Nonlinear stability results for kinetic equations

with

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1. Vlasov-Poisson: Nonlinear stability

Distribution function $f(t,x,v) \geq 0$
density of particles at $(x,v,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$.
Liouville equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t,x) \cdot \nabla_v f = 0$$

$$F(t,x) \;=\; -q\left(\nabla_x \phi(t,x) + \nabla_x \phi_e(x)\right) .$$

Self-consistent potential: $\phi = K * \rho(f)$
$K = \frac{q}{4\pi\epsilon_0}|x|^{-1}$, $\rho(f)(t,x) \;=\; \int_{\mathbb{R}^3} f(t,x,v)\; dv$.

(H1) f_0 *is a nonnegative function in* $L^1(\mathbb{R}^6)$

Theorem 1 $\phi_e(x) \;\to\; \infty$ *as* $|x| \;\to\; +\infty$, *s.t.* $(x,s) \mapsto s^{3/2-1}\gamma(s+\phi_e(x)) \in L^1 \cap L^\infty(\mathbb{R}^3, L^1(\mathbb{R}))$. *Let* f *be a weak solution of Vlasov-Poisson with* f_0 *in* $L^1 \cap L^{p_0}$, $p_0 = (12+3\sqrt{5})/11$, *s.t.* $\sigma(f_0)$, $(|\phi_e| + |v|^2) f_0 \in L^1(\mathbb{R}^3)$. *If for some* $p \in [1,2]$,

$$\inf_{s \in (0,+\infty)} \sigma''(s)/s^{p-2} > 0$$

$$\|f - f_\infty\|_{L^p}^2 \leq \frac{1}{2}\int_{\mathbb{R}^3} |\nabla(\phi_0 - \phi_\infty)|^2 \, dx$$
$$+C\int_{\mathbb{R}^6}[\sigma(f_0)-\sigma(f_\infty)-\sigma'(f_\infty)(f_0-f_\infty)]d(x,v)$$

where $\Big(f_\infty(x,v) = \gamma(\frac{1}{2}|v|^2 + \phi_e(x) + \phi_\infty(x)), \phi_\infty\Big)$. *Here* $\gamma^{-1} = -\sigma'$.

Value of p_0 [Hörst and Hunze] (weak solutions). Renormalization [DiPerna and Lions, Mischler]. [Braasch, Rein and Vukadinović, 1998]. Csiszár-Kullback inequality: [AMTU, 2000]

Theorem 2 $p_0 = 2$ *and* $\sigma(s) = s\log s - s$. *There exists a convex functional* $\mathcal{F}$ *reaching its minimum at* $f_\infty(x,v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}}\rho_\infty(x)$ *s.t.*

$$\|f(t,\cdot) - f_\infty\|_{L^2}^2 \leq \mathcal{F}[f_0]\,.$$

Here $p=1$, $\gamma(s)=e^{-s}$

$$-\Delta\phi_\infty = \rho_\infty = \|f_0\|_{L^1}\frac{e^{-(\phi_\infty+\phi_e)}}{\int e^{-(\phi_\infty+\phi_e)}\,dx}$$

Properties of the solutions

1. f *is nonnegative* for all $t \geq 0$.
2. *Conservation of mass:*

$$\int_{\mathbb{R}^6} f(t,x,v)\, d(x,v) = \int_{\mathbb{R}^6} f_0(x,v)\, d(x,v) = M\ .$$

3. *Finite energy and entropy:*

$$\int_{\mathbb{R}^6} \left(\frac{1}{2}|v|^2 + \phi_e(x) + \phi(x)\right) f\ d(x,v)$$
$$\leq \int_{\mathbb{R}^6} \left(\frac{1}{2}|v|^2 + \phi_e(x) + \phi_0(x)\right) f_0\ d(x,v)$$

and $\int_{\mathbb{R}^6} f \log f\ d(x,v) \leq \int_{\mathbb{R}^6} f_0 \log f_0\ d(x,v)$,

(equality in the case of classical solutions).

4. $f(t,\cdot)\|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)}$.
5. Moreover, if we assume that

$$\textbf{(H2)} \qquad \int_{\mathbb{R}^6} \sigma(f_0)\ d(x,v) < \infty$$

for some strictly convex continuous function $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$, then for any $t \geq 0$,

$$\int_{\mathbb{R}^6} \sigma(f)\ d(x,v)\ \leq\ \int_{\mathbb{R}^6} \sigma(f_0)\ d(x,v)\ ,$$

(equality in the case of classical solutions).

Stationary solutions, entropy functionals

Let $f \in L^1$: $\phi = \phi[f]$ s.t. $-\Delta\phi = \int_{\mathbb{R}^3} f \; dv$.

$$f_{\infty,\sigma}(x,v) = \gamma\left(\frac{1}{2}|v|^2 + \phi[f_{\infty,\sigma}](x) + \phi_e(x) - \alpha\right)$$

is a stationary solution of Vlasov-Poisson $\Longleftrightarrow$

$$-\Delta\phi_{\infty,\sigma} = G_\sigma(\phi_{\infty,\sigma} + \phi_e - \alpha)$$

with $G_\sigma(\phi) = 4\pi\sqrt{2}\int_0^{+\infty}\sqrt{s}\;\gamma(s+\phi)ds$, has a solution $\phi_{\infty,\sigma} = \phi[f_{\infty,\sigma}]$ s.t. $\int f_{\infty,\sigma} d(x,v) = M$.

Confinement conditions :

(H3) $\sigma \in C^2(\mathbb{R}^+) \cap C^0(\mathbb{R}_0^+)$ *is a bounded from below strictly convex function such that*

$$\lim_{s\to+\infty} \frac{\sigma(s)}{s} = +\infty\;.$$

(H4) $\phi_e : \mathbb{R}^3 \to \mathbb{R}$ *is a measurable bounded from below function such that* $\lim \phi_e(x) = +\infty$ *and* $x \mapsto G_\sigma(\phi_e(x)) = 4\pi\sqrt{2}\int_0^{+\infty}\sqrt{s}\,\gamma(s+\phi_e(x))\,ds$ *is bounded in* $L^1 \cap L^\infty(\mathbb{R}^3)$.

On $L^1_M(\mathbb{R}^6) = \{f \in L^1(\mathbb{R}^6) : f \geq 0 \ \|f\|_{L^1} = M\}$,

$$K_\sigma[f] = \int_{\mathbb{R}^6} \left[\sigma(f) + \left(\tfrac{1}{2}|v|^2 + \phi_e(x)\right) f \right] \, d(x,v) + \tfrac{1}{2} \int_{\mathbb{R}^3} |\, \nabla \phi[f] \,|^2 \, dx \, .$$

Lemma 3 *Under Assumptions* (H3)-(H4), K_σ *is a strictly convex bounded from below functional on* $L^1_M(\mathbb{R}^6)$. *It has a unique global minimum,* $f_{\infty,\sigma}$, *which is a stationary solution.*

$$\Sigma_\sigma[f|f_{\infty,\sigma}] = K_\sigma[f] - K_\sigma[f_{\infty,\sigma}]$$

$$= \int_{\mathbb{R}^6} [\sigma(f) - \sigma(f_{\infty,\sigma}) - \sigma'(f_{\infty,\sigma})(f - f_{\infty,\sigma})] \, d(x,v)$$

$$+ \tfrac{1}{2} \int_{\mathbb{R}^3} |\nabla_x (\phi - \phi_{\infty,\sigma})|^2 \, dx$$

The quantity $\Sigma_\sigma[f|f_{\infty,\sigma}]$ will be called the *relative entropy of* f *with respect to* $f_{\infty,\sigma}$.

Corollary 4 *Consider a renormalized or weak solution f under Assumptions (H1), (H2), (H3) and (H4). Then $\Sigma_\sigma[f(t)|f_{\infty,\sigma}] \le \Sigma_\sigma[f_0|f_{\infty,\sigma}]$.*

Basic examples:
$\sigma_q(s) = s^q$, $\gamma_q(s) = (-s/q)_+{}^{1/(q-1)}$, $q > 1$.
$\sigma_1(s) = s \log s - s$ and $\gamma_1(s) = e^{-s}$.

L^p**-nonlinear stability** – Proof of Theorem 1

Proposition 5 *Let f and g be two nonnegative functions in $L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6)$, $p \in [1,2]$ and consider a strictly convex function $\sigma : \mathbb{R}_0^+ \to \mathbb{R}$ in $C^2(\mathbb{R}^+) \cap C^0(\mathbb{R}_0^+)$. Let*

$$A = \inf \left\{ \sigma''(s)/s^{p-2} : \ s \in (0, \infty) \right\}$$

If $A > 0$, then the following inequality holds:

$$\Sigma_\sigma[f|g] \ge \frac{2^{-2/p} A}{\left[\max\left(\|f\|_{L^p}^{2-p}, \|g\|_{L^p}^{2-p}\right)\right]} \|f - g\|_{L^p}^2 + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x (\phi[f] - \phi[g])|^2 \, dx$$

L^2-Nonlinear stability of maxwellian states

Lemma 6 *Assume that* $e^{-\beta\,\phi_e}$ *is bounded in* $L^1(\mathbb{R}^3)$ *for some* $\beta > 0$. *Let* f *be a nonnegative function in* $L^1 \cap L^q(\mathbb{R}^6)$, $q > 1$, *such that* $(x, v) \mapsto (|v|^2 + \phi_e(x))f(x, v)$ *is bounded in* L^1. *Then* $f \log f$ *is also bounded in* $L^1(\mathbb{R}^6)$.

Proof of Theorem 2. Use a cut-off functional. Let $E_1(x, v) := \frac{1}{2}|v|^2 + \phi_{\infty,1}(x) + \phi_e(x)$.

$$E_{min} := \inf\{E_1(x, v)\} \geq \inf\{\phi_e(x)\} > -\infty\ .$$

Define $m = \varphi \circ E_1$, $\varphi(s) = \kappa\, e^{-s}$,
$\kappa = \frac{M}{(2\pi)^{3/2}}[\int e^{-\phi_{1,\infty}-\phi_e}\, dx]^{-1}$, $\bar{s} = \varphi(E_{min})$ and

$$\tau_1(s) := \begin{cases} s \log s - s & \text{if } s \in [0, \bar{s}] \\ \frac{1}{2\kappa}\, e^{E_{min}}\, (s-\bar{s})^2 - (E_{min} - \log \kappa)(s-\bar{s}) \\ \qquad +\bar{s} \log \bar{s} - \bar{s} & \text{if } s \in (\bar{s}, +\infty) \end{cases}$$

$\square$

2. Vlasov-Poisson system with injection boundary conditions

ω is a bounded domain in $\mathbb{R}^d$, $\partial\omega$ is of class C^1 $\Omega = \omega \times \mathbb{R}^d$ and $\Gamma = \partial\Omega = \partial\omega \times \mathbb{R}^d$ are the phase space and its boundary respectively.

$$\Sigma^{\pm}(x) = \{v \in \mathbb{R}^d \ : \ \pm\, v \cdot \nu(x) > 0\}$$

$$\Gamma^{\pm} = \{(x,v) \in \Gamma \ : \ v \in \Sigma^{\pm}(x)\}\ .$$

Vlasov-Poisson system with injection boundary conditions

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v f = 0 \\ f_{|t=0} = f_0\,,\ f_{|\Gamma^- \times \mathbf{R}^+}(x,v,t) = \gamma(\frac{1}{2}|v|^2 + \phi_0(x))\,, \\ -\Delta\phi = \rho = \displaystyle\int_{\mathbf{R}^d} f\, dv\,, (x,t) \in \omega \times \mathbb{R}^+, \\ \qquad \text{and} \quad \phi(x,t) = 0\,, (x,t) \in \partial\omega \times \mathbb{R}^+. \end{cases}$$

Assumptions:

(H5) *The initial condition* f_0 *is a nonnegative bounded function.*

(H6) *The external electrostatic potential is nonnegative,* $C^2(\overline{\omega})$*.*

(H7) *The function* γ *has* Property $\mathcal{P}$*: it is defined on* $(\min_{x\in\omega}\phi_0(x), +\infty)$*, bounded, smooth, strictly decreasing with values in* $\mathbb{R}_*^+$*, and rapidly decreasing at infinity, so that*

$$\sup_{x\in\omega}\int_0^{+\infty} s^{d/2}\gamma(s+\phi_0(x))\, ds < +\infty\ .$$

γ is strictly decreasing: $\beta_\gamma = -\int_0^g \gamma^{-1}(z)\, dz$ is strictly convex.

Let $U : L^1(\Omega) \to W_0^{1,d/(d-1)}(\omega)$: $U[g] = u$ the unique solution in $W_0^{1,d/(d-1)}(\omega)$ of

$$-\Delta u = \int_{\mathbb{R}^d} g(x,v)\, dv \;.$$

The operator U is linear and satisfies

$$\textstyle\int_\Omega g\, U[f]\, dxdv = \int_\Omega f\, U[g]\, dxdv \;,$$

$$\textstyle\int_\Omega f\, U[f]\, dxdv = \int_\omega |\, \nabla_x U[f]\, |^2 \; dx \;.$$

$M(x,v) = \gamma\left(\frac{1}{2}|v|^2 + U[M](x) + \phi_0(x)\right)$ is a stationary solution:

$$v \cdot \nabla_x M - (\nabla_x \phi_0 + \nabla_x U[M]) \cdot \nabla_v M = 0$$

It is the unique critical point in $H_0^1(\omega)$ of the strictly convex coercive functional

$$U \mapsto \frac{1}{2}\int_\omega |\nabla U|^2\, dx - \int_\omega G(U+\phi_0)\, dx \;,$$

where $G'(u) = g(u) = \int_{\mathbb{R}^d} \gamma\left(\frac{1}{2}|v|^2 + u\right)\; dv$

$$= 2^{d/2-1}|S^{d-1}| \cdot \textstyle\int_0^{+\infty} s^{d/2-1}\gamma(s+u)\, ds$$

Relative entropy

$$\Sigma_\gamma[g|h] = \int_\Omega \Big(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h)\Big)\ dxdv$$

$$+\tfrac{1}{2}\int_\omega |\nabla U[g-h]|^2\ dx$$

β_γ is the real function defined by

$$\beta_\gamma(g) = -\int_0^g \gamma^{-1}(z)\ dz\ .$$

$$\int_\Omega f(\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[f] + \phi_0)\,dxdv$$

$$= \int_\Omega \Big[\tfrac{1}{2}(f-M)U[f-M] - \tfrac{1}{2}MU[M] - f\beta'_\gamma(M)\Big]\,dxdv$$

Thus

$$\begin{aligned}&\Sigma_\gamma[f|M]\\ &= \int_\Omega \Big(\beta_\gamma(f) + (\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[f] + \phi_0)f\Big)dxdv\\ &\quad - \int_\Omega \Big(\beta_\gamma(M) + (\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[M] + \phi_0)M\Big)dxdv\end{aligned}$$

Relative entropy and irreversibility

Theorem 7 *Assume that* $f_0 \in L^1 \cap L^\infty$ *is a nonnegative function such that* $\Sigma_\gamma[f_0|M] < +\infty$*. Under Assumptions (H5)-(H7),*

$$\frac{d}{dt}\Sigma_\gamma[f(t)|M] = -\Sigma_\gamma^+[f(t)|M]$$

Σ_γ^+ *is the boundary relative entropy flux*

$$\Sigma_\gamma^+[g|h] = \int_{\Gamma^+} \Big(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta_\gamma'(h)\Big)\, d\sigma\ .$$

Proof.

$$\frac{d}{dt}\int_\Omega \beta(f)\, dxdv = \sum_{\pm} \mp \int_{\Gamma^\mp} \beta(f)\, d\sigma$$

$$\frac{d}{dt}\int_\Omega f\left(\frac{1}{2}|v|^2 + \frac{1}{2}U[f] + \phi_0\right)\, dxdv$$

$$= \sum_\pm \pm \int_{\Gamma^\mp} f\left(\frac{1}{2}|v|^2 + \phi_0\right)\, d\sigma$$

$$\int_{\Gamma^\pm} f\left(\frac{1}{2}|v|^2 + \phi_0(x)\right)\, d\sigma = -\int_{\Gamma^\pm} f\beta_\gamma'(M)\, d\sigma \quad \square$$

The large time limit

$$\boxed{\Sigma_\gamma[f(t)|M] + \int_0^t \Sigma_\gamma^+[f(s)|M]ds \le \Sigma_\gamma[f_0|M]}$$

$$(f^n(x,v,t), \phi^n(x,t)) = (f(x,v,t+t_n), \phi(x,t+t_n))$$

$$\lim_{n\to+\infty} \int_{\mathbf{R}^+} \Sigma_\gamma^+[f^n(s)|M]\, ds = 0$$

$$\sup_{t>0} \Sigma_\gamma[f^n(t)|M] \le C\ .$$

$+$ Dunford-Pettis criterion: $(f^n, \phi^n) \rightharpoonup (f^\infty, \phi^\infty)$ weakly in $L^1_{\mathrm{loc}}(dt, L^1(\Omega)) \times L^1_{\mathrm{loc}}(dt, H^1_0(\omega))$.

$$\boxed{f^\infty \equiv M \text{ on } \Gamma}$$

Is f^∞ stationary ? A partial answer for $d = 1$.

Theorem 8 *Assume that γ satisfies Property ($\mathcal{P}$) and consider a solution (f^∞, ϕ^∞) such that $f^\infty \equiv M$ on Γ on the interval $\omega = (0,1)$ ($d = 1$). If ϕ_0 is analytic in x with C^∞ (in time) coefficients and if ϕ_0 is analytic with $-\frac{d^2\phi_0}{dx^2} \geq 0$ on ω, then (f, ϕ) is the unique stationary solution, given by: $f = M$, $\phi = U[M]$.*

<u>*Proof.*</u> Let $\phi_0(0) = 0$, $\phi_0(1) \geq 0$: $\phi_0'(0) \geq 0$. Characteristics

$$\frac{\partial X}{\partial t} = V \;, \qquad \frac{\partial V}{\partial t} = -\frac{\partial \phi}{\partial x}(X,t) - \frac{d\phi_0}{dx}(X) \;,$$

$$X(s; x, v, s) = x \;, \quad V(s; x, v, s) = v$$

are defined on $(\mathcal{T}_{in}(s;x,v), \mathcal{T}_e(s;x,v))$: either $\mathcal{T}_{in}(s;x,v) = -\infty$ or $(X_{in}, V_{in})(s;x,v) \in \Gamma^-$; either $\mathcal{T}_e(s;x,v) = +\infty$ or $(X_e, V_e)(s;x,v) \in \Gamma^+$.
Step 1: the electric field is repulsive at $x = 0$.

Step 2 : Analysis of the characteristics in a neighborhood of $(0, 0, t)$.

$$f(X_{in}, V_{in}, \mathcal{T}_{in}) = \gamma\left(\frac{1}{2}\,|V_{in}|^2\right)$$
$$f(X_e, V_e, \mathcal{T}_e) = \gamma\left(\frac{1}{2}\,|V_e|^2\right)\ .$$

The function γ is strictly decreasing:

$$|V_{in}(t_0, x_0, 0)| = |V_e(t_0, x_0, 0)|$$

Characteristics are parametrized by

$$\frac{dt^{\pm}}{dX} = \frac{1}{V}\ , \quad \frac{dV}{dX} = -\frac{1}{V}\frac{\partial \Phi}{\partial x}(X, t^{\pm})\ .$$

Let $e_{\pm}(X) = \frac{1}{2}V^2(X)$.

$$t^{\pm}(X) = t_0 \mp \int_{x_0}^{X} \frac{dY}{\sqrt{2e_{\pm}(Y)}} \quad \forall\ X \in [0, x_0]\ ,$$

$$\frac{de_{\pm}}{dX} = -\frac{\partial \Phi}{\partial x}(X, t^{\pm}(X))\ , \quad e_{\pm}(x_0) = 0\ .$$

Rescaling: $x_0 = \varepsilon^2$, $X = \varepsilon^2(1 - x)$ and $e_{\pm}(X) := \frac{\varepsilon^2}{2}\, e_{\pm}^{\varepsilon}(x)$

$$\frac{de^{\varepsilon}_{\pm}}{dx} = 2\,\frac{\partial \Phi}{\partial x}\left(\varepsilon^2(1-x), t_0 \pm \varepsilon \int_0^x \frac{dy}{\sqrt{e^{\varepsilon}_{\pm}(y)}}\right)$$

with the condition $e^{\varepsilon}_{+}(1) = e^{\varepsilon}_{-}(1)$ for any $\varepsilon > 0$ small enough.

$$e^{\varepsilon}_{\pm} = \sum_{n=0}^{+\infty} \varepsilon^n e^{\pm}_n \, .$$

Lemma 9 *With the above notations, for all* $n \in \mathbb{N}$, *we have the following identities:*

(i) $\dfrac{de^{\pm}_{2n}}{dx}(x) = \dfrac{2(1-x)^n}{n!}\,\partial_x^{n+1}\Phi(0, t_0)$,

(ii) $\dfrac{de^{\pm}_{2n+1}}{dx}(x) = \dfrac{\pm 2(1-x)^n}{(n+1)!}\left(\displaystyle\int_0^x \frac{dy}{\sqrt{e_0(y)}}\right)$

$\partial_t\,\partial_x^{n+1}\Phi(0, t_0)$,

(iii) $\partial_t\,\partial_x^{n+1}\Phi(0, t_0) = 0$.

$\square$

3. With collision kernels

Vlasov-Poisson-Boltzmann system with injection boundary conditions

$$
\begin{cases}
\partial_t f + v\cdot\nabla_x f - (\nabla_x\phi + \nabla_x\phi_0)\cdot\nabla_v f = Q(f) \\
f_{|t=0} = f_0\,,\ f_{|\Gamma^-\times\mathbb{R}^+}(x,v,t) = \gamma(\tfrac{1}{2}|v|^2 + \phi_0(x))\,, \\
-\Delta\phi = \rho = \int_{\mathbb{R}^d} f\,dv\,, (x,t)\in\omega\times\mathbb{R}^+, \\
\qquad \text{and} \quad \phi(x,t) = 0\,, (x,t)\in\partial\omega\times\mathbb{R}^+.
\end{cases}
$$

Additional assumptions:

(H8) *The collision operator Q preserves the mass $\int_{\mathbb{R}^d} Q(g)\,dv = 0$, and satisfies the H-theorem*

$$
D[g] = -\int_{\mathbb{R}^d} Q(g)\left[\frac{1}{2}|v|^2 - \gamma^{-1}(g)\right]\,dv \geq 0\,,
$$

for any nonnegative function g in $L^1(\mathbb{R}^d)$.

(H9) $D[g] = 0 \Longleftrightarrow Q(g) = 0$

Examples

1. Pure Vlasov-Poisson system.

The inflow function γ is arbitrary.

2. The Vlasov-Poisson-Fokker-Planck system.

$$Q_{FP}(f) = \mathrm{div}_v(vf + \theta \nabla_v f)$$

$$Q_{FP,\alpha}(f) = \mathrm{div}_v(vf(1-\alpha f) + \theta \nabla_v f)$$

Take $M_\theta = (2\pi\theta)^{-d/2} e^{-|v|^2/(2\theta)}$

$$\gamma(u) = \frac{1}{\alpha + e^{(u-\mu)/\theta}}$$

Lemma 10 *[H-theorem] $\alpha \geq 0$, $0 \leq f \leq \alpha^{-1}$.*

$$\mathcal{H}(f) = \int_{\mathbf{R}^d} Q_{FP,\alpha}(f) \log\left(\frac{f}{(1-\alpha f)M_\theta}\right) dv \leq 0$$

and $\mathcal{H}(f) = 0 \iff Q_{FP,\alpha(f)} = 0 \iff f(v) = \left(\alpha + e^{(|v|^2/2-\mu)/\theta}\right)^{-1}$.

3. BGK approx. of the Boltzmann operator.

$$Q_\alpha(f) = \int_{\mathbf{R}^d} \sigma(v,v')[M_\theta(v) f(v')(1-\alpha f(v)) - M_\theta(v') f(v)(1-\alpha f(v'))]\, dv'$$

H-theorem: same as for the Vlasov-Poisson-Fokker-Planck system.

4. Linear elastic collision operator.

$$Q_E(f) = \int_{\mathbf{R}^d} \chi\, (f(v') - f(v))\, \delta(|v'|^2 - |v|^2)\, dv'\ ,$$

$\chi(v,v')$ is a symmetric positive cross-section. Let $\lambda(v) = \int_{\mathbf{R}^d} \chi(v,v')\, \delta(|v|^2 - |v'|^2)\, dv' \in L^\infty$.

Lemma 11 *Q_E is bounded on $L^1 \cap L^\infty(\mathbb{R}^d)$. For any function ψ and H (increasing)*

$$\int_{\mathbf{R}^d} Q_E(f) \cdot \psi(|v|^2)\, dv = 0$$
$$\mathcal{H}(f) = \int_{\mathbf{R}^d} Q_E(f) \cdot H(f)\, dv \le 0\ .$$

If H is strictly increasing, $\mathcal{H}(f) = 0 \Longleftrightarrow Q_E(f) = 0 \Longleftrightarrow f(v) = \psi(|v|^2)$. γ is arbitrary.

5. Electron-Electron collision operator.

Boltzmann collision operator $Q^B_{ee,\alpha}$ or Fokker-Planck-Landau collision operator $Q^L_{ee,\alpha}$.

Relative entropy and irreversibility

Theorem 12 *Assume that* $f_0 \in L^1 \cap L^\infty$ *is a nonnegative function such that* $\Sigma_\gamma[f_0|M] < +\infty$. *Under Assumptions (H5)-(H7),*

$$\frac{d}{dt}\Sigma_\gamma[f(t)|M] = -\Sigma_\gamma^+[f(t)|M] - \int_\omega D[f](x,t)\,dx$$

Σ_γ^+ *is the boundary relative entropy flux*

$$\Sigma_\gamma^+[g|h] = \int_{\Gamma^+} \Big(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta_\gamma'(h)\Big)\,d\sigma\;.$$

<u>*Proof.*</u>

$$\begin{aligned}&\tfrac{d}{dt}\textstyle\int_\Omega \beta(f)\,dxdv\\ &= \textstyle\sum_\pm \mp \int_{\Gamma^\mp} \beta(f)\,d\sigma + \int_\Omega \beta'(f)\,Q(f)\,dxdv\end{aligned}$$

$$\begin{aligned}&\tfrac{d}{dt}\textstyle\int_\Omega f\left(\frac{1}{2}|v|^2 + \frac{1}{2}U[f] + \phi_0\right)dxdv\\ &= \textstyle\sum_\pm \pm \int_{\Gamma^\mp} f\left(\frac{1}{2}|v|^2 + \phi_0\right)d\sigma + \int_\Omega \frac{1}{2}|v|^2 Q(f)\,dxdv\end{aligned}$$

$$\textstyle\int_{\Gamma^\pm} f\left(\frac{1}{2}|v|^2 + \phi_0(x)\right)d\sigma = -\int_{\Gamma^\pm} f\beta_\gamma'(M)\,d\sigma \quad \square$$

The large time limit

$$\Sigma_\gamma[f(t)|M] + \int_0^t \Sigma_\gamma^+[f(s)|M]ds + \int_0^t\int_\omega D[f(x,\cdot,s]\,dx\,ds \;\le\; \Sigma_\gamma[f_0|M]$$

$$(f^n(x,v,t),\phi^n(x,t)) = (f(x,v,t+t_n),\phi(x,t+t_n))$$

$$\lim_{n\to+\infty}\int_{\mathbf{R}^+}\Sigma_\gamma^+[f^n(s)|M]\;ds \;= 0$$

$$\lim_{n\to+\infty}\int_{\mathbf{R}^+}\int_\omega D[f^n(x,\cdot,s)]\,dx\,ds = 0$$

$$\sup_{t>0}\Sigma_\gamma[f^n(t)|M] \le C\ .$$

+ Dunford-Pettis criterion: $(f^n,\phi^n) \rightharpoonup (f^\infty,\phi^\infty)$ weakly in $L^1_{\text{loc}}(dt, L^1(\Omega)) \times L^1_{\text{loc}}(dt, H^1_0(\omega))$.

$$f^\infty \equiv M \text{ on } \Gamma$$

Ex. 2, 3, 4: $f^\infty \in \mathrm{Ker\,Q}$ depends only on $|v|^2$

Lemma 13 *Let $f \in L^1_{loc}$ be a solution. If f is even (or odd) with respect to the v variable, then it does not depend on t.*

Proof. The operator ∂_t conserves the v parity while $v \cdot \nabla_x - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v$ transforms the v parity into its opposite. □

Corollary 14 *Let f be a solution s.t. $f \equiv M$ on Γ, with $Q = Q_E$, $Q_{FP,\alpha}$, Q_α, $Q_{ee,\alpha} + Q_E$, $Q_{ee,\alpha} + Q_\alpha$, $Q_{ee,\alpha} + Q_{FP,\alpha}$ or a combination of these operators. Then on Ω, $f \equiv M$ is stationary if there are no closed characteristics.*

Theorem 15 *Assume that $\alpha \geq 0$. Let $0 \leq f \leq F_D(x,v) = \left(\alpha + e^{(\frac{1}{2}|v|^2 + \phi_0(x) - \mu)/\theta}\right)^{-1}$ on Γ^-.*

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_0 \cdot \nabla_v f = Q_\alpha(f)$$
$$f(x,v) = g(x,v)\ , \quad (x,v) \in \Gamma^-$$

Any solution such that $0 \leq f_0 \leq F_D$ converges to the unique stationary solution.

4. Other boundary conditions: entropies

Diffuse reflection boundary conditions for f. For any $(x,t) \in \partial\omega \times \mathbb{R}^+$, let

$$\rho_+(x,t) := \int_{\Sigma^+(x)} f(x,v,t)\, v \cdot \nu(x)\, dv \ .$$

There exists a unique function $\mu : \partial\omega \times \mathbb{R}^+ \to \mathbb{R}$ s.t.

$$\rho_+ = \int_{\sum^-(x)} \gamma \left(\frac{1}{2}|v|^2 + \phi_0(x) - \mu(x,t) \right) |v \cdot \nu(x)|\, dv$$

With the notation

$$m_f(x,v,t) := \gamma \left(\frac{1}{2}|v|^2 + \phi_0(x) - \mu(x,t) \right) \ ,$$

we shall say that f is subject to *diffuse reflection boundary conditions* (DRBC) iff

$$f(x,v,t) = m_f(x,v,t)\ , \quad \forall\ (x,v,t) \in \Gamma^- \times \mathbb{R}^+ \ .$$

Note that under this condition, the total mass is preserved: $\frac{d}{dt} \int_\Omega f(x,v,t)\, dx dv = 0$.

Lemma 16 *For any $\mathcal{M} > 0$, the system with diffuse reflection boundary conditions has at least one nonnegative stationary solution*

$$M(x,v) = \gamma\left(\frac{1}{2}|v|^2 + \phi_0(x) + U[M] - \mu_{\mathcal{M}}\right)$$

such that $\|M\|_{L^1(\Omega)} = \mathcal{M}$.

$$\begin{aligned}\Sigma_\gamma[g|h] = \textstyle\int_\Omega(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h))dxdv \\ +\tfrac{1}{2}\textstyle\int_\omega |\nabla U[g-h]|^2\, dx \\ \Sigma_\gamma^+[g|h] = \textstyle\int_{\Gamma^+}(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h))\, d\sigma\end{aligned}$$

Theorem 17 *Let (f,ϕ) be a solution with diffuse reflection boundary conditions, $\mathcal{M} = \|f_0\|_{L^1}$. Under (H5)-(H8) hold, if $\lim_{s\to-\infty}\gamma(s) = +\infty$. Then the relative entropy satisfies*

$$\frac{d}{dt}\Sigma_\gamma[f(t)|M] = -\Sigma_\gamma^+[f|m_f] + \int_\omega D[f(x,\cdot,s]\, dx$$

If there are no closed trajectories and if ω is connected , then there exists a unique continuous nonnegative stationary solution.

5. Relation with nonlinear diffusion equations

Consider a solution in $C^0(\mathbb{R}^+, L^1_+(\mathbb{R}^d))$ of

$$\rho_t = \Delta\,\nu(\rho) + \nabla\,(\rho\nabla\phi_0) + \nabla\cdot(\rho\nabla\phi)$$

$\phi = V[\rho] = |S^{d-1}|^{-1}\,|x|^{-(d-2)} * \rho$
$\Gamma^{-1}(u) = -\nu'(u)/u$ is nonnegative decreasing
$\beta'(u) = -\int_0^u \Gamma^{-1}(z)\,dz = \int_0^u \frac{\nu'(z)}{z}\,dz$

$$\rho(t,\cdot) \to \rho_\infty(x) = \Gamma(\phi_0 + V[\rho_\infty] - \mu)$$

$$\Sigma[\rho|\rho_\infty] = \int_{\mathbf{R}^d} \left(\beta(\rho) - \beta(\rho_\infty) - \beta'(\rho_\infty)(\rho-\rho_\infty)\right) dx + \frac{1}{2}\int_{\mathbf{R}^d} |\nabla V[\rho-\rho_\infty]|^2\,dx$$

$$\frac{d}{dt}\Sigma[\rho|\rho_\infty] = -I[\rho]$$

$I[\rho] = \int_{\mathbf{R}^d} \rho \left|\nabla\phi_0 + \nabla V[\rho] - \nabla(\Gamma^{-1}(\rho))\right|^2\,dx$
Csiszár-Kullback inequality:

$$\Sigma[\rho|\rho_\infty] \geq C\,\|\rho-\rho_\infty\|^2_{L^1(\mathbf{R}^d)} + \frac{1}{2}\,\|\nabla U[\rho-\rho_\infty]\|^2_{L^2(\mathbf{R}^d)}$$

The nonlinear diffusion equation can be obtained from a singular perturbation of a kinetic equation. Let γ satisfy Property $\mathcal{P}$ and consider a function $g(v)$:

$$m_g(v) = \gamma\left(\frac{1}{2}|v|^2 + a_g\right)$$

such that $\int_{\mathbf{R}^d} m_g(v)\,dv = \int_{\mathbf{R}^d} g(v)\,dv$.

Consider the Vlasov-Poisson-Boltzmann system

$$\partial_t f^\eta + \frac{1}{\eta}[v\cdot\nabla_x f^\eta - (\nabla_x\phi_0 + \varepsilon\nabla_x\phi)\cdot\nabla_v f^\eta] = \frac{1}{\eta^2}Q(f^\eta)$$

where $Q(g) = m_g - g$. H-Theorem

$$\frac{d}{dt}\Sigma_\gamma[f^\eta(\cdot,\cdot,t)|M] = -\frac{1}{\eta^2}\int_{\mathbf{R}^d} D[f^\eta(x,\cdot,t)]\,dx\ .$$

$$\begin{aligned} -D[g] &= \int_{\mathbf{R}^d} Q(g)\left[\tfrac{1}{2}|v|^2 - \gamma^{-1}(g)\right]\,dv \\ &= \int_{\mathbf{R}^d}(m_g - g)\left(\gamma^{-1}(m_g) - \gamma^{-1}(g)\right)\,dv \le 0 \end{aligned}$$

Formal limit as $\eta\to 0$: $f^0 = \gamma(\frac{1}{2}|v|^2 + a_{f_0}(x,t))$

$\rho(x,t) = \int_{\mathbf{R}^d} f^0(x,v,t)\,dv$ *satisfies the nonlinear diffusion equation with*

$$\nu(u) = -\int_0^u s\,(\Gamma^{-1})'(s)\,ds$$
$$\Gamma(u) = \int_{\mathbf{R}^d} \gamma\left(\tfrac{1}{2}|v|^2 + u\right)\,dv$$
$$\rho(x,t) = \Gamma(a_{f^0}(x,t)).$$

Proof.

$$\partial_t\,\rho^\eta + \mathrm{div}_x j^\eta = 0$$

with $\rho^\eta = \int_{\mathbf{R}^d} f^\eta\,dv$ and $j^\eta = \frac{1}{\eta}\int_{\mathbf{R}^d} v f^\eta\,dv$

$$-j^\eta = \eta^2\partial_t\,j^\eta + \mathrm{div}_x\left[\int v\otimes v\;f^\eta\,dv\right] + (\nabla\phi_0 + \nabla\phi^\eta)\,\rho^\eta$$

$$\to \mathrm{div}_x\left[\int v\otimes\;\gamma\Big(\tfrac{1}{2}|v|^2 + a_{f^0}(x,t)\Big)\,dv\right] + (\nabla\phi_0 + \nabla\phi)\,\rho$$
$$= \frac{1}{d}\,\nabla_x \int_{\mathbf{R}^d} |v|^2\;\gamma\Big(\tfrac{1}{2}|v|^2 + a_{f^0}\Big)\;dv$$
$$= -(\Gamma\circ a_{f^0})\,\nabla_x\,a_{f^0} = \nabla_x\,\nu(\rho).$$

This formal limit holds not only at the level of the equations but also for the *relative entropy*.

$$\Sigma_\gamma[f^0|M] = \Sigma[\rho|\rho_\infty]$$
$$\frac{1}{\eta^2}\int_{\mathbf{R}^d} D[f^\eta]\,dx \to I[\rho]$$

Hilbert expansion for f^η:

$$f^\eta = f^0 + \eta f_1 + O(\eta^2)\ .$$

$$D_{f^0}Q(f_1) = v\cdot\nabla_x f^0 - (\nabla_x\phi_0 + \nabla_x\phi)\cdot\nabla_v f^0 = -f_1$$

$$\begin{aligned}
&\frac{1}{\eta^2}\int D[f^\eta]\,dx = -\frac{1}{\eta^2}\int Q(f^\eta)\left[\frac{1}{2}|v|^2 - \gamma^{-1}(f^\eta)\right]dvdx\\
&\to \int D_{f^0}Q(f_1)\cdot(\gamma^{-1})'(f^0)\,f_1\,dv\,dx\\
&= -\int (f_1)^2(\gamma^{-1})'(f^0)\,dv\,dx\\
&= -\int\{\int \gamma'\left(\frac{1}{2}|v|^2 + a\right)\\
&\qquad |(-\nabla_x a_{f^0} + \nabla_x\phi_0 + \nabla_x\phi)\cdot v|^2\,dv\}\,dx\\
&= -\frac{1}{d}\int\left\{\int |v|^2\gamma'\left(\frac{1}{2}|v|^2 + a\right)\,dv\right\}\\
&\qquad |-\nabla_x a_{f^0} + \nabla_x\phi_0 + \nabla_x\phi|^2\,dx\\
&= -\int \rho\,|-\nabla_x a_{f^0} + \nabla_x\phi_0 + \nabla_x\phi|^2\,dx = I[\rho]
\end{aligned}$$

Example. Fermi-Dirac statistics

$$f_\infty(x,v) = \gamma\left(\frac{1}{2}|v|^2 + \phi_0 + \phi - \mu\right)$$

$\gamma(u) = (\alpha + e^u)^{-1}$, where $\alpha > 0$ is a parameter related to Planck's constant.

$$\rho_\infty(x) = \Gamma(\phi_0 + \phi - \mu)$$

is the unique equilibrium density of

$$\rho_t = \nabla \cdot (\nabla \nu(\rho) + \rho \nabla \phi)\ ,$$

where $\nu(u) = -\int_0^u s\,(\Gamma^{-1})'(s)\,ds$ and

$$\Gamma(u) = |S^{d-1}| \int_0^{+\infty} (2s)^{d/2-1}\,\gamma\,(s+u)\,ds\ .$$

Proof. Taylor development at order two

$$\Sigma_\sigma[f|g] = \frac{1}{2}\int_{\mathbb{R}^6} \sigma''(\xi)|f-g|^2 \; d(x,v) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x(\phi[$$

$$\geq \frac{A}{2}\int_{\mathbb{R}^6} \xi^{p-2}|f-g|^2 \; d(x,v) + \frac{1}{2}\int_{\mathbb{R}^3} |\nabla_x(\phi[f$$

Hölder's inequality, for any $h > 0$ and for any measurable set $\mathcal{A} \subset \mathbb{R}^6$,

$$\int_{\mathcal{A}} |f-g|^p \, h^{-\alpha} \, h^{\alpha} \; d(x,v) \leq \left(\int_{\mathcal{A}} |f-g|^2 \, h^{p-2} \; d(x,v)\right)^{p/2}$$

with $\alpha = p\,(2-p)/2$, $s = 2/(2-p)$.

$$\left(\int_{\mathcal{A}} |f-g|^2 \, h^{p-2} \; d(x,v)\right)^{p/2} \geq \left(\int_{\mathcal{A}} |f-g|^p \; d(x,v)\right) \left(\int\right.$$

[i)] On $\mathcal{A} = \mathcal{A}_1 = \{(x,v) \in \mathbb{R}^6 : f(x,v) > g(x,v)\}$, use $\xi^{p-2} > f^{p-2}$ and take $h = f$:

$$\left(\int_{\mathcal{A}_1} |f-g|^2 \xi^{p-2}\ d(x,v)\right)^{p/2} \geq \left(\int_{\mathcal{A}_1} |f-g|^p\ d(x,v)\right)$$

[ii)] On $\mathcal{A} = \mathcal{A}_2 = \{(x,v) \in \mathbb{R}^6 : f(x,v) \leq g(x,v)\}$, use $\xi^{p-2} \geq g^{p-2}$ and take $h = g$:

$$\left(\int_{\mathcal{A}_2} |f-g|^2 \xi^{p-2}\ d(x,v)\right)^{p/2} \geq \left(\int_{\mathcal{A}_2} |f-g|^p\ d(x,v)\right)$$

$\square$