{p-2}}$?

A typical *rigidity result* is: there exists $\lambda_0 > 0$ such that $v \equiv \lambda^{\frac{1}{p-2}}$ if $\lambda \in (0, \lambda_0]$

Assumptions ?
Optimal λ_0 ?

Scope (2/3): interpolation inequalities

Still on a smooth compact connected Riemannian manifold $(\mathfrak{M}, g)$
we assume that $\text{vol}_g(\mathfrak{M}) = 1$

For any $p \in (1, 2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$, consider the
interpolation inequality

$$\|\nabla v\|_{L^2(\mathfrak{M})}^2 \geq \frac{\lambda}{p-2} \left[\|v\|_{L^p(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right] \quad \forall v \in H^1(\mathfrak{M})$$

What is the largest possible value of λ ?

- using $u = 1 + \varepsilon \varphi$ as a test function proves that $\lambda \leq \lambda_1$
- the minimum of $v \mapsto \|\nabla v\|_{L^2(\mathfrak{M})}^2 - \frac{\lambda}{p-2} \left[\|v\|_{L^p(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right]$
under the constraint $\|v\|_{L^p(\mathfrak{M})} = 1$ is negative if λ is above the rigidity
threshold
- the threshold case $p = 2$ is the *logarithmic Sobolev inequality*

$$\|\nabla u\|_{L^2(\mathfrak{M})}^2 \geq \lambda \int_{\mathfrak{M}} u^2 \log \left(\frac{u^2}{\|u\|_{L^2(\mathfrak{M})}^2} \right) dv_g \quad \forall u \in H^1(\mathfrak{M})$$

Scope (3/3): flows

We shall consider a flow of porous media / fast diffusion type

$$u_t = u^{2-2\beta} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right), \quad \kappa = 1 + \beta(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt} \|v\|_{L^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 d\nu_g + \frac{\lambda}{p-2} \left[\int_{\mathfrak{M}} u^{2\beta} d\nu_g - \left(\int_{\mathfrak{M}} u^{\beta p} d\nu_g \right)^{2/p} \right]$$

is monotone decaying as long as λ is not too big. Hence, if the limit as $t \rightarrow \infty$ is 0 (convergence to the constants), we know that $\mathcal{F}[u] \geq 0$

Structure ? Link with computations in the rigidity approach

Some references (1/2)

Some references (incomplete) and *goals*

- 1 rigidity results and elliptic PDEs: [Gidas-Spruck 1981], [Bidaut-Véron & Véron 1991], [Licois & Véron 1995]
→ *systematize and clarify the strategy*
- 2 semi-group approach and Γ_2 or *carré du champ* method: [Bakry-Emery 1985], [Bakry & Ledoux 1996], [Bentaleb et al., 1993-2010], [Fontenas 1997], [Brouttelande 2003], [Demange, 2005 & 2008]
→ *emphasize the role of the flow, get various improvements*
→ *get rid of pointwise constraints on the curvature, discuss optimality*
- 3 harmonic analysis, duality and spectral theory: [Lieb 1983], [Beckner 1993]
→ *apply results to get new spectral estimates*

Outline

- 1 The case of the sphere
 - 1 Inequalities on the sphere
 - 1 Flows on the sphere
 - 1 Spectral consequences
 - 1 Improved inequalities
- 2 The case of Riemannian manifolds
 - 1 Flows
 - 1 Spectral consequences
- 3 Inequalities on the line
 - 1 Variational approaches
 - 1 Mass transportation
 - 1 Flows

Joint work with:

Maria J. Esteban, Michal Kowalczyk, Ari Laptev and Michael Loss

These slides can be found at

<http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/>
▷ Lectures

The sphere

- The case of the sphere as a simple example

Inequalities on the sphere

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere:

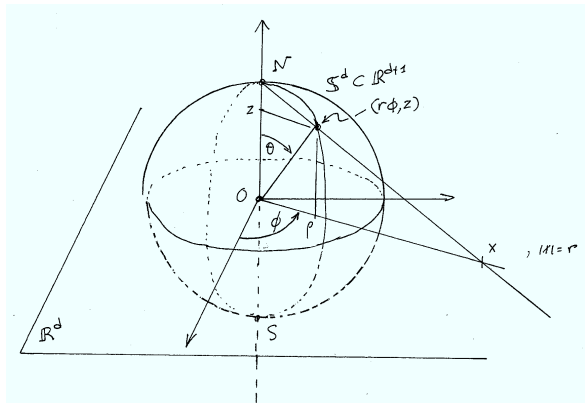
$$\frac{p-2}{d} \int_{\mathbb{S}^d} |\nabla u|^2 d\nu_g + \int_{\mathbb{S}^d} |u|^2 d\nu_g \geq \left(\int_{\mathbb{S}^d} |u|^p d\nu_g \right)^{2/p} \quad \forall u \in H^1(\mathbb{S}^d, d\nu_g)$$

- for any $p \in (2, 2^*]$ with $2^* = \frac{2d}{d-2}$ if $d \geq 3$
- for any $p \in (2, \infty)$ if $d = 2$

Here $d\nu_g$ is the uniform probability measure: $\nu_g(\mathbb{S}^d) = 1$

- 1 is the optimal constant, equality achieved by constants
- $p = 2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho \phi, z)$ onto $\mathbb{R}^d$:
to $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such
that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}, \quad \rho = \frac{2r}{r^2 + 1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\frac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}} v(x)$$

• $p = 2^*$, $S_d = \frac{1}{4} d(d-2) |\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2 dx \geq S_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2d}{d-2}} dx \right]^{\frac{d-2}{d}} \quad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Extended inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 d\nu_g \geq \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p d\nu_g \right)^{2/p} - \int_{\mathbb{S}^d} |u|^2 d\nu_g \right] \quad \forall u \in H^1(\mathbb{S}^d, d\mu)$$

is valid

• for any $p \in (1, 2) \cup (2, \infty)$ if $d = 1, 2$

• for any $p \in (1, 2) \cup (2, 2^*]$ if $d \geq 3$

• Logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 d\nu_g \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 d\nu_g} \right) d\nu_g \quad \forall u \in H^1(\mathbb{S}^d, d\mu)$$

• case $p = 2$

• Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 d\nu_g \geq d \int_{\mathbb{S}^d} |u - \bar{u}|^2 d\nu_g \quad \text{with} \quad \bar{u} := \int_{\mathbb{S}^d} u d\nu_g \quad \forall u \in H^1(\mathbb{S}^d, d\mu)$$

• case $p = 1$

A spectral approach when $p \in (1, 2)$ – 1st step

[Dolbeault-Esteban-Kowalczyk-Loss] adapted from [Beckner] (case of Gaussian measures).

Nelson's hypercontractivity result. Consider the heat equation

$$\frac{\partial f}{\partial t} = \Delta_g f$$

with initial datum $f(t=0, \cdot) = u \in L^{2/p}(\mathbb{S}^d)$, for some $p \in (1, 2]$, and let $F(t) := \|f(t, \cdot)\|_{L^{p(t)}(\mathbb{S}^d)}$. The key computation goes as follows.

$$\frac{F'}{F} = \frac{p'}{p^2 F^p} \left[\int_{\mathbb{S}^d} v^2 \log \left(\frac{v^2}{\int_{\mathbb{S}^d} v^2 d v_g} \right) d v_g + 4 \frac{p-1}{p'} \int_{\mathbb{S}^d} |\nabla v|^2 d v_g \right]$$

with $v := |f|^{p(t)/2}$. With $4 \frac{p-1}{p'} = \frac{2}{d}$ and $t_* > 0$ e such that $p(t_*) = 2$, we have

$$\|f(t_*, \cdot)\|_{L^2(\mathbb{S}^d)} \leq \|u\|_{L^{2/p}(\mathbb{S}^d)} \quad \text{if} \quad \frac{1}{p-1} = e^{2dt_*}$$

A spectral approach when $p \in (1, 2)$ – 2nd step

Spectral decomposition. Let $u = \sum_{k \in \mathbb{N}} u_k$ be a spherical harmonics decomposition, $\lambda_k = k(d + k - 1)$, $a_k = \|u_k\|_{L^2(\mathbb{S}^d)}^2$ so that

$$\|u\|_{L^2(\mathbb{S}^d)}^2 = \sum_{k \in \mathbb{N}} a_k \text{ and } \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 = \sum_{k \in \mathbb{N}} \lambda_k a_k$$

$$\|f(t_*, \cdot)\|_{L^2(\mathbb{S}^d)}^2 = \sum_{k \in \mathbb{N}} a_k e^{-2\lambda_k t_*}$$

$$\begin{aligned} \frac{\|u\|_{L^2(\mathbb{S}^d)}^2 - \|u\|_{L^p(\mathbb{S}^d)}^2}{2-p} &\leq \frac{\|u\|_{L^2(\mathbb{S}^d)}^2 - \|f(t_*, \cdot)\|_{L^2(\mathbb{S}^d)}^2}{2-p} \\ &= \frac{1}{2-p} \sum_{k \in \mathbb{N}^*} \lambda_k a_k \frac{1 - e^{-2\lambda_k t_*}}{\lambda_k} \\ &\leq \frac{1 - e^{-2\lambda_1 t_*}}{(2-p)\lambda_1} \sum_{k \in \mathbb{N}^*} \lambda_k a_k = \frac{1 - e^{-2\lambda_1 t_*}}{(2-p)\lambda_1} \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 \end{aligned}$$

The conclusion easily follows if we notice that $\lambda_1 = d$, and $e^{-2\lambda_1 t_*} = p - 1$ so that $\frac{1 - e^{-2\lambda_1 t_*}}{(2-p)\lambda_1} = \frac{1}{d}$

Optimality: a perturbation argument

- The optimality of the constant can be checked by a Taylor expansion of $u = 1 + \varepsilon v$ at order two in terms of $\varepsilon > 0$, small
- For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or 2 , it is remarkable that

$$Q[u] := \frac{(p-2) \|\nabla u\|_{L^2(\mathbb{S}^d)}^2}{\|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2} \geq \inf_{u \in H^1(\mathbb{S}^d, d\mu)} Q[u] = \frac{1}{d}$$

is achieved by $Q[1 + \varepsilon v]$ as $\varepsilon \rightarrow 0$ and v is an eigenfunction associated with the first nonzero eigenvalue of Δ_g

- $p > 2$ no simple proof based on spectral analysis: [Beckner], an approach based on Lieb's duality, the Funk-Hecke formula and some (non-trivial) computations
- elliptic methods / Γ_2 formalism of Bakry-Emery / flow... they are the same (main contribution) and can be simplified (!) As a side result, you can go beyond these approaches and discuss optimality

Schwarz foliated symmetry and the ultraspherical setting

$(\xi_0, \xi_1, \dots, \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on ξ_d

- Let $d\sigma(\theta) := \frac{(\sin \theta)^{d-1}}{Z_d} d\theta$, $Z_d := \sqrt{\pi} \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall v \in H^1([0, \pi], d\sigma)$

$$\frac{p-2}{d} \int_0^\pi |v'(\theta)|^2 d\sigma + \int_0^\pi |v(\theta)|^2 d\sigma \geq \left(\int_0^\pi |v(\theta)|^p d\sigma \right)^{\frac{2}{p}}$$

- Change of variables $z = \cos \theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d} \int_{-1}^1 |f'|^2 \nu d\nu_d + \int_{-1}^1 |f|^2 d\nu_d \geq \left(\int_{-1}^1 |f|^p d\nu_d \right)^{\frac{2}{p}}$$

where $\nu_d(z) dz = d\nu_d(z) := Z_d^{-1} \nu^{\frac{d}{2}-1} dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1} \nu^{\frac{d}{2}-1} dz$, $\nu(z) := 1 - z^2$, consider the space $L^2((-1, 1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^1 f_1 f_2 d\nu_d, \quad \|f\|_p = \left(\int_{-1}^1 f^p d\nu_d \right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}f := (1 - z^2) f'' - dz f' = \nu f'' + \frac{d}{2} \nu' f'$$

which satisfies $\langle f_1, \mathcal{L} f_2 \rangle = - \int_{-1}^1 f_1' f_2' \nu d\nu_d$

Proposition

Let $p \in [1, 2) \cup (2, 2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L} f \rangle = \int_{-1}^1 |f'|^2 \nu d\nu_d \geq d \frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} \quad \forall f \in H^1([-1, 1], d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Heat flow and the Bakry-Emery method

With $g = f^p$, i.e. $f = g^\alpha$ with $\alpha = 1/p$

$$(\text{Ineq.}) \quad -\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L} g^\alpha \rangle =: \mathcal{I}[g] \geq d \frac{\|g\|_1^{2\alpha} - \|g^{2\alpha}\|_1}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L} g$$

$$\frac{d}{dt} \|g\|_1 = 0, \quad \frac{d}{dt} \|g^{2\alpha}\|_1 = -2(p-2) \langle f, \mathcal{L} f \rangle = 2(p-2) \int_{-1}^1 |f'|^2 \nu \, d\nu_d$$

which finally gives

$$\frac{d}{dt} \mathcal{F}[g(t, \cdot)] = -\frac{d}{p-2} \frac{d}{dt} \|g^{2\alpha}\|_1 = -2d \mathcal{I}[g(t, \cdot)]$$

$$\text{Ineq.} \iff \frac{d}{dt} \mathcal{F}[g(t, \cdot)] \leq -2d \mathcal{F}[g(t, \cdot)] \iff \frac{d}{dt} \mathcal{I}[g(t, \cdot)] \leq -2d \mathcal{I}[g(t, \cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + (p-1) \frac{|f'|^2}{f} \nu$$

$$-\frac{1}{2} \frac{d}{dt} \int_{-1}^1 |f'|^2 \nu d\nu_d = \frac{1}{2} \frac{d}{dt} \langle f, \mathcal{L} f \rangle = \langle \mathcal{L} f, \mathcal{L} f \rangle + (p-1) \langle \frac{|f'|^2}{f} \nu, \mathcal{L} f \rangle$$

$$\begin{aligned} \frac{d}{dt} \mathcal{I}[g(t, \cdot)] + 2 d \mathcal{I}[g(t, \cdot)] &= \frac{d}{dt} \int_{-1}^1 |f'|^2 \nu d\nu_d + 2 d \int_{-1}^1 |f'|^2 \nu d\nu_d \\ &= -2 \int_{-1}^1 \left(|f''|^2 + (p-1) \frac{d}{d+2} \frac{|f'|^4}{f^2} - 2(p-1) \frac{d-1}{d+2} \frac{|f'|^2 f''}{f} \right) \nu^2 d\nu_d \end{aligned}$$

is nonpositive if

$$|f''|^2 + (p-1) \frac{d}{d+2} \frac{|f'|^4}{f^2} - 2(p-1) \frac{d-1}{d+2} \frac{|f'|^2 f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1) \frac{d-1}{d+2} \right]^2 \leq (p-1) \frac{d}{d+2} \iff p \leq \frac{2d^2+1}{(d-1)^2} < \frac{2d}{d-2} = 2^*$$

... up to the critical exponent: a proof on two slides

$$\left[\frac{d}{dz}, \mathcal{L} \right] u = (\mathcal{L} u)' - \mathcal{L} u' = -2z u'' - d u'$$

$$\begin{aligned} \int_{-1}^1 (\mathcal{L} u)^2 d\nu_p &= \int_{-1}^1 |u''|^2 \nu^2 d\nu_p + d \int_{-1}^1 |u'|^2 \nu d\nu_p \\ \int_{-1}^1 (\mathcal{L} u) \frac{|u'|^2}{u} \nu d\nu_p &= \frac{d}{d+2} \int_{-1}^1 \frac{|u'|^4}{u^2} \nu^2 d\nu_p - 2 \frac{d-1}{d+2} \int_{-1}^1 \frac{|u'|^2 u''}{u} \nu^2 d\nu_p \end{aligned}$$

On $(-1, 1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t = u^{2-2\beta} \left(\mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right)$$

If $\kappa = \beta(p-2) + 1$, the L^p norm is conserved

$$\frac{d}{dt} \int_{-1}^1 u^{\beta p} d\nu_p = \beta p (\kappa - \beta(p-2) - 1) \int_{-1}^1 u^{\beta(p-2)} |u'|^2 \nu d\nu_p = 0$$

$$f = u^\beta, \|f'\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \left(\|f\|_{L^2(\mathbb{S}^d)}^2 - \|f\|_{L^p(\mathbb{S}^d)}^2 \right) \geq 0 ?$$

$$\begin{aligned} \mathcal{A} &:= -\frac{1}{2\beta^2} \frac{d}{dt} \int_{-1}^1 \left(|(u^\beta)'|^2 \nu + \frac{d}{p-2} (u^{2\beta} - \bar{u}^{2\beta}) \right) d\nu_p \\ &= \int_{-1}^1 \left(\mathcal{L} u + (\beta-1) \frac{|u'|^2}{u} \nu \right) \left(\mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right) d\nu_p \\ &\quad + \frac{d}{p-2} \frac{\kappa-1}{\beta} \int_{-1}^1 |u'|^2 \nu d\nu_p \\ &= \int_{-1}^1 |u''|^2 \nu^2 d\nu_p - 2 \frac{d-1}{d+2} (\kappa + \beta - 1) \int_{-1}^1 u'' \frac{|u'|^2}{u} \nu^2 d\nu_p \\ &\quad + \left[\kappa(\beta-1) + \frac{d}{d+2} (\kappa + \beta - 1) \right] \int_{-1}^1 \frac{|u'|^4}{u^2} \nu^2 d\nu_p \\ &= \int_{-1}^1 \left| u'' - \frac{p+2}{6-p} \frac{|u'|^2}{u} \right|^2 \nu^2 d\nu_p \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p} \end{aligned}$$

$$\mathcal{A} \text{ is nonnegative for some } \beta \text{ if } \frac{8d^2}{(d+2)^2} (p-1)(2^*-p) \leq 0$$

the rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta} \left(\mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right)$

$$- \mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p-2} u = \frac{\lambda}{p-2} u^\kappa$$

Multiply by $\mathcal{L} u$ and integrate

$$\dots \int_{-1}^1 \mathcal{L} u u^\kappa d\nu_p = - \kappa \int_{-1}^1 u^\kappa \frac{|u'|^2}{u} d\nu_p$$

Multiply by $\kappa \frac{|u'|^2}{u}$ and integrate

$$\dots = + \kappa \int_{-1}^1 u^\kappa \frac{|u'|^2}{u} d\nu_p$$

The two terms cancel and we are left only with the two-homogenous terms.

Spectral consequences

- 🟢 A quantitative deviation with respect to the semi-classical regime

Some references (2/2)

Consider the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$ and denote by $(\lambda_k)_{k \geq 1}$ its eigenvalues

• Euclidean case [Keller, 1961]

$$|\lambda_1|^\gamma \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} V_+^{\gamma + \frac{d}{2}}$$

[Lieb-Thirring, 1976]

$$\sum_{k \geq 1} |\lambda_k|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma + \frac{d}{2}}$$

$\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

• Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rosenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

An interpolation inequality (I)

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (2, 2^*)$. Then there exists a concave increasing function $\mu : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ with the following properties

$$\mu(\alpha) = \alpha \quad \forall \alpha \in \left[0, \frac{d}{q-2}\right] \quad \text{and} \quad \mu(\alpha) < \alpha \quad \forall \alpha \in \left(\frac{d}{q-2}, +\infty\right)$$

$$\mu(\alpha) = \mu_{\text{asympt}}(\alpha) (1 + o(1)) \quad \text{as} \quad \alpha \rightarrow +\infty, \quad \mu_{\text{asympt}}(\alpha) := \frac{K_{q,d}}{\kappa_{q,d}} \alpha^{1-\vartheta}$$

such that

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \alpha \|u\|_{L^2(\mathbb{S}^d)}^2 \geq \mu(\alpha) \|u\|_{L^q(\mathbb{S}^d)}^2 \quad \forall u \in H^1(\mathbb{S}^d)$$

If $d \geq 3$ and $q = 2^*$, the inequality holds with $\mu(\alpha) = \min \{\alpha, \alpha_*\}$,
 $\alpha_* := \frac{1}{4} d(d-2)$

• $\mu_{\text{asympt}}(\alpha) := \frac{K_{q,d}}{\kappa_{q,d}} \alpha^{1-\vartheta}$, $\vartheta := d \frac{q-2}{2q}$ corresponds to the *semi-classical regime* and $K_{q,d}$ is the optimal constant in the *Euclidean* Gagliardo-Nirenberg-Sobolev inequality

$$K_{q,d} \|v\|_{L^q(\mathbb{R}^d)}^2 \leq \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in H^1(\mathbb{R}^d)$$

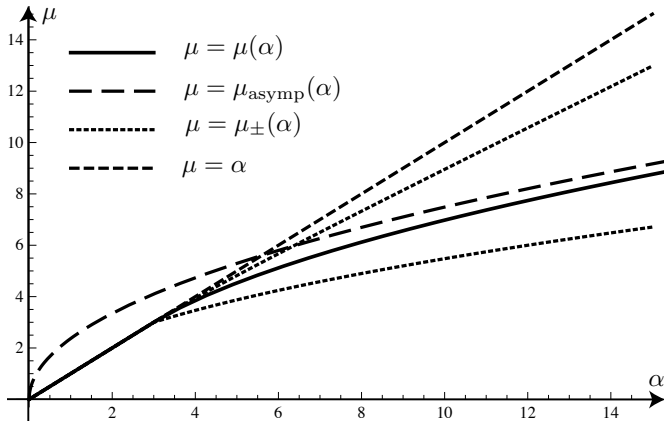
• Let φ be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

$$-\Delta \varphi = d \varphi$$

Consider $u = 1 + \varepsilon \varphi$ as $\varepsilon \rightarrow 0$ Taylor expand $\mathcal{Q}_\alpha$ around $u = 1$

$$\mu(\alpha) \leq \mathcal{Q}_\alpha[1 + \varepsilon \varphi] = \alpha + [d + \alpha(2 - q)] \varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2 dv_g + o(\varepsilon^2)$$

By taking ε small enough, we get $\mu(\alpha) < \alpha$ for all $\alpha > d/(q-2)$
Optimizing on the value of $\varepsilon > 0$ (not necessarily small) provides an interesting test function...



Consider the Schrödinger operator $-\Delta - V$ and the energy

$$\begin{aligned}\mathcal{E}[u] &:= \int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V |u|^2 \\ &\geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu \|u\|_{L^q(\mathbb{S}^d)}^2 \geq -\alpha(\mu) \|u\|_{L^2(\mathbb{S}^d)}^2 \quad \text{if } \mu = \|V_+\|_{L^p(\mathbb{S}^d)}\end{aligned}$$

Theorem (Dolbeault-Esteban-Laptev)

Let $d \geq 1$, $p \in (\max\{1, d/2\}, +\infty)$. Then there exists a convex increasing function α s.t. $\alpha(\mu) = \mu$ if $\mu \in [0, \frac{d}{2}(p-1)]$ and $\alpha(\mu) > \mu$ if $\mu \in (\frac{d}{2}(p-1), +\infty)$

$$|\lambda_1(-\Delta - V)| \leq \alpha(\|V\|_{L^p(\mathbb{S}^d)}) \quad \forall V \in L^p(\mathbb{S}^d)$$

For large values of μ , we have $\alpha(\mu)^{p-\frac{d}{2}} = L_{p-\frac{d}{2}, d}^1(\kappa_{q,d} \mu)^p (1 + o(1))$
and the above estimate is optimal

If $p = d/2$ and $d \geq 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \alpha_*]$

A Keller-Lieb-Thirring inequality

Corollary (Dolbeault-Esteban-Laptev)

Let $d \geq 1, \gamma = p - d/2$

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim L_{\gamma,d}^1 \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}} \quad \text{as } \mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \rightarrow \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ or $\gamma = 1/2$ and $d = 1$

However, if $\mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4} d (2\gamma + d - 2)$, then we have

$$|\lambda_1(-\Delta - V)|^{\gamma + \frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ and this estimate is optimal

$L_{\gamma,d}^1$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} \phi_+^{\gamma + \frac{d}{2}} dx$$

Another interpolation inequality (II)

Let $d \geq 1$ and $\gamma > d/2$ and assume that $L^1_{-\gamma,d}$ is the optimal constant in

$$\lambda_1(-\Delta + \phi)^{-\gamma} \leq L^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2}-\gamma} dx$$

$$q = 2 \frac{2\gamma - d}{2\gamma - d + 2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}$$

Theorem (Dolbeault-Esteban-Laptev)

$$(\lambda_1(-\Delta + W))^{-\gamma} \lesssim L^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma} \quad \text{as} \quad \beta = \|W^{-1}\|_{L^{\gamma-\frac{d}{2}}(\mathbb{S}^d)}^{-1} \rightarrow \infty$$

However, if $\gamma \geq \frac{d}{2} + 1$ and $\beta = \|W^{-1}\|_{L^{\gamma-\frac{d}{2}}(\mathbb{S}^d)}^{-1} \leq \frac{1}{4} d (2\gamma - d + 2)$

$$(\lambda_1(-\Delta + W))^{\frac{d}{2}-\gamma} \leq \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}$$

and this estimate is optimal

$K_{q,d}^*$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$K_{q,d}^* \|v\|_{L^2(\mathbb{R}^d)}^2 \leq \|\nabla v\|_{L^2(\mathbb{R}^d)}^2 + \|v\|_{L^q(\mathbb{R}^d)}^2 \quad \forall v \in H^1(\mathbb{R}^d)$$

and $\mathcal{L}_{-\gamma,d}^1 := \left(K_{q,d}^*\right)^{-\gamma}$ with $q = 2 \frac{2\gamma-d}{2\gamma-d+2}$, $\delta := \frac{2q}{2d-q(d-2)}$

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (0, 2)$ and $d \geq 1$. There exists a concave increasing function ν

$$\nu(\beta) \leq \beta \quad \forall \beta > 0 \quad \text{and} \quad \nu(\beta) < \beta \quad \forall \beta \in \left(\frac{d}{2-q}, +\infty\right)$$

$$\nu(\beta) = \beta \quad \forall \beta \in \left[0, \frac{d}{2-q}\right] \quad \text{if} \quad q \in [1, 2)$$

$$\nu(\beta) = K_{q,d}^* (\kappa_{q,d} \beta)^\delta (1 + o(1)) \quad \text{as} \quad \beta \rightarrow +\infty$$

such that

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 + \beta \|u\|_{L^q(\mathbb{S}^d)}^2 \geq \nu(\beta) \|u\|_{L^2(\mathbb{S}^d)}^2 \quad \forall u \in H^1(\mathbb{S}^d)$$

The threshold case: $q = 2$

Lemma (Dolbeault-Esteban-Laptev)

Let $p > \max\{1, d/2\}$. There exists a concave nondecreasing function ξ

$$\xi(\alpha) = \alpha \quad \forall \alpha \in (0, \alpha_0) \quad \text{and} \quad \xi(\alpha) < \alpha \quad \forall \alpha > \alpha_0$$

for some $\alpha_0 \in [\frac{d}{2}(p-1), \frac{d}{2}p]$, and $\xi(\alpha) \sim \alpha^{1-\frac{d}{2p}}$ as $\alpha \rightarrow +\infty$

such that, for any $u \in H^1(\mathbb{S}^d)$ with $\|u\|_{L^2(\mathbb{S}^d)} = 1$

$$\int_{\mathbb{S}^d} |u|^2 \log |u|^2 \, d\nu_g + p \log \left(\frac{\xi(\alpha)}{\alpha} \right) \leq p \log \left(1 + \frac{1}{\alpha} \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 \right)$$

Corollary (Dolbeault-Esteban-Laptev)

$$e^{-\lambda_1(-\Delta-W)/\alpha} \leq \frac{\alpha}{\xi(\alpha)} \left(\int_{\mathbb{S}^d} e^{-pW/\alpha} \, d\nu_g \right)^{1/p}$$

Improvements of the inequalities (subcritical range)

as long as the exponent is either in the range $(1, 2)$ or in the range $(2, 2^*)$, one can establish *improved inequalities*

[Dolbeault-Esteban-Kowalczyk-Loss]

What does “improvement” mean ?

An *improved* inequality is

$$d \|u\|_{L^2(\mathbb{S}^d)}^2 \Phi\left(\frac{e}{\|u\|_{L^2(\mathbb{S}^d)}^2}\right) \leq i \quad \forall u \in H^1(\mathbb{S}^d)$$

for some function Φ such that $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi' > 0$ and $\Phi(s) > s$ for any s . With $\Psi(s) := s - \Phi^{-1}(s)$

$$i - de \geq d \|u\|_{L^2(\mathbb{S}^d)}^2 (\Psi \circ \Phi)\left(\frac{e}{\|u\|_{L^2(\mathbb{S}^d)}^2}\right) \quad \forall u \in H^1(\mathbb{S}^d)$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\begin{aligned} & \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - \frac{d}{p-2} \left[\|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right] \\ & \geq d \|u\|_{L^2(\mathbb{S}^d)}^2 (\Psi \circ \Phi)\left(C \frac{\|u\|_{L^{\frac{p}{2}}(\mathbb{S}^d)}^{2(1-r)}}{\|u\|_{L^2(\mathbb{S}^d)}^2} \|u^r - \bar{u}^r\|_{L^q(\mathbb{S}^d)}^2\right) \quad \forall u \in H^1(\mathbb{S}^d) \end{aligned}$$

$s(p) := \max\{2, p\}$ and $p \in (1, 2)$: $q(p) := 2/p$, $r(p) := p$; $p \in (2, 4)$:
 $q = p/2$, $r = 2$; $p \geq 4$: $q = p/(p-2)$, $r = p-2$

Linear flow: improved Bakry-Emery method

$$w_t = \mathcal{L} w + \kappa \frac{|w'|^2}{w}$$

With $2^\# := \frac{2d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2} \right)^2 (p-1)(2^\# - p) \quad \text{if } d > 1, \quad \gamma_1 := \frac{p-1}{3} \quad \text{if } d = 1$$

If $p \in [1, 2) \cup (2, 2^\#]$ and w is a solution to (2), then

$$\frac{d}{dt} (i - d e) \leq -\gamma_1 \int_{-1}^1 \frac{|w'|^4}{w^2} d\nu_d \leq -\gamma_1 \frac{|e'|^2}{1 - (p-2)e}$$

Recalling that $e' = -i$, we get a differential inequality

$$e'' + d e' \geq \gamma_1 \frac{|e'|^2}{1 - (p-2)e}$$

which after integration implies an inequality of the form

$$d \Phi(e(0)) \leq i(0)$$

$$w_t = w^{2-2\beta} \left(\mathcal{L} w + \kappa \frac{|w'|^2}{w} \right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta(p-2) + 1$, $\frac{d}{dt} \int_{-1}^1 w^{\beta p} d\nu_d = 0$

$$-\frac{1}{2\beta^2} \frac{d}{dt} \int_{-1}^1 \left(|(w^\beta)'|^2 \nu + \frac{d}{p-2} (w^{2\beta} - \bar{w}^{2\beta}) \right) d\nu_d \geq \gamma \int_{-1}^1 \frac{|w'|^4}{w^2} \nu^2 d\nu_d$$

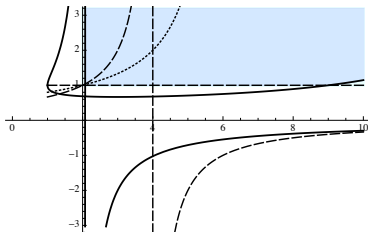
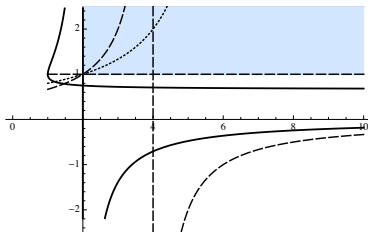
Lemma

For all $w \in H^1((-1, 1), d\nu_d)$, such that $\int_{-1}^1 w^{\beta p} d\nu_d = 1$

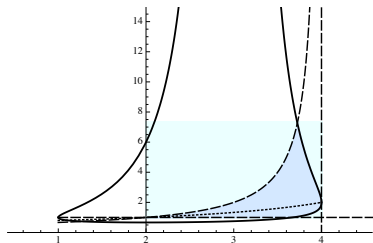
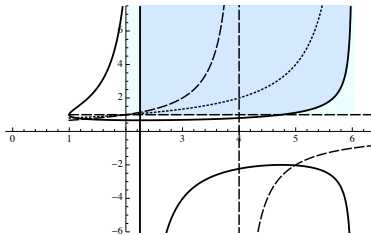
$$\int_{-1}^1 \frac{|w'|^4}{w^2} \nu^2 d\nu_d \geq \frac{1}{\beta^2} \frac{\int_{-1}^1 |(w^\beta)'|^2 \nu d\nu_d \int_{-1}^1 |w'|^2 \nu d\nu_d}{\left(\int_{-1}^1 w^{2\beta} d\nu_d \right)^\delta}$$

.... but there are conditions on β

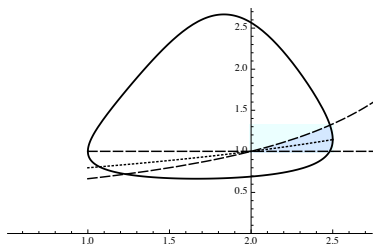
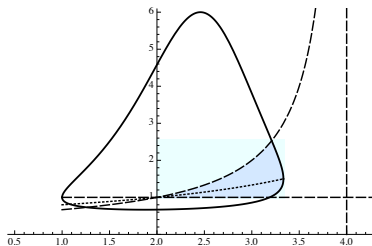
Admissible (p, β) for $d = 1, 2$



Admissible (p, β) for $d = 3, 4$



Admissible (p, β) for $d = 5, 10$



Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established
- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold dimension d , no boundary, Δ_g is the Laplace-Beltrami operator $\text{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi, \xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \quad \text{where} \quad \theta = \frac{(d - 1)^2 (p - 1)}{d(d + 2) + p - 1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\Delta_g v + \frac{\lambda}{p - 2} (v - v^{p-1}) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1, 2) \cup (2, 2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in H^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1 - \theta) (\Delta_g u)^2 + \frac{\theta d}{d-1} \Re(\nabla u, \nabla u) \right] d\nu_g}{\int_{\mathfrak{M}} |\nabla u|^2 d\nu_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$: $u \equiv 1$

$\lim_{\rho \rightarrow 1_+} \theta(\rho) = 0 \implies \lim_{\rho \rightarrow 1_+} \lambda_\star(\rho) = \lambda_1$ if ρ is bounded
 $\lambda_\star = \lambda_1 = d\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1 - \theta) \lambda_1 + \theta \frac{d\rho}{d-1} \leq \lambda_\star \leq \lambda_1$$

Riemannian manifolds: second improvement

$H_g u$ denotes Hessian of u and $\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1}$

$$Q_g u := H_g u - \frac{g}{d} \Delta_g u - \frac{(d-1)(p-1)}{\theta(d+3-p)} \left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]$$

$$\Lambda_\star := \inf_{u \in H^2(\mathfrak{M}) \setminus \{0\}} \frac{(1-\theta) \int_{\mathfrak{M}} (\Delta_g u)^2 d\nu_g + \frac{\theta d}{d-1} \int_{\mathfrak{M}} [\|Q_g u\|^2 + \Re(\nabla u, \nabla u)]}{\int_{\mathfrak{M}} |\nabla u|^2 d\nu_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1, 2) \cup (2, 2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0, \Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1, 2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|_{L^2(\mathfrak{M})}^2 \geq \frac{\lambda}{p-2} \left[\|v\|_{L^p(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right] \quad \forall v \in H^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon \varphi$ as a test function where φ we get $\lambda \leq \lambda_1$

A minimum of

$$v \mapsto \|\nabla v\|_{L^2(\mathfrak{M})}^2 - \frac{\lambda}{p-2} \left[\|v\|_{L^p(\mathfrak{M})}^2 - \|v\|_{L^2(\mathfrak{M})}^2 \right]$$

under the constraint $\|v\|_{L^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tools the flow

$$u_t = u^{2-2\beta} \left(\Delta_g u + \kappa \frac{|\nabla u|^2}{u} \right), \quad \kappa = 1 + \beta(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt} \|v\|_{L^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2 dv_g + \frac{\lambda}{p-2} \left[\int_{\mathfrak{M}} u^{2\beta} dv_g - \left(\int_{\mathfrak{M}} u^{\beta p} dv_g \right)^{2/p} \right]$$

is monotone decaying

🟢 J. Demange, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611. Also see C. Villani, *Optimal Transport, Old and New*

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 d\nu_g = \frac{d}{d-1} \int_{\mathfrak{M}} \|L_g u\|^2 d\nu_g + \frac{d}{d-1} \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) d\nu_g$$

Based on the Bochner-Lichnerovitz-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|H_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \Re(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\begin{aligned} \int_{\mathfrak{M}} \Delta_g u \frac{|\nabla u|^2}{u} d\nu_g \\ = \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} d\nu_g - \frac{2d}{d+2} \int_{\mathfrak{M}} [L_g u] \cdot \left[\frac{\nabla u \otimes \nabla u}{u} \right] d\nu_g \end{aligned}$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 d\nu_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 d\nu_g \quad \forall u \in H^2(\mathfrak{M})$$

and λ_1 is the optimal constant in the above inequality

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta (\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{|\nabla u|^2}{u} + \kappa (\beta - 1) \frac{|\nabla u|^4}{u^2} \right] d\nu_g$$

Lemma

$$\frac{1}{2\beta^2} \frac{d}{dt} \mathcal{F}[u] = - (1 - \theta) \int_{\mathfrak{M}} (\Delta_g u)^2 d\nu_g - \mathcal{G}[u] + \lambda \int_{\mathfrak{M}} |\nabla u|^2 d\nu_g$$

$$Q_g^\theta u := L_g u - \frac{1}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta d}{d-1} \left[\int_{\mathfrak{M}} \|Q_g^\theta u\|^2 d\nu_g + \int_{\mathfrak{M}} \Re(\nabla u, \nabla u) d\nu_g \right] - \mu \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} d\nu_g$$

$$\text{with } \mu := \frac{1}{\theta} \left(\frac{d-1}{d+2} \right)^2 (\kappa + \beta - 1)^2 - \kappa (\beta - 1) - (\kappa + \beta - 1) \frac{d}{d+2}$$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall p \in (1, 2^*)$$

where $\beta_{\pm} := \frac{b \pm \sqrt{b^2 - a}}{2a}$ with $a = 2 - p + \left[\frac{(d-1)(p-1)}{d+2} \right]^2$ and $b = \frac{d+3-p}{d+2}$

Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1, 2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\beta^2} \frac{d}{dt} \mathcal{F}[u] \leq (\lambda - \Lambda_*) \int_{\mathfrak{M}} |\nabla u|^2 dv_g$$

The line

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{L^p(\mathbb{R})} \leq C_{\text{GN}}(p) \|f'\|_{L^2(\mathbb{R})}^\theta \|f\|_{L^2(\mathbb{R})}^{1-\theta} \quad \text{if } p \in (2, \infty)$$

$$\|f\|_{L^2(\mathbb{R})} \leq C_{\text{GN}}(p) \|f'\|_{L^2(\mathbb{R})}^\eta \|f\|_{L^p(\mathbb{R})}^{1-\eta} \quad \text{if } p \in (1, 2)$$

with $\theta = \frac{p-2}{2p}$ and $\eta = \frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p \rightarrow 2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2 \log \left(\frac{u^2}{\|u\|_{L^2(\mathbb{R})}^2} \right) dx \leq \frac{1}{2} \|u\|_{L^2(\mathbb{R})}^2 \log \left(\frac{2}{\pi e} \frac{\|u'\|_{L^2(\mathbb{R})}^2}{\|u\|_{L^2(\mathbb{R})}^2} \right)$$

If $p > 2$, $u_*(x) = (\cosh x)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2 u'' + 4u - 2p|u|^{p-2}u = 0$$

If $p \in (1, 2)$ consider $u_*(x) = (\cos x)^{\frac{2}{2-p}}$, $x \in (-\pi/2, \pi/2)$

Mass transportation

Theorem (Dolbeault-Esteban-Laptev-Loss)

If $p \in (2, \infty)$, we have

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{p+2}{3p-2}} dy}{\left(\int_{\mathbb{R}} G |y|^2 dy\right)^{\frac{p-2}{3p-2}} \left(\int_{\mathbb{R}} G dy\right)^{\frac{4}{3p-2}}} = c_p \inf_f \frac{\|f'\|_{L^2(\mathbb{R})}^{\frac{2(p-2)}{3p-2}} \|f\|_{L^2(\mathbb{R})}^{\frac{2(p+2)}{3p-2}}}{\|f\|_{L^p(\mathbb{R})}^{\frac{4p}{3p-2}}}$$

and if $p \in (1, 2)$, we obtain

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{2}{4-p}} dy}{\left(\int_{\mathbb{R}} G |y|^2 dy\right)^{\frac{2-p}{2(4-p)}} \left(\int_{\mathbb{R}} G dy\right)^{\frac{p+2}{2(4-p)}}} = c_p \inf_f \frac{\|f'\|_{L^2(\mathbb{R})}^{\frac{2-p}{4-p}} \|f\|_{L^p(\mathbb{R})}^{\frac{2p}{4-p}}}{\|f\|_{L^2(\mathbb{R})}^{\frac{p+2}{4-p}}}$$

for some explicit numerical constant c_p

Flow

Let us define on $H^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|_{L^2(\mathbb{R})}^2 + \frac{4}{(p-2)^2} \|v\|_{L^2(\mathbb{R})}^2 - C \|v\|_{L^p(\mathbb{R})}^2 \quad \text{s.t. } \mathcal{F}[u_*] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}} \left[v'' + \frac{2p}{p-2} z v' + \frac{p}{2} \frac{|v'|^2}{v} + \frac{2}{p-2} v \right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2, \infty)$. Then

$$\frac{d}{dt} \mathcal{F}[v(t)] \leq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \mathcal{F}[v(t)] = 0$$

$$\frac{d}{dt} \mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_*(x - x_0)$$

Similar result for $p \in (1, 2)$

The inequality ($p > 2$) and the ultraspherical operator

🟢 *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}} |v'|^2 dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 dx \geq C \left(\int_{\mathbb{R}} |v|^p dx \right)^{\frac{2}{p}}$$

With

$$z(x) = \tanh x, \quad v_\star = (1 - z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x) f(z(x))$$

equality is achieved for $f = 1$ and, if we let $\nu(z) := 1 - z^2$, then

$$\int_{-1}^1 |f'|^2 \nu \, d\nu_p + \frac{2p}{(p-2)^2} \int_{-1}^1 |f|^2 \, d\nu_p \geq \frac{2p}{(p-2)^2} \left(\int_{-1}^1 |f|^p \, d\nu_p \right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z) := \frac{1}{\zeta_p^2} \nu^{\frac{2}{p-2}} dz$

$$d = \frac{2p}{p-2} \iff p = \frac{2d}{d-2}$$

The change of variables amounts to the stereographic projection composed with the Emden-Fowler transformation

A summary

- the sphere: the flow tells us what to do, and provides a simple proof (*choice of the exponents*) once the problem is reduced to the ultraspherical setting
- the spectral point of view on the inequality: how to measure the deviation with respect to the *semi-classical* estimates, a nice example of bifurcation (and *symmetry breaking*)
- Riemannian manifolds*: no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The flow explores the energy landscape... and generically shows the non-optimality of the improved criterion
- the flow is a nice way of exploring an energy space. *Rigidity* result tell you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal

<http://www.ceremade.dauphine.fr/~dolbeaul>

▷ Preprints (or arxiv, or HAL)

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Thank you for your attention !