Travelling fronts in stochastic Stokes' drifts and Brownian ratchets: homogenized functional inequalities and large time behaviour of the solutions

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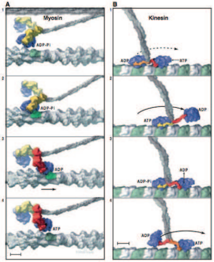
Outline

- Introduction to ratchets models
- Homogenized functional inequalities
- Traveling and tilted ratchets: speed of the center of mass
- Rescaling and formal asymptotic expansion: effective diffusion
- Results
- Physical interpretation
- Concluding remarks

Introduction to ratchets models

- Molecular motors: how to produce motion at 1μm scale ? "life at low Rayleigh numbers" [Purcell], modelling in biology [Vale-Milligan]



- Physics of brownian ratchets [Reimann]: $f_t = \Delta f + \nabla \cdot \big(f \, \nabla \psi(t,x)\big)$

Ratchet models: ψ is t-periodic

$$f_t = \Delta f + \nabla \cdot \big(f \, \nabla \psi(t,x) \big)$$

Flashing ratchet: a model case

$$f_t = \varepsilon(t) \, \Delta f + (1 - \varepsilon(t) \,) \nabla \cdot \big(f \, \nabla \psi() \big)$$

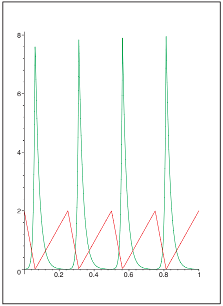


Figure: Potential and Gibbs state: $e^{-\psi}$ when $\varepsilon(t) \equiv 1/2$

Flashing ratchet: mass transport

Assume now that $t \mapsto \varepsilon(t)$ is 1-periodic

$$\varepsilon \equiv 1 \text{ if } t \in [0, 1/2) \text{ and } \varepsilon \equiv 0 \text{ if } t \in [1/2, 1)$$

$$f_t = \varepsilon(t)\,\Delta f + (1 - \varepsilon(t)\,)\nabla \cdot \big(f\,\nabla\psi(x)\big)$$

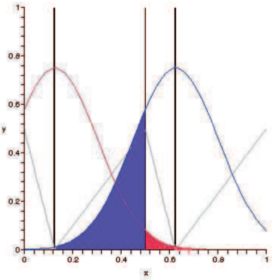


Figure: The mechanism of transport

[Chipot-Hastings-Kinderlehrer, Kinderlehrer-Kowlczyk, JD-Kinderlehrer-Kowlczyk]

1d case, no flux boundary conditions: $\varepsilon(t)\, f_x + f\, \psi(t,x)_x = 0$ on the boundary

$$f_t = \varepsilon(t)\,\Delta f + \nabla \cdot \big(f\, \nabla\psi(t,x)\big)$$

The solution converges to a unique time-periodic solution

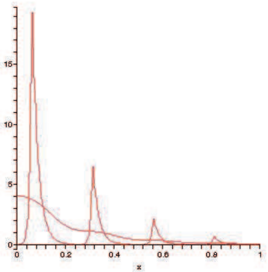


Figure: The periodic solution at $t = 1/2$ and $t = 1$

Mass has been transported (to the left) of the interval

Consider a solution of

$$g_t = \Delta g + \nabla \cdot \big(g \, \nabla \psi(t,x)\big) \quad x \in \mathbb{T}^d \,, \; t > 0$$

with initial datum $g_0 \in L^1_+(\mathbb{T}^d)$, $\|g_0\|_{L^1(\mathbb{T}^d)} = 1$ and assume that ψ is doubly periodic: $\psi(t+T,x) = \psi(t,x)$

- Conservation of mass
- Contraction in relative entropy [Bartier-JD-Illner-Kowalczyk]

$$\frac{d}{dt} \int_{\mathbb{T}^d} \varphi\left(\frac{g_1}{g_2}\right) g_2 \, dx = -\int_{\mathbb{T}^d} \varphi''\left(\frac{g_1}{g_2}\right) \left|\nabla\left(\frac{g_1}{g_2}\right)\right|^2 g_2 \, dx$$

- Existence of a (unique) doubly periodic solution: $g_2 = e^{-\psi} / \int_{\mathbb{T}^d} e^{-\psi} \, dx$, $\varphi(s) = s \log s$

$$\frac{d}{dt} \int_{\mathbb{T}^d} g_1 \log\left(\frac{g_1}{g_2}\right) dx \le -C_{\mathrm{LS}} \int_{\mathbb{T}^d} g_1 \log\left(\frac{g_1}{g_2}\right) dx + \left\| \frac{(g_2)_t}{g_2} \right\|_{L^\infty}$$

The doubly periodic case and entropy methods: results

- Existence of a (unique) doubly periodic solution: entropy estimate + fixed-point methods: existence of a doubly periodic solution
- Contraction: the doubly periodic solution attracts all other solutions
- If there is a logarithmic Sobolev inequality, then there is an exponential convergence in L^1 (Csiszár-Kullback inequality)

[JD-Kinderlehrer-Kowlczyk]
[Bartier-JD-Illner-Kowalczyk]

Good reasons to use entropy methods

- Easy estimates (compared to L^∞ or L^2 / Fourier estimates)
- Go well with mass conservation; gradient flow structure (Wasserstein distance [Jordan-Kinderlehrer-Otto])
- Robust (1): allows for not too smooth potentials
- Robust (2): easy to generalize to nonlinear models
- Robust (3): ok even if the asymptotic state is not known
- Give nice results in messy problems (with various time and length scales): "strong" two-scale convergence

But require a detailed knowledge of tricky functional inequalities

Stochastic Stokes' drift model

$$f_t = f_{xx} + \big(\psi'(x - \omega\, t)\, f\big)_x \tag{1}$$

$\psi'(x - \omega\, t)$ is a traveling potential moving at constant speed ω, ψ is 1−periodic: $\psi(x+1) = \psi(x)$
conservation of mass: $\int_{\mathbb{R}} f(t,x)\, dx = 1$ for any $t \geq 0$

- Position of the center of mass: $\bar{x}(t) := \int_{\mathbb{R}} x\, f(t,x)\, dx$
  There exists a *drift velocity* or *ballistic velocity* c_ω such that

$$\Big|\frac{d}{dt}\bar{x}(t) - c_\omega\Big| = O\big(e^{-t/\gamma}\big) \quad \text{as} \quad t \to \infty$$

- A diffusive traveling front appears: effective diffusion coefficient ? asymptotic profile ?
  Ansatz: equation in self-similar variables

$$f(t,x) = \frac{1}{R(t)}\, u\left(\log R(t), \frac{x - c_\omega\, t}{R(t)}\right) \quad \text{with} \quad R(t) := \sqrt{1+2t}$$

... unlimited motion of brownian ratchets

- position of the center of mass ?
- profile of the solutions for large times ?
- rate of convergence towards the asymptotic profile ?

Tools:

- homogenized functional inequalities (logarithmic Sobolev inequalities)
- cell problem
- time-dependent asymptotic expansions (time-dependent homogenization): effective diffusion
- logarithmic Sobolev inequalities control the convergence

Physics: efficiency

Homogenized functional inequalities

Consider the Fokker-Planck equation with a drift corresponding to a harmonic potential modified by a periodic perturbation in the limit $\varepsilon \to 0_+$

$$u_t^\varepsilon = \Delta u^\varepsilon + \nabla \cdot \left[x\, u^\varepsilon + \frac{1}{\varepsilon} \nabla\phi \left(\frac{x}{\varepsilon}\right)\, u^\varepsilon \right] \quad x \in \mathbb{R}^d \;\; t > 0 \tag{2}$$

It has a unique stationary solution with mass 1

$$u_\infty^\varepsilon(x) := \, Z_\varepsilon^{-1}\, e^{-\frac{1}{2}|x|^2 - \phi(x/\varepsilon)}$$

Homogenization and turbulence, weak 2-scale convergence: [Goudon-Poupaud]
How to study the convergence of $u(t,\cdot)$ to u_∞ ? Poincaré inequality / logarithmic Sobolev inequalities / *entropy methods*
[Bakry-Emery, Arnold-Markowich-Toscani-Unterreiter]

$$d\mu_0 := Z_0^{-1}\, e^{-|x|^2/2}\, dx\,, \quad d\mu_\varepsilon := Z_\varepsilon^{-1}\, e^{-\phi(x/\varepsilon)}\, d\mu_0(x) = u_\infty^\varepsilon(x)\, dx$$

Generalized entropies, generalized Fisher information

For any $p \in (1,2]$, consider for $v = u/u^\varepsilon_\infty$

$$\mathsf{E}^{(p)}_\varepsilon[u] := \frac{1}{p-1}\int_{\mathbb{R}^d}\left[v^p - 1 - p\,(v-1)\right]\,d\mu_\varepsilon$$

$$\mathsf{E}^{(1)}_\varepsilon[u] := \int_{\mathbb{R}^d} v\,\log v\,d\mu_\varepsilon$$

If u is a solution of (2), v solves the Ornstein-Uhlenbeck equation

$$v^\varepsilon_t = \Delta v^\varepsilon - \left[x + \frac{1}{\varepsilon}\,\nabla\phi\left(\frac{x}{\varepsilon}\right)\right]\cdot\nabla v$$

Generalized Fisher information: $\mathsf{I}^{(p)}_\varepsilon[u] := p\int_{\mathbb{R}^d} v^{p-2}\,|\nabla v|^2\,d\mu_\varepsilon$

$$\frac{d}{dt}\,\mathsf{E}^{(p)}_\varepsilon[u^\varepsilon(t,\cdot)] = -\,\mathsf{I}^{(p)}_\varepsilon[u^\varepsilon(t,\cdot)]$$

The strategy of entropy methods

- $\frac{d}{dt} \mathsf{E}_\varepsilon^{(p)}[u^\varepsilon(t,\cdot)] = -\,\mathsf{I}_\varepsilon^{(p)}[u^\varepsilon(t,\cdot)]$
- Functional inequality: for some $\mathcal{C}_\varepsilon^{(p)} > 0$

$$\frac{4}{p}\,\mathcal{C}_\varepsilon^{(p)}\,\mathsf{E}_\varepsilon^{(p)}[u] \le \mathsf{I}_\varepsilon^{(p)}[u]$$

- decay of the entropy

$$\mathsf{E}_\varepsilon^{(p)}[u^\varepsilon(t,\cdot)] \le \mathsf{E}_\varepsilon^{(p)}[u^\varepsilon(0,\cdot)]\, e^{-\frac{4}{p}\,\mathcal{C}_\varepsilon^{(p)}\,t} \quad t \ge 0$$

- Generalized Csiszár-Kullback inequalities

$$\|u^\varepsilon(t,\cdot) - u_\infty^\varepsilon\|_{L^p(\mathbb{R}^d,u_\infty^\varepsilon dx)} \le C\; e^{-\frac{2}{p}\,\mathcal{C}_\varepsilon^{(p)}\,t}$$

One step further: Bakry-Emery method

Optimal constants

$u^p \mapsto u^2$, $u \mapsto u^{2/p}$

Logarithmic Sobolev inequality

$$\mathcal{C}_\varepsilon^{(1)} := \inf_{\substack{\nabla u \neq 0 \; d\mu_\varepsilon \text{ a.e.} \\ u \in H^1(d\mu_\varepsilon)}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 \, d\mu_\varepsilon}{\int_{\mathbb{R}^d} |u|^2 \log\left(\frac{|u|^2}{\int_{\mathbb{R}^d} |u|^2 \, d\mu_\varepsilon}\right) \, d\mu_\varepsilon}$$

Generalized Poincaré (Beckner) inequalities

$$\mathcal{C}_\varepsilon^{(p)} := \inf_{\substack{\nabla u \neq 0 \; d\mu_\varepsilon \text{ a.e.} \\ u \in H^1(d\mu_\varepsilon)}} \frac{(p-1) \int_{\mathbb{R}^d} |\nabla u|^2 \, d\mu_\varepsilon}{\int_{\mathbb{R}^d} |u|^2 \, d\mu_\varepsilon - \left(\int_{\mathbb{R}^d} |u|^{2/p} \, d\mu_\varepsilon\right)^p}$$

A (rough) upper estimate

Assume: $\int_{\mathbb{T}^d} e^{-\phi(y)}\, dy = 1$ and use $u_{\mathrm{e}}(x) = x \cdot \mathrm{e}$ as a test function

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |u_{\mathrm{e}}|^2 \, d\mu_\varepsilon = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |\nabla u_{\mathrm{e}}|^2 \, d\mu_\varepsilon = 1$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |u_{\mathrm{e}}|^{2/p} \, d\mu_\varepsilon = \frac{2^{1/p}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{1}{p}\right)$$

$$\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} |u_{\mathrm{e}}|^2 \log |u_{\mathrm{e}}|^2 \, d\mu_\varepsilon = \log 2 - 2 + \gamma \approx -0.729637$$

where $\gamma \approx 0.577216$ is Euler's constant

Lemma

$$\lim_{\varepsilon \to 0} \mathcal{C}_\varepsilon^{(p)} \le \kappa(p) := \frac{p-1}{1 - \frac{2^{1/p}}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{1}{p}\right)}$$

Convex Sobolev inequality [Arnold et al.] or φ-*entropy* [Chafaï]

$$\int \left[\varphi(u) - \varphi(\bar{u}) - \varphi'(\bar{u})(u - \bar{u})\right] d\mu \leq \mathcal{C}_\varphi \int \varphi''(u)|\nabla u|^2 \, d\mu$$

$\bar{u}$ is the average of u with respect to $d\mu$: $\bar{u} := \int u \, d\mu$ and let $d\tilde{\mu}$ be a measure which is absolutely continuous with respect to $d\mu$

$$e^{-b} \, d\mu \leq d\tilde{\mu} \leq e^{-a} \, d\mu \quad \mu \text{ a.e.}$$

Lemma

If $\varphi \in C^3$ is a convex positive function, then

$$\int \left[\varphi(u) - \varphi(\tilde{u}) - \varphi'(\tilde{u})(u - \tilde{u})\right] d\tilde{\mu} \leq e^{b-a} \, \mathcal{C}_\varphi \int \varphi''(u) \, |\nabla u|^2 \, d\tilde{\mu}$$

$$\text{Consequence}: \quad \mathcal{C}_\varepsilon^{(p)} \geq \frac{p}{2} \, e^{-\mathrm{Osc}(\phi)}$$

$$u_t^\varepsilon = \Delta u^\varepsilon + \nabla \cdot \left[x\, u^\varepsilon + \frac{1}{\varepsilon} \nabla\phi\left(\frac{x}{\varepsilon}\right)\, u^\varepsilon \right] \quad x \in \mathbb{R}^d \;\; t > 0$$

Normalization: $\int_{\mathbb{T}^d} e^{-\phi}\, dy = 1$ and $\int_{\mathbb{T}^d} y \cdot \nabla_y \left(e^{-\phi}\right)\, dy = 0$
Assume that the solution can be written as

$$u^\varepsilon(t,x) = u^{(0)}\left(t,x,\tfrac{x-x_0}{\varepsilon}\right) + \varepsilon\, u^{(1)}\left(t,x,\tfrac{x-x_0}{\varepsilon}\right) + \varepsilon^2\, u^{(2)}\left(t,x,\tfrac{x-x_0}{\varepsilon}\right) + O(\varepsilon^3)$$

where $y \mapsto u^{(i)}(t,x,y) =: v^{(i)}(t,x,y)\, e^{-\phi(y)}$ is periodic...

At order ε^{-2} :

$$\Delta_y u^{(0)} + \nabla_y \cdot \left(u^{(0)}\, \nabla_y \phi(y)\right) = 0$$

$v^{(0)}$ does not depend on y

At order ε^{-1} :

$$\Delta_y u^{(1)} + \nabla_y \cdot \Big(u^{(1)}\, \nabla_y \phi(y)\Big) = -\nabla_x \cdot \Big(2\, \nabla_y u^{(0)} + \nabla_y \phi(y)\, u^{(0)}\Big)$$
$$= \nabla_y \phi(y) \cdot \nabla_x u^{(0)}$$

that is

$$\nabla_y \cdot \Big(e^{-\phi(y)} \Big(\nabla_y v^{(1)} + \nabla_x v^{(0)}\Big)\Big) = 0$$

With $v^{(1)}(t,x,y) = \nabla_x v^{(0)}(t,x) \cdot w(t,y)$, $w(t,y) = (w_j(t,y))_{j=1}^{d}$ solves *cell equation*

$$\nabla_y \cdot \Big(e^{-\phi(y)}\, (\nabla_y w_j + \mathrm{e}_j)\Big) = 0$$

Thus we have obtained that

$$\nabla_y w_j = \left[\frac{e^{\phi}}{\int_{\mathbb{T}^d} e^{\phi(y)}\, dy} - 1\right] \mathrm{e}_j\,, \;\; u^{(1)}(t,x,y) = \nabla_x v^{(0)}(x) \cdot w(t,y)\, e^{-\phi(y)}$$

At order $\varepsilon^0 = 1$:

$$u_t^{(0)} = \nabla_y \cdot \left(e^{-\phi(y)} \, \nabla_y v^{(2)} \right) + \nabla_x \cdot \left(\nabla_x v^{(0)} + x \, v^{(0)} \right) \, e^{-\phi(y)}$$
$$+ \nabla_x \cdot \left(2 \, \nabla_y u^{(1)} + \nabla_y \phi(y) \, u^{(1)} \right) + y \, v^{(0)} \cdot \nabla_y \left(e^{-\phi(y)} \right) \, .$$

Solvability condition: formally integrate with respect to $y \in \mathbb{T}^d$...

$$v_t^{(0)} = \mathsf{K} \, \Delta v^{(0)} + \nabla \cdot \left(x \, v^{(0)} \right)$$

$$\mathsf{K} := \frac{1}{\int_{\mathbb{T}^d} e^{\phi(y)} \, dy \int_{\mathbb{T}^d} e^{-\phi(y)} \, dy}$$

The solution $u^\varepsilon(t,x)$ of $u_t^\varepsilon = \Delta u^\varepsilon + \nabla \cdot \left[x\, u^\varepsilon + \frac{1}{\varepsilon} \nabla \phi \left(\frac{x}{\varepsilon} \right) \, u^\varepsilon \right]$ has been written as

$$u^\varepsilon(t,x) = \left(v^{(0)}(t,x) + \varepsilon \, \nabla_x v^{(0)}(t,x) \cdot w \left(t, \frac{x}{\varepsilon} \right) + O(\varepsilon^2) \right) e^{-\phi(\frac{x}{\varepsilon})}$$

where w is a solution of the cell problem and $v^{(0)}$ is a solution of a Fokker-Planck equation with diffusion coefficient K. As $t \to \infty$

$$u(t,x,y) = v^{(0)}(x,t) \, e^{-\phi(y)} (1 + O(\varepsilon))$$

$$v^{(0)}(t,x) \to v_\infty^{(0)}(x) = \frac{M}{(2\,\pi\,\mathrm{K})^{d/2}} \, e^{-\frac{|x|^2}{2\,\mathrm{K}}}$$

A summary

$$u^{\varepsilon}(t,x) \quad \xrightarrow[t\to\infty]{L^1\cap L^2} \quad u^{\varepsilon}_{\infty}(x) = M\,\frac{e^{-\frac{1}{2}|x|^2-\phi(x/\varepsilon)}}{\int_{\mathbb{R}^d} e^{-\frac{1}{2}|z|^2-\phi(z/\varepsilon)}\,dz} \xrightarrow[\varepsilon\to 0]{\text{two-scale}} \frac{M}{(2\,\pi)^{d/2}}\,e^{-\frac{|x|^2}{2}}\,e^{-\phi(y)}$$

$$\| \qquad\qquad\qquad\qquad \not\|$$

$$u^{\varepsilon}(t,x) \quad \xrightarrow[\varepsilon\to 0]{\text{two-scale}} \quad v^{(0)}(t,x)\,e^{-\phi(y)} \quad \xrightarrow[t\to\infty]{L^1\cap L^2} \quad \frac{M}{(2\,\pi\,\mathrm{K})^{d/2}}\,e^{-\frac{|x|^2}{2\,\mathrm{K}}}\,e^{-\phi(y)}$$

Theorem

Assume that ϕ is a C^2 function on $\mathbb{T}^d$

$$\forall\, p \in (1,2] \quad \lim_{\varepsilon\to 0_+} \mathcal{C}_\varepsilon^{(p)} = \mathrm{K}\,\mathcal{C}_0^{(p)}$$

Moreover, $\lim_{\varepsilon\to 0_+} \mathcal{C}_\varepsilon^{(1)} \in [\mathrm{k}\,\mathcal{C}_0^{(1)}, \mathrm{K}\,\mathcal{C}_0^{(1)}]$ *with* $\mathrm{k} = e^{-\mathrm{Osc}(\phi)}$

$$\mathrm{K}^{-1} = \int_{\mathbb{T}^d} e^{\phi(y)} \int_{\mathbb{T}^d} e^{-\phi(y)}\, dy$$

$\mathcal{C}_0^{(p)} = p/2$; it is an open question to determine whether $\lim_{\varepsilon\to 0_+} \mathcal{C}_\varepsilon^{(1)} = \mathrm{K}\,\mathcal{C}_0^{(1)}$ or not

Corollary

Assume that ϕ is a $C^2(\mathbb{T}^d)$. If u is a smooth solution of (2), $\exists\, \mathrm{A}[u_0]$

$$\|u^\varepsilon - u_\infty^\varepsilon\|^2_{L^p(\mathbb{R}^d,(u_\infty^\varepsilon)^{1-p}dx)} \le \mathrm{A}\, e^{-4\,\mathcal{C}_\varepsilon^{(p)} t/p} \quad \forall\, t > 0$$

$\lim_{\varepsilon\to 0_+} 4\,\mathcal{C}_\varepsilon^{(p)}/p = 2\,\mathrm{K} < 2$ *if* $p \in (1,2]$, $\lim_{\varepsilon\to 0_+} 4\,\mathcal{C}_\varepsilon^{(1)} \le 2\,\mathrm{K} < 2$

Proposition ([Allaire] – definition of "two-scale convergence")

Let Ω be an open set in $\mathbb{R}^d$. If $(u_\varepsilon)_{\varepsilon>0}$ is a bounded sequence in $L^2(\Omega)$, then there exists $u_0 \in L^2(\Omega \times \mathbb{T}^d)$ such that, up to subsequences,

$$\lim_{\varepsilon\to 0} \int_\Omega u_\varepsilon(x)\,\varphi\left(x, \frac{x}{\varepsilon}\right)\,dx = \int\int_{\Omega\times\mathbb{T}^d} u_0(x,y)\,\varphi(x,y)\,dx\,dy \qquad (3)$$

for all smooth $y-$periodic function φ. Moreover, $(u_\varepsilon)_{\varepsilon>0}$ weakly converges in $L^2(\Omega)$ to $u_*(x) := \int_{\mathbb{T}^d} u_0(x,y)\,dy$ and

$$\lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^2(\Omega)} \geq \|u_0\|_{L^2(\Omega\times\mathbb{T}^d)} \geq \|u_*\|_{L^2(\Omega)}\,.$$

Proposition ([Allaire])

Assume that $u_\varepsilon \rightharpoonup u_*$ in $H^1(\Omega)$. Then there exist a subsequence of $(u_\varepsilon)_{\varepsilon>0}$, still denoted $(u_\varepsilon)_{\varepsilon>0}$, which two-scale converges to $u_* = u_*(x)$ Moreover, there exists a function $u_1 \in L^2\left(\Omega, H^1(\mathbb{T}^d)\right)$ such that

$$\nabla u_\varepsilon \xrightarrow[\varepsilon\to 0]{\text{two-scale}} \nabla_x u_*(x) + \nabla_y u_1(x,y)$$

Lemma

The embedding $H^1(\mathbb{R}^d, d\mu_0) \hookrightarrow L^2(\mathbb{R}^d, d\mu_0)$ is compact

Proof. $\|u_n\|^2_{H^1(\mathbb{R}^d,d\mu_0)} \leq 1,\ u_n := v_n/\sqrt{\mu_0}$

$$\int_{\mathbb{R}^d} |\nabla u_n|^2 \, d\mu_0 = \int_{\mathbb{R}^d} |\nabla v_n|^2 \, dx + \frac{1}{4}\int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, dx - \frac{d}{2}\|v_n\|^2_{L^2}$$

$$\int_{\mathbb{R}^d} |u_n|^2 \log |u_n|^2 \, d\mu_0 = \int_{\mathbb{R}^d} |v_n|^2 \log |v_n|^2 \, dx + \log Z_0 \|v_n\|^2_{L^2} + \frac{1}{2}\int_{\mathbb{R}^d} |x|^2 \, |v_n|^2 \, dx$$

... logarithmic Sobolev inequality, with $\mathcal{C}_0^{(1)} = 1/2$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, d\mu_0 \geq \mathcal{C}_0^{(1)} \int_{\mathbb{R}^d} |u|^2 \, \log\left(\frac{|u|^2}{\int_{\mathbb{R}^d} |u|^2 \, d\mu_0}\right) \, d\mu_0 \quad \forall\, u \in H^1(\mathbb{R}^d, d\mu_0)$$

Conclusion by Dunford-Pettis' theorem $\square$

Also valid for $d\mu_\varepsilon$... uniformly with respect to ε

Lemma

For any $p \in [1,2]$, $\mathcal{C}_\varepsilon^{(p)} \le \frac{p}{2}\,\mathcal{C}_\varepsilon^{(2)}$

Theorem ([Beckner, Arnold-Bartier-JD])

$$\forall\ p \in (1,2] \quad \mathcal{C}_\varepsilon^{(2)} \le \frac{1}{p-1}\left[1-\left(\frac{2-p}{p}\right)^{\alpha}\right]\mathcal{C}_\varepsilon^{(p)} \quad \textit{with} \quad \alpha := \frac{\mathcal{C}_\varepsilon^{(2)}}{2\,\mathcal{C}_\varepsilon^{(1)}}$$

Corollary

If $\mathcal{C}_\varepsilon^{(1)} = \frac{1}{2}\,\mathcal{C}_\varepsilon^{(2)}$, *then* $\mathcal{C}_\varepsilon^{(p)} = \frac{p}{2}\,\mathcal{C}_\varepsilon^{(2)}$ *for any* $p \in [1,2]$

The case $p = 2$ (1/3)

There is a non-trivial minimizer u_ε to $\mathcal{C}_\varepsilon^{(2)}$ such that $\int_{\mathbb{R}^d} u_\varepsilon \, d\mu_\varepsilon = 0$, $\int_{\mathbb{R}^d} |u_\varepsilon|^2 \, d\mu_\varepsilon = 1$ and

$$-\nabla \cdot \left(e^{-\frac{1}{2}|x|^2 - \phi(x/\varepsilon)} \, \nabla u_\varepsilon(x) \right) = \mathcal{C}_\varepsilon^{(2)} \, u_\varepsilon(x) \, e^{-\frac{1}{2}|x|^2 - \phi(x/\varepsilon)} .$$

Let $\varphi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi_1 \in \mathcal{D}(\mathbb{R}^d, C^\infty(\mathbb{T}^d))$

$$\int_{\mathbb{R}^d} \nabla_x u_\varepsilon \left[\nabla_x \varphi(x) + \varepsilon \, \nabla_x \varphi_1 \left(x, \frac{x}{\varepsilon} \right) + \nabla_y \varphi_1 \left(x, \frac{x}{\varepsilon} \right) \right] d\mu_\varepsilon$$
$$= \mathcal{C}_\varepsilon^{(2)} \int_{\mathbb{R}^d} u_\varepsilon \left[\varphi(x) + \varepsilon \, \varphi_1 \left(x, \frac{x}{\varepsilon} \right) \right] d\mu_\varepsilon$$

$\nabla u_\varepsilon \xrightarrow[\varepsilon \to 0]{\text{two-scale}} \nabla_x u_*(x) + \nabla_y u_1(x, y)$, $\mathcal{K}_0^{(2)} := \lim_{\varepsilon \to 0_+} \mathcal{C}_\varepsilon^{(2)}$
a *two-scale homogenized equation:*

$$\int \int_{\mathbb{R}^d \times \mathbb{T}^d} \Big[\nabla_x u_*(x) + \nabla_y u_1(x, y) \Big] \Big[\nabla_x \varphi(x) + \nabla_y \varphi_1 (x, y) \Big] \, e^{-\frac{1}{2}|x|^2 - \phi(y)} \, dx \, dy$$
$$= \mathcal{K}_0^{(2)} \int \int_{\mathbb{R}^d \times \mathbb{T}^d} u_*(x) \, \varphi(x) \, e^{-\frac{1}{2}|x|^2 - \phi(y)} \, dx \, dy$$

An evaluation with $\varphi = 0$ shows that u_1 is given as a solution of

$$\nabla_y \cdot \left[e^{-\phi(y)} \left(\nabla_y u_1(x,y) + \nabla_x u_*(x) \right) \right] = 0 \ ,$$

$$u_1(x,y) = \nabla_x u_*(x) \cdot w(y)$$

where $w = (w_j)_{j=1}^d$ is the solution of the *cell equation*

$$\nabla_y u_1(x,y) = \left[\frac{e^{\phi}}{\int_{\mathbb{T}^d} e^{\phi(y)}\, dy} - 1 \right] \nabla_x u_*(x)$$

Test with $\varphi = u_*$ (up to an appropriate regularization) and $\varphi_1 = 0$

$$\int_{\mathbb{R}^d} \frac{|\nabla_x u_*|^2}{\int_{\mathbb{T}^d} e^{\phi(y)}\, dy}\, d\mu_0 = \mathcal{K}_0^{(2)} \int_{\mathbb{R}^d} |u_*|^2 \, d\mu_0$$

We can also observe that

$$\int_{\mathbb{R}^d} u_* \, d\mu_0 = \lim_{\varepsilon \to 0_+} \int_{\mathbb{R}^d} u_\varepsilon \, d\mu_\varepsilon = 0$$

Altogether this proves that

$$\mathcal{K}_0^{(2)} \geq \frac{\mathcal{C}_0^{(2)}}{\int_{\mathbb{T}^d} e^{\phi(y)}\, dy} = \mathsf{K}\, \mathcal{C}_0^{(2)}$$

To prove the reverse inequality, $\mathcal{K}_0^{(2)} \leq \mathsf{K}\, \mathcal{C}_0^{(2)}$, consider

$$\tilde{u}_\varepsilon(x) := u_{\mathrm{e}}(x) + \varepsilon\, \nabla_x u_{\mathrm{e}}(x)\, w\left(\frac{x}{\varepsilon}\right)$$

where $u_{\mathrm{e}}(x) = x \cdot \mathrm{e}$ and w is the solution to the *cell problem*

$$\mathcal{K}_0^{(2)} \leq \lim_{\varepsilon \to 0_+} \frac{\int_{\mathbb{R}^d} |\nabla \tilde{u}_\varepsilon|^2\, d\mu_\varepsilon}{\int_{\mathbb{R}^d} |\tilde{u}_\varepsilon|^2\, d\mu_\varepsilon - \left(\int_{\mathbb{R}^d} \tilde{u}_\varepsilon\, d\mu_\varepsilon\right)^2} = \mathsf{K}\, \mathcal{C}_0^{(2)}$$

This completes the proof in case $p = 2$ □

A result inspired by [Rothaus]

Proposition

Let ϕ be a continuous function on $\mathbb{T}^d$ and take $p \in (1,2)$, $\varepsilon > 0$. Either

$$\mathcal{C}_\varepsilon^{(p)} \leq \frac{p}{2}\,\mathcal{C}_\varepsilon^{(2)}$$

is achieved by some non trivial function, or

$$\mathcal{C}_\varepsilon^{(p)} = \frac{p}{2}\,\mathcal{C}_\varepsilon^{(2)}$$

is not achieved by any non trivial function

No minimizer: result follows from the case $p = 2$
Otherwise, consider sequences of minimizers and apply the two-scale convergence approach

Traveling and tilted ratchets: speed of the center of mass

If f is a solution of (1)

$$f_t = f_{xx} + \big(\psi'(x - \omega\, t)\, f\big)_x$$

we observe that $\tilde{f}(t,x) = f(t, x - \omega\, t)$ is a solution of

$$\tilde{f}_t = \tilde{f}_{xx} + \Big((\omega + \psi')\, \tilde{f}\,\Big)_x$$

a problem which is known as the *tilted Smoluchowski-Feynman ratchet*
Tilted Brownian ratchets are actually much more general, since in the equation for $\tilde{f}$, ψ may still depend on t (flow reversals)

Tilted potential

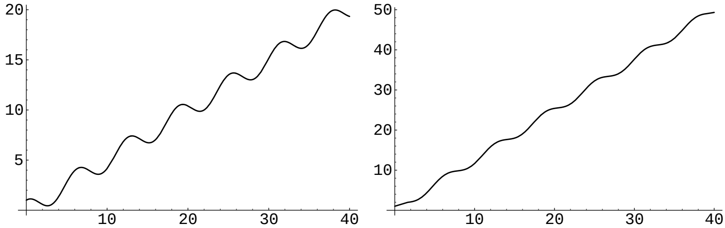


Figure: $x \mapsto \omega\, x + \psi(x)$, $\psi(x) = \cos x$, and left: $\omega = 0.5$, right $\omega = 1.25$
The *critical tilt:* corresponds to $\omega = 1$

The doubly periodic case

Instead of considering (1) $f_t = f_{xx} + \big(\psi'(x-\omega\, t)\, f\big)_x$, consider

$$\begin{cases} g_t = g_{xx} + \big(g\,\psi'(x-\omega\, t)\big)_x & x \in S^1\,,\ t>0\,, \\ g(t=0,x) = g_0(x) = \sum_{k\in\mathbb{Z}} f_0(x+k) & x\in S^1\,, \end{cases} \tag{4}$$

for which, by linearity of the equations, we get

$$g(t,x) = \sum_{k\in\mathbb{Z}^d} f(t,x+k) \quad \forall\,(t,x)\in\mathbb{R}^+\times S^1$$

With $\int_{\mathbb{R}^d} f_0\, dx = 1 = \int_{\mathbb{R}^d} f(t,\cdot)\, dx = 1$ for any $t\geq 0$, we can define the position of the center of mass by

$$\bar{x}(t) := \int_{\mathbb{R}^d} x\, f(t,x)\, dx$$

$$\frac{d\bar{x}}{dt} = \int_{\mathbb{R}^d} x\, f_t\; dx \quad = \quad -d \int_{\mathbb{R}^d} \psi'(x - \omega\, t)\, f(t,x)\; dx$$

$$= \quad -d \sum_{k \in \mathbb{Z}} \int_{S^1} \psi'(x - \omega\, t)\, f(t, x + k)\; dx$$

$$= \quad -d \int_{S^1} \psi'(x - \omega\, t)\, g(t,x)\; dx$$

$$\underset{t \to \infty}{\sim} \quad -d \int_{S^1} \psi'(x - \omega\, t)\, g_\infty(t,x)\; dx =: c_\omega$$

The time-periodic solution $g_\infty(t,x) = g_\omega(x - \omega\, t)$ solves the equation

$$(g_\omega)_{xx} + \big((\omega + \psi')\, g_\omega\big)_x = 0$$

with periodic boundary conditions

Take a primitive: $x \mapsto (g_\omega)_x + (\omega + \psi')\, g_\omega =: A(\omega)$ is constant and

$$\omega - c_\omega = \omega \int_0^1 g_\omega \, dx + \int_0^1 \psi' \, g_\omega \, dx = A(\omega)$$

Some elementary but tedious computations show that $c_\omega < \omega$, $\lim_{\omega \to 0_+} c_\omega/\omega > 0$, c_ω is positive for large values of ω, and $\lim_{\omega \to \infty} c_\omega = 0$

Two potentials

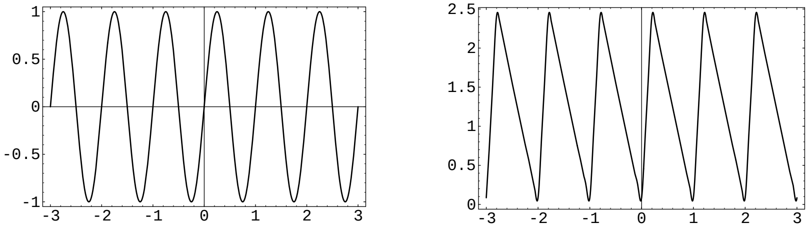


Figure: Plots of the potential $\psi(x) = \sin x$ (left) and ψ: asymmetric smooth sawtooth potential (right)

Velocity of the center of mass

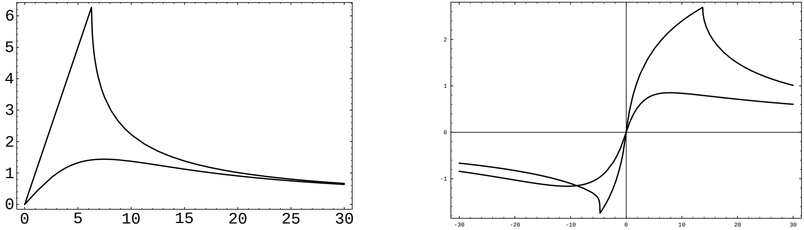


Figure: Plots of c_ω and c_ω^0 (no diffusion) as functions of ω in the sinusoidal case (left) and in the case of the asymmetric smooth sawtooth potential (right, in logarithmic coordinates). In the sinusoidal case, the symmetry is reflected by the fact that $c_{-\omega} = -c_\omega$ (values corresponding to $\omega < 0$ are not represented). This is not true in the sawtooth case.

A characteristic property of the curve $\omega \mapsto c_\omega^0$ is the *critical tilt*, which is still present when diffusion is added

Rescaling and formal asymptotic expansion: effective diffusion

If f is a solution of (1) $f_t = f_{xx} + \big(\psi'(x - \omega\, t)\, f\big)_x$ and

$$f(t,x) = \frac{1}{R(t)}\, u\left(\log R(t), \frac{x - c_\omega\, t}{R(t)}\right) \quad \text{with} \quad R(t) := \sqrt{1+2t}$$

In the new variables: $R = e^t$ and $z = R\, x - \frac{1}{2}\,(R^2 - 1)\,(\omega - c_\omega)$,

$$u_t = u_{xx} + (x\, u)_x + R\,\big((\psi'(z) + c_\omega)\, u\big)_x$$

We will prove that

$$u(t,x) = g_\omega(z)\, h - \frac{1}{R}\, g_\omega^{(1)}(z)\, h_x + O\left(R^{-2}\right)$$

g_ω depends only on the fast oscillating variable z, κ_ω is an effective diffusion constant

$$h_t = \kappa_\omega\, h_{xx} + (x\, h)_x$$

Formal asymptotic expansion (1/2)

$u_t = u_{xx} + (x\,u)_x + R\left[\left(\psi'\left(R\,x - \frac{1}{2}\,(R^2-1)\,A(\omega)\right) + c_\omega\right)\,u\right]_x,$
$R(t) = e^t\,A(\omega) \neq 0\ldots$ two-scale function U

$$u(t,x) = \mathrm{U}(t,x;z) \quad \text{with} \quad z := R\,x - \tfrac{1}{2}\,(R^2-1)\,A(\omega)\,,$$

$$\mathrm{U}_t - \mathrm{U}_{xx} - (x\,\mathrm{U})_x = R^2\Big(\mathrm{U}_{zz} + \big((\omega+\psi'(z))\,\mathrm{U}\,\big)_z\Big) + R\Big(2\,\mathrm{U}_z + \big(\psi'(z)+c_\omega\big)\,\mathrm{U}\Big)$$

We will formally solve the equation order by order.

(i) To cancel the terms of order R^2 in the equation, U has to be proportional to g_ω

$$\mathrm{U}(t,x;z) = g_\omega(z)\,h(t,x) + R^{-1}\mathrm{U}^{(1)}(t,x;z) + O\left(R^{-2}\right)$$

(ii) At order R, we find that

$$\mathrm{U}^{(1)}(t,x;z) = g_\omega^{(1)}(z)\,h_x(t,x)$$

where $g_\omega^{(1)}$ is given as a solution of the equation

$$(g_\omega^{(1)})_{zz} + \Big(\big(\omega+\psi'(z)\big)\,g_\omega^{(1)}\Big)_z = -2\,(g_\omega)_z - \big(\psi'(z)+c_\omega\big)\,g_\omega$$

Formal asymptotic expansion (2/2)

There is a *solvability condition* at order R: the average on $(0,1)$ of the right hand side of the equation is 0. Since all functions are periodic and $\int_0^1 g_\omega(z)\, dz = 1$, we recover the condition

$$\int_0^1 \psi'(z)\, g_\omega(z)\, dz + c_\omega = 0$$

Uniqueness under proper normalization $\int_0^1 g_\omega^{(1)}(z)\, dz = 0$
(iii) At order $R^0 = 1$, the solvability condition is:

$$h_t - h_{xx} - (x\,h)_x = h_{xx} \int_0^1 \big(\psi'(z) + c_\omega\big)\, g_\omega^{(1)}(z)\, dz$$

Hence we obtain a modified Fokker-Planck equation

$$h_t = \kappa_\omega\, h_{xx} + (x\,h)_x$$

where the *effective diffusion coefficient* is given by

$$\kappa_\omega := 1 + \int_0^1 \psi'(z)\, g_\omega^{(1)}(z)\, dz$$

Lemma

Let χ be the unique periodic solution of

$$\chi'' - (\psi' + \omega)\ \chi' = \psi' + c_\omega$$

such that $\int_0^1 \chi\, dz = 0$. Then

$$\kappa_\omega = \int_0^1 |1 + \chi'|^2\, g_\omega\ dz\ > 0$$

$\kappa_{\omega_{|\omega=0}} = \left(\int_0^1 e^{\psi}\, dz \int_0^1 e^{-\psi}\, dz\right)^{-1} = \mathsf{K}^{-1} < 1$ *and* $\lim_{\omega\to\infty} \kappa_\omega = 1$

A proof by duality [Goudon-Poupaud]

$$\mathrm{U}(t,x;z) = g_\omega(z)\, h(t,x) + g_\omega^{(1)}(z)\, h_x(t,x) + O\left(R^{-2}\right),$$

The solution of $h_t = \kappa_\omega\, h_{xx} + (x\, h)_x$ converges to $h_\infty(x) := \frac{e^{-\frac{|x|^2}{2\,\kappa_\omega}}}{(2\,\pi\,\kappa_\omega)^{1/2}}$

Lemma

If $\int_{\mathbb{R}} h_0\, dx = 1$ *and* $\int_{\mathbb{R}} h_0\, \log(h_0/h_\infty)\, dx < \infty$, *then*

$$\|h(t,\cdot) - h_\infty\|_{L^1(\mathbb{R})} = O\left(e^{-t}\right) \quad \text{as} \quad t \to \infty$$

Proof. The proof is based on the *logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^d} h\, \log\left(\frac{h}{h_\infty}\right)\, dx \le \frac{\kappa_\omega}{2} \int_{\mathbb{R}^d} h\left|\left(\log\left(\frac{h}{h_\infty}\right)\right)_x\right|^2 dx$$

and on the *Csiszár-Kullback inequality*

$$\|h - h_\infty\|^2_{L^1(\mathbb{R}^d)} \le \frac{1}{4}\int_{\mathbb{R}^d} h\, \log\left(\frac{h}{h_\infty}\right)\, dx$$

and $\frac{d}{dt}\int_{\mathbb{R}^d} h\, \log\left(\frac{h}{h_\infty}\right)\, dx = -\kappa_\omega \int_{\mathbb{R}^d} h\left|\log\left(\frac{h}{h_\infty}\right)_x\right|^2 dx$ □

Heuristics (completed)

Summarizing

$$u(t,x) = \mathrm{U}(t,x;z) = \left(g_\omega(z) - \frac{x}{\kappa_\omega\, R}\, g_\omega^{(1)}(z) \right) h_\infty(x) \Big(1 + o(1) \Big)$$

with $R = e^t$ and $z = R\,x - \frac{1}{2}\,(R^2 - 1)\,A(\omega)$

$$f(t,x) = \left[g_\omega(x - \omega\, t) - \frac{x - c(\omega)\, t}{\kappa_\omega\, \sqrt{1+2\,t}}\, g_\omega^{(1)}(x - \omega\, t) \right] \frac{h_\infty\left(\frac{x - c(\omega)\, t}{\sqrt{1+2\,t}} \right)}{\sqrt{1+2\,t}} \Big(1 + o(1) \Big)$$

Results

$$u_t = u_{xx} + (x\,u)_x + e^t \left[\left(\psi'\left(e^t x - \tfrac{1}{2}\left(e^{2t}-1\right) A(\omega)\right) + c_\omega(\omega) \right) \, u \right]_x$$

Theorem

Let $d = 1$, $\omega > 0$, and assume that ψ is C^2, periodic + a technical condition. For any $\delta > 0$,

$$\limsup_{t\to\infty} e^{(\min(1,1/\mathtt{k})-\delta))\,t} \, \|u(t) - u_\infty(t)\|_{L^1(\mathbb{R})} < \infty$$

$\mathtt{k}$ to be specified... In the original variables

$$f_t = f_{xx} + \left(\psi'(x - \omega\,t)\,f\right)_x$$

Corollary

For any $\delta > 0$,

$$\limsup_{t\to\infty} t^{(\min(1,1/\mathtt{k})-\delta))/2} \, \|f(t) - f_\infty(t)\|_{L^1(\mathbb{R})} < \infty$$

Expanding a solution: remainder terms

First step: $u(t,x) = \mathrm{U}(t,x;z)$ is a solution if and only if $\mathrm{L}\,\mathrm{U} = 0$

$$
\begin{aligned}
\mathrm{L}_0\,\mathrm{U} &:= \mathrm{U}_{zz} + \big((\omega + \psi'(z))\,\mathrm{U}\,\big)_z \\
\mathrm{L}_1\,\mathrm{U} &:= \Big(2\,\mathrm{U}_{zz} + (\psi'(z) + c_\omega)\,\mathrm{U}\,\Big)_x \\
\mathrm{L}_2\,\mathrm{U} &:= \mathrm{U}_{xx} + \big(x\,\mathrm{U}\big)_x - \mathrm{U}_t \\
\mathrm{L}\,\mathrm{U} &:= -\Big(R^2\,\mathrm{L}_0\,\mathrm{U} + R\,\mathrm{L}_1\,\mathrm{U} + \mathrm{L}_2\,\mathrm{U}\Big)
\end{aligned}
$$

and $\mathrm{U} := \mathrm{U}_0 + R^{-1}\,\mathrm{U}_1 + R^{-2}\,\mathrm{U}_2$

$$
\begin{aligned}
\mathrm{U}_0(t,x;z) &:= g_\omega(z)\,h(t,x) \\
\mathrm{U}_1(t,x;z) &:= g_\omega^{(1)}(z)\,h_x(t,x) \\
\mathrm{U}_2(t,x;z) &:= g_\omega^{(2)}(z)\,h_{xx}(t,x)
\end{aligned}
$$

$$0 = \mathrm{L}\,\mathrm{U} = \tfrac{1}{R}\Big(\mathrm{L}_1\,\mathrm{U}_2 + \mathrm{L}_2\,\mathrm{U}_1\Big) + \tfrac{1}{R^2}\,\mathrm{L}_2\,\mathrm{U}_2$$

Second step: Consider $\mathsf{U}_\infty := \frac{1}{Z(t)}\Big(\mathsf{U}_{\infty,0} + R^{-1}\,\mathsf{U}_{\infty,1} + R^{-2}\,\mathsf{U}_{\infty,2}\Big)$
$\mathsf{U}_{\infty,0}(t,x;z) := g_\omega(z)\,h_\infty(x)$, $\mathsf{U}_{\infty,1}(t,x;z) := g_\omega^{(1)}(z)\,h_{\infty,x}(x)\,\chi(e^{-t}x)$,
$\mathsf{U}_{\infty,2}(t,x;z) := g_\omega^{(2)}(z)\,h_{\infty,xx}(x)\,\chi(e^{-t}x)$
U_∞ is only an approximate solution, we have

$$\mathsf{L}\,\mathsf{U}_\infty = \frac{\dot{Z}}{Z}\,\mathsf{U}_\infty + \frac{1}{R}\,\mathsf{F}$$

$\mathsf{F}/\mathsf{U}_\infty$ is a polynomial of order four in x

$$u_\infty(t,x) := \mathsf{U}_\infty\left(t,x;e^t\,x - \tfrac{1}{2}\,(e^{2t}-1)\,A(\omega)\right)$$
$$\mathsf{f}(t,x) := \mathsf{F}\left(t,x;e^t\,x - \tfrac{1}{2}\,(e^{2t}-1)\,A(\omega)\right)$$

Relative entropy production term

$$u_t = u_{xx} + (\varphi'(t,x)\,u)_x$$

and

$$(u_\infty)_t = (u_\infty)_{xx} + (\varphi'(t,x)\,u_\infty)_x - \frac{\dot{Z}}{Z}\,u_\infty + e^{-t}\,\mathsf{f}$$

with $\varphi'(t,x) = \psi(x - \omega\,t)$

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} u \log\left(\frac{u}{u_\infty}\right)\,dx &= \int_{\mathbb{R}}\left[1 + \log\left(\frac{u}{u_\infty}\right)\right] u_t\,dx - \int_{\mathbb{R}} \frac{u}{u_\infty}\,(u_\infty)_t\,dx \\ &= -\int_{\mathbb{R}}\left|\left(\log\left(\frac{u}{u_\infty}\right)\right)_x\right|^2 u\,dx + \frac{\dot{Z}}{Z} + e^{-t}\int_{\mathbb{R}} \frac{\mathsf{f}}{u_\infty}\,u\,dx \end{aligned}$$

Corollary (homogenized logarithmic Sobolev inequality)

Let $u_\infty^\varepsilon(x) := c_\varepsilon \, e^{-\phi(x/\varepsilon)-|x|^2/(2\kappa_\omega)}\,dx$ *such that* $\int_{\mathbb{R}} u_\infty^\varepsilon\,dx = 1$*. For any* $\varepsilon > 0$*, there exists a positive constant* $\mathcal{K}_\varepsilon$ *such that, for any nonnegative* $u \in L^1(\mathbb{R})$ *satisfying* $\int_{\mathbb{R}} u\,dx = 1$*,*

$$\int_{\mathbb{R}} u \,\log\left(\frac{u}{u_\infty^\varepsilon}\right)\,dx \le \mathcal{K}_\varepsilon \int_{\mathbb{R}} \left|\left(\log\left(\frac{u}{u_\infty^\varepsilon}\right)\right)_x\right|^2 u\,dx$$

Moreover, $\limsup_{\varepsilon\to 0} \mathcal{K}_\varepsilon =: \mathtt{k}/2$ *satisfies*

$$\kappa_\omega/\mathsf{K} \le \mathtt{k} \le \kappa_\omega \max_{[0,1]} g_\omega \cdot \left(\min_{[0,1]} g_\omega\right)^{-1}$$

$$\mathsf{K}^{-1} = \int_0^1 g_\omega\,dz \int_0^1 g_\omega^{-1}\,dz \quad \textit{and} \quad \lim_{\omega\to 0} \mathsf{K}/\kappa_\omega = 1$$

Choice: $\phi(z) = \log g_\omega(z)$, which is itself computed in terms of ψ

To control the error terms, we need to control fourth order moments in the rescaled variables

Proposition ([Dalibard])

If there exists $m > d + 8$ such that

$$\int_{\mathbb{R}^d} (1 + |x|^2)^{m/2} \, u_0 \, dx < \infty$$

then

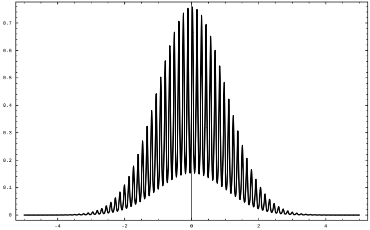
$$\limsup_{t \to +\infty} \int_{\mathbb{R}^d} |x|^4 \, u(t,x) \, dx < \infty$$

Physical interpretation

At first order, $u(t,x)$ behaves for large values of t like

$$u_\infty(t,x) = g_\omega(z)\, h_\infty(x)\ , \quad h_\infty(x) := \frac{e^{-\frac{|x|^2}{2\kappa_\omega}}}{\sqrt{2\pi\kappa_\omega}}$$

where $z = e^t\, x - \frac{1}{2}\,(e^{2t} - 1)\,(\omega - c_\omega)$



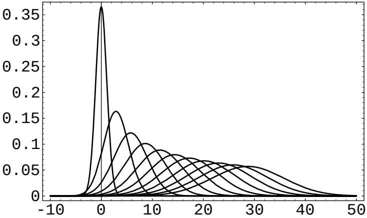
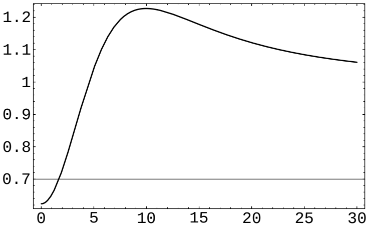


Figure: In the sinusoidal case, the limiting function u_∞ is shown on the left, in self-similar-variables, while on the right, the diffuse, traveling front F_∞ is plotted in the original variables for $t = 0,\ 1,\ \ldots 20$. Here we take $\omega = 5$ and (left) $u_\infty(t,x)$ is shown as a function of x for $t = 2$.

Effective diffusion coefficient



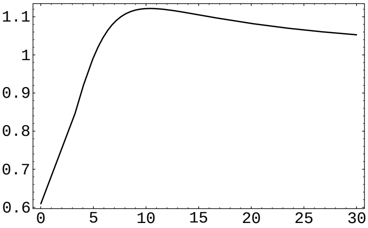


Figure: Plot of the diffusion coefficient κ_ω as a function of ω in the sinusoidal case (left) and in the smooth sawtooth potential case (right).

The Péclet number Pe describes the competition between the directional drift and the stochastic diffusion of the particle

$$\mathrm{Pe} := \frac{c_\omega \, \ell}{\kappa_\omega}$$

where ℓ is a typical length scale. Larger Pe number means that the drift predominates over diffusion

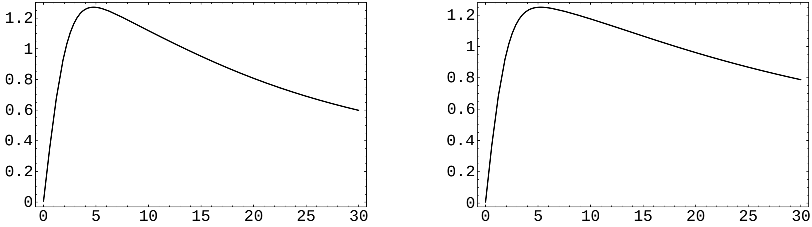


Figure: Plot of the Péclet number Pe as a function of ω in the sinusoidal case (left) and in the smooth sawtooth potential case (right).

The *characteristic length scale*

$$\mathrm{L} := \frac{\ell}{\mathsf{Pe}}$$

can be compared with the diffusion scale at characteristic time T such that $\sqrt{\kappa_\omega \mathrm{T}} = c_\omega \mathrm{T} = \mathrm{L}$: at that time, the percentage of the initial distribution which is still in the $x < 0$ region is given by $\int_{-\infty}^{0} \exp\left[-|x - \mathrm{L}|^2/(2\kappa_\omega \mathrm{T})\right] dx = \frac{1}{2}\mathrm{Erf}(1/\sqrt{2}) \approx 16\%$.

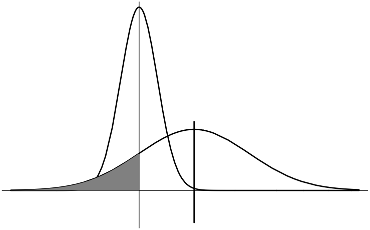


Figure: Definition of L and T can be understood as follow. If one starts with a Gaussian distribution centered at $x = 0$ and evolve it according to the effective Fokker-Planck eqaution, T is the time for which the solution (centered at L in the above plot) has a variance equal to L. The grey area represents 16% of the area below the solution at time $t = \mathrm{T}$.

The *characteristic time scale* $\mathrm{T} = \kappa_\omega / c_\omega^2$ is related with the Péclet number

$$\mathrm{T} = \frac{\ell}{c_\omega \, \mathsf{Pe}} \, .$$

and can be compared to the time period of the potential $\mathrm{T}_0 := \ell/\omega$. Hence it is meaningful to consider

$$\mathrm{N} := \frac{\mathrm{T}}{\mathrm{T}_0} = \frac{\omega \, \kappa_\omega}{\ell \, c_\omega^2} = \frac{\omega}{c_\omega \, \mathsf{Pe}}$$

which measures the "time" in takes to achieve the equality $\sqrt{\kappa_\omega \, \mathrm{T}} = c_\omega \, \mathrm{T}$ in natural units, and to define the efficiency of the transport by

$$\mathsf{E} := \frac{1}{\mathrm{N}} = \frac{\ell \, c_\omega^2}{\omega \, \kappa_\omega} = \mathsf{Pe} \, \frac{c_\omega}{\omega}$$

Efficiency: plots

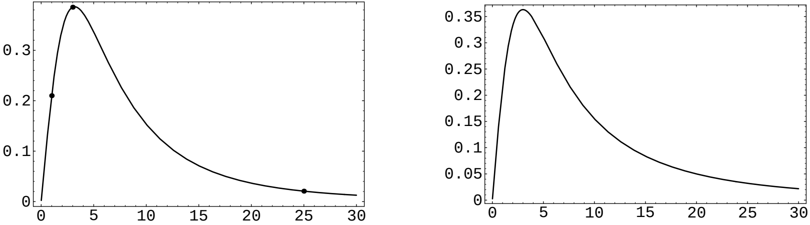


Figure: Plot of the efficiency E as a function of ω in the sinusoidal case (left) and in the smooth sawtooth potential case (right). We observe that in both cases, the maximum is extremely well defined. Dots (left) correspond $(\omega, \mathrm{E}(\omega))$ taking the values $(1, 0.210)$, $(3, 0.385)$, $(25, 0.021)$

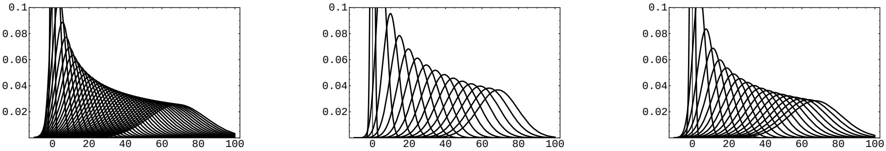


Figure: The effective profile F_∞ is represented for ω taking the values 1, 3 and 25, which correspond to the dots in the above plots. Curves are plotted for $\omega = 1$ (left), 3 (center), 25 (right) for $t = 0$, 5, 10, etc, as long as $c_\omega\ t \leq 70$. The curve corresponding to $\omega = 3$ (center) is the most efficient, in the sense that $c_\omega\ t \approx 70$ is reached for a smaller value of t than for the other curves and the solution is kept more peaked. Computations are done in the case of the sinusoidal potential.

Concluding remarks

- Even for the simplest model, get sharp results: optimal constant in the homogenized Logarithmic Sobolev inequality, higher order expansions in the regime $\kappa_\omega < 1$
- Qualitative issues: prove $\kappa_\omega > \mathsf{K}$, characterize flux reversal
- Numerical challenges (1): proof of the convergence of Monte-Carlo schemes
- Numerical challenges (2): beyond Monte-Carlo schemes: track intermediate regimes and measure the convergence at micro /macro levels

Main idea: by entropy methods, reduce the study of large time behaviours to functional inequalities in a singular limit, that can be studied using variational tools

Further perspectives

Extend the methods to carefully selected models

- Nonlinear diffusion
- Tilted periodic channel subject to gravitation
- ...

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