Nonlinear stability results for kinetic equations

with

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Bilbao, november 2001
1. Vlasov-Poisson: Nonlinear stability

Distribution function \( f(t, x, v) \geq 0 \)
density of particles at \((x, v, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+ \).

Liouville equation

\[
\frac{\partial f}{\partial t} + v \cdot \nabla_x f + F(t, x) \cdot \nabla_v f = 0
\]

\[
F(t, x) = -q(\nabla_x \phi(t, x) + \nabla_x \phi_e(x)).
\]

Self-consistent potential: \( \phi = K * \rho(f) \)

\[
K = \frac{q}{4\pi\varepsilon_0}|x|^{-1}, \quad \rho(f)(t, x) = \int_{\mathbb{R}^3} f(t, x, v) \, dv.
\]

(H1) \( f_0 \) is a nonnegative function in \( L^1(\mathbb{R}^6) \)

Theorem 1 \( \phi_e(x) \to \infty \) as \( |x| \to +\infty \), s.t.

\((x, s) \mapsto s^{3/2-1}\gamma(s+\phi_e(x)) \in L^1 \cap L^\infty(\mathbb{R}^3, L^1(\mathbb{R}))\).

Let \( f \) be a weak solution of Vlasov-Poisson with \( f_0 \) in \( L^1 \cap L^{p_0} \), \( p_0 = \frac{(12 + 3\sqrt{5})}{11} \), s.t.

\( \sigma(f_0), (|\phi_e| + |v|^2)f_0 \in L^1(\mathbb{R}^3) \).

If for some \( p \in [1, 2] \),

\[
\inf_{s \in (0, +\infty)} \frac{\sigma''(s)}{s^{p-2}} > 0
\]
\[ \| f - f_\infty \|_{L^p}^2 \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla (\phi_0 - \phi_\infty)|^2 \, dx \]
\[ + C \int_{\mathbb{R}^6} [\sigma(f_0) - \sigma(f_\infty) - \sigma'(f_\infty)(f_0 - f_\infty)] d(x, v) \]

where \( f_\infty(x, v) = \gamma(\frac{1}{2}|v|^2 + \phi_e(x) + \phi_\infty(x)), \phi_\infty \). Here \( \gamma^{-1} = -\sigma' \).


**Theorem 2** \( p_0 = 2 \) and \( \sigma(s) = s \log s - s \).

There exists a convex functional \( F \) reaching its minimum at \( f_\infty(x, v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}} \rho_\infty(x) \) s.t.

\[ \| f(t, \cdot) - f_\infty \|_{L^2}^2 \leq F[f_0] \]

Here \( p=1, \gamma(s) = e^{-s} \)

\[-\Delta \phi_\infty = \rho_\infty = \| f_0 \|_{L^1} \frac{e^{-(\phi_\infty + \phi_e)}}{\int e^{-(\phi_\infty + \phi_e)} \, dx} \]
Properties of the solutions

1. \( f \) is nonnegative for all \( t \geq 0 \).
2. Conservation of mass:
   \[
   \int_{\mathbb{R}^6} f(t, x, v) \, d(x, v) = \int_{\mathbb{R}^6} f_0(x, v) \, d(x, v) = M.
   \]
3. Finite energy and entropy:
   \[
   \int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + \phi_e(x) + \phi(x) \right) f \, d(x, v) \\
   \leq \int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + \phi_e(x) + \phi_0(x) \right) f_0 \, d(x, v)
   \]
   and
   \[
   \int_{\mathbb{R}^6} f \log f \, d(x, v) \leq \int_{\mathbb{R}^6} f_0 \log f_0 \, d(x, v),
   \]
   (equality in the case of classical solutions).
4. \( f(t, \cdot) \|_{L^\infty(\mathbb{R}^6)} \leq \|f_0\|_{L^\infty(\mathbb{R}^6)} \).
5. Moreover, if we assume that
   \[
   (H2) \quad \int_{\mathbb{R}^6} \sigma(f_0) \, d(x, v) < \infty
   \]
   for some strictly convex continuous function \( \sigma : \mathbb{R}^+_0 \to \mathbb{R} \), then for any \( t \geq 0 \),
   \[
   \int_{\mathbb{R}^6} \sigma(f) \, d(x, v) \leq \int_{\mathbb{R}^6} \sigma(f_0) \, d(x, v),
   \]
   (equality in the case of classical solutions).
Stationary solutions, entropy functionals

Let \( f \in L^1 \): \( \phi = \phi[f] \) s.t. \(-\Delta \phi = \int_{\mathbb{R}^3} f \ dv\).

\( f_{\infty,\sigma}(x,v) = \gamma \left( \frac{1}{2} |v|^2 + \phi[f_{\infty,\sigma}](x) + \phi_e(x) - \alpha \right) \)

is a stationary solution of Vlasov-Poisson \( \iff \)

\(-\Delta \phi_{\infty,\sigma} = G_\sigma(\phi_{\infty,\sigma} + \phi_e - \alpha) \)

with \( G_\sigma(\phi) = 4\pi \sqrt{2} \int_0^{+\infty} \sqrt{s} \gamma(s + \phi) ds \), has a solution \( \phi_{\infty,\sigma} = \phi[f_{\infty,\sigma}] \) s.t. \( \int f_{\infty,\sigma} d(x,v) = M \).

**Confinement conditions:**

(H3) \( \sigma \in C^2(\mathbb{R}^+) \cap C^0(\mathbb{R}_0^+) \) is a bounded from below strictly convex function such that

\[
\lim_{s \to +\infty} \frac{\sigma(s)}{s} = +\infty.
\]

(H4) \( \phi_e : \mathbb{R}^3 \to \mathbb{R} \) is a measurable bounded from below function such that \( \lim \phi_e(x) = +\infty \)

and \( x \mapsto G_\sigma(\phi_e(x)) = 4\pi \sqrt{2} \int_0^{+\infty} \sqrt{s} \gamma(s + \phi_e(x)) \) ds

is bounded in \( L^1 \cap L^\infty(\mathbb{R}^3) \).
On $L^1_M(\mathbb{R}^6) = \{ f \in L^1(\mathbb{R}^6) : f \geq 0, \|f\|_{L^1} = M \}$,

$$K_\sigma[f] = \int_{\mathbb{R}^6} \left[ \sigma(f) + \left( \frac{1}{2}|v|^2 + \phi_e(x) \right) f \right] d(x, v)$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi[f]|^2 \, dx.$$

**Lemma 3** Under Assumptions (H3)-(H4), $K_\sigma$ is a strictly convex bounded from below functional on $L^1_M(\mathbb{R}^6)$. It has a unique global minimum, $f_{\infty, \sigma}$, which is a stationary solution.

\[
\Sigma_\sigma[f|f_{\infty, \sigma}] = K_\sigma[f] - K_\sigma[f_{\infty, \sigma}]
\]

\[
= \int_{\mathbb{R}^6} [\sigma(f) - \sigma(f_{\infty, \sigma}) - \sigma'(f_{\infty, \sigma})(f - f_{\infty, \sigma})] \, d(x, v)
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla x (\phi - \phi_{\infty, \sigma})|^2 \, dx.
\]

The quantity $\Sigma_\sigma[f|f_{\infty, \sigma}]$ will be called the relative entropy of $f$ with respect to $f_{\infty, \sigma}$. 

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Corollary 4 Consider a renormalized or weak solution $f$ under Assumptions (H1), (H2), (H3) and (H4). Then $\Sigma_\sigma [f(t) | f_\infty, \sigma] \leq \Sigma_\sigma [f_0 | f_\infty, \sigma]$.

Basic examples:
$\sigma_q(s) = s^q, \gamma_q(s) = (-s/q)^{1/(q-1)}, q > 1$.
$\sigma_1(s) = s \log s - s$ and $\gamma_1(s) = e^{-s}$.

$\mathbb{L}^p$-nonlinear stability – Proof of Theorem 1

Proposition 5 Let $f$ and $g$ be two nonnegative functions in $L^1(\mathbb{R}^6) \cap L^p(\mathbb{R}^6), p \in [1, 2]$ and consider a strictly convex function $\sigma : \mathbb{R}_0^+ \rightarrow \mathbb{R}$ in $C^2(\mathbb{R}^+)) \cap C^0(\mathbb{R}_0^+)$. Let

$$A = \inf \left\{ \sigma''(s)/s^{p-2} : s \in (0, \infty) \right\}$$

If $A > 0$, then the following inequality holds:

$$\Sigma_\sigma [f|g] \geq \frac{2^{-2/p} A}{\max \left( \|f\|^{2-p}_{L^p}, \|g\|^{2-p}_{L^p} \right)} \|f - g\|^2_{L^p}$$

$$+ \frac{1}{2} \int_{\mathbb{R}^3} \|\nabla x (\phi[f] - \phi[g])\|^2 \, dx$$
**L²-Nonlinear stability of maxwellian states**

**Lemma 6** Assume that $e^{-\beta \phi_e}$ is bounded in $L^1(\mathbb{R}^3)$ for some $\beta > 0$. Let $f$ be a nonnegative function in $L^1 \cap L^q(\mathbb{R}^6)$, $q > 1$, such that $(x,v) \mapsto (|v|^2 + \phi_e(x))f(x,v)$ is bounded in $L^1$. Then $f \log f$ is also bounded in $L^1(\mathbb{R}^6)$.

**Proof of Theorem 2.** Use a cut-off functional.
Let $E_1(x,v) := \frac{1}{2}|v|^2 + \phi_{\infty,1}(x) + \phi_e(x)$.

\[ E_{min} := \inf\{E_1(x,v)\} \geq \inf\{\phi_e(x)\} > -\infty \]  

Define $m = \varphi \circ E_1$, $\varphi(s) = \kappa e^{-s}$,  
\[ \kappa = \frac{M}{(2\pi)^{3/2}} \left[ \int e^{-\phi_{1,\infty} - \phi_e} \, dx \right]^{-1} \]  
\[ \bar{s} = \varphi(E_{min}) \]  
and  
\[ \tau_1(s) := \begin{cases}  
  s \log s - s & \text{if } s \in [0, \bar{s}] \\
  \frac{1}{2\kappa} e^{E_{min}} (s-\bar{s})^2 - (E_{min} - \log \kappa)(s-\bar{s}) + \bar{s} \log \bar{s} - \bar{s} & \text{if } s \in (\bar{s}, +\infty) 
\end{cases} \]
2. Vlasov-Poisson system with injection boundary conditions

\( \omega \) is a bounded domain in \( \mathbb{R}^d \), \( \partial \omega \) is of class \( C^1 \)
\( \Omega = \omega \times \mathbb{R}^d \) and \( \Gamma = \partial \Omega = \partial \omega \times \mathbb{R}^d \) are the phase space and its boundary respectively.

\[
\Sigma^\pm(x) = \{ v \in \mathbb{R}^d : \pm v \cdot \nu(x) > 0 \}
\]

\[
\Gamma^\pm = \{ (x,v) \in \Gamma : v \in \Sigma^\pm(x) \}.
\]

Vlasov-Poisson system with injection boundary conditions

\[
\begin{aligned}
\partial_t f + v \cdot \nabla_x f - (\nabla x \phi + \nabla x \phi_0) \cdot \nabla_v f &= 0 \\
f|_{t=0} &= f_0, \\ f|_{\Gamma^- \times \mathbb{R}^+}(x,v,t) &= \gamma \left( \frac{1}{2} |v|^2 + \phi_0(x) \right), \\
-\Delta \phi &= \rho = \int_{\mathbb{R}^d} f \, dv, (x,t) \in \omega \times \mathbb{R}^+, \\
\text{and} \quad \phi(x,t) &= 0, (x,t) \in \partial \omega \times \mathbb{R}^+.
\end{aligned}
\]
Assumptions:

(H5) The initial condition $f_0$ is a nonnegative bounded function.

(H6) The external electrostatic potential is nonnegative, $C^2(\overline{\omega})$.

(H7) The function $\gamma$ has Property $P$: it is defined on $(\min_{x \in \omega} \phi_0(x), +\infty)$, bounded, smooth, strictly decreasing with values in $\mathbb{R}_+^*$, and rapidly decreasing at infinity, so that

$$\sup_{x \in \omega} \int_0^{+\infty} s^{d/2} \gamma(s + \phi_0(x)) \, ds < +\infty.$$ 

$\gamma$ is strictly decreasing: $\beta_\gamma = -\int_0^g \gamma^{-1}(z) \, dz$ is strictly convex.
Let $U : L^1(\Omega) \rightarrow W^{1,d/(d-1)}_0(\omega)$: $U[g] = u$ the unique solution in $W^{1,d/(d-1)}_0(\omega)$ of

$$-\Delta u = \int_{\mathbb{R}^d} g(x,v) \, dv.$$ 

The operator $U$ is linear and satisfies

$$\int_{\Omega} g U[f] \, dx dv = \int_{\Omega} f U[g] \, dx dv,$$

$$\int_{\Omega} f U[f] \, dx dv = \int_{\omega} |\nabla_x U[f]|^2 \, dx.$$ 

$M(x,v) = \gamma \left(\frac{1}{2}|v|^2 + U[M](x) + \phi_0(x)\right)$ is a stationary solution:

$$v \cdot \nabla_x M - (\nabla_x \phi_0 + \nabla_x U[M]) \cdot \nabla_v M = 0$$

It is the unique critical point in $H^1_0(\omega)$ of the strictly convex coercive functional

$$U \mapsto \frac{1}{2} \int_{\omega} |\nabla U|^2 \, dx - \int_{\omega} G(U + \phi_0) \, dx,$$

where $G''(u) = g(u) = \int_{\mathbb{R}^d} \gamma \left(\frac{1}{2}|v|^2 + u\right) \, dv$

$$= 2^{d/2-1}|S^{d-1}| \cdot \int_0^{+\infty} s^{d/2-1} \gamma(s + u) \, ds.$$
Relative entropy

\[ \Sigma_\gamma[g|h] = \int_\Omega \left( \beta_\gamma(g) - \beta_\gamma(h) - (g - h)\beta'_\gamma(h) \right) \, dx \, dv \]
\[ + \frac{1}{2} \int_\omega |\nabla U[g - h]|^2 \, dx \]

\( \beta_\gamma \) is the real function defined by

\[ \beta_\gamma(g) = -\int_0^g \gamma^{-1}(z) \, dz . \]

\[ \int_\Omega f \left( \frac{1}{2} |v|^2 + \frac{1}{2} U[f] + \phi_0 \right) \, dx \, dv \]
\[ = \int_\Omega \left[ \frac{1}{2} (f - M)U[f - M] - \frac{1}{2} MU[M] - f \beta'_\gamma(M) \right] \, dx \, dv \]

Thus

\[ \Sigma_\gamma[f|M] \]
\[ = \int_\Omega \left( \beta_\gamma(f) + \left( \frac{1}{2} |v|^2 + \frac{1}{2} U[f] + \phi_0 \right) f \right) \, dx \, dv \]
\[ - \int_\Omega \left( \beta_\gamma(M) + \left( \frac{1}{2} |v|^2 + \frac{1}{2} U[M] + \phi_0 \right) M \right) \, dx \, dv \]
Relative entropy and irreversibility

**Theorem 7** Assume that \( f_0 \in L^1 \cap L^\infty \) is a nonnegative function such that \( \Sigma \gamma [f_0|M] < +\infty \). Under Assumptions (H5)-(H7),

\[
\frac{d}{dt} \Sigma \gamma [f(t)|M] = -\Sigma^+ \gamma [f(t)|M]
\]

\( \Sigma^+ \gamma \) is the boundary relative entropy flux

\[
\Sigma^+ \gamma [g|h] = \int_{\Gamma^+} (\beta \gamma (g) - \beta \gamma (h) - (g - h) \beta' \gamma (h)) \ d\sigma.
\]

**Proof.**

\[
\frac{d}{dt} \int_{\Omega} \beta (f) \ dx dv = \sum_{\pm} \mp \int_{\Gamma^\pm} \beta (f) \ d\sigma
\]

\[
\frac{d}{dt} \int_{\Omega} f \left( \frac{1}{2} |v|^2 + \frac{1}{2} U[f] + \phi_0 \right) \ dx dv
\]

\[
= \sum_{\pm} \pm \int_{\Gamma^\pm} f \left( \frac{1}{2} |v|^2 + \phi_0 \right) \ d\sigma
\]

\[
\int_{\Gamma^\pm} f \left( \frac{1}{2} |v|^2 + \phi_0(x) \right) \ d\sigma = -\int_{\Gamma^\pm} f \beta'_\gamma (M) \ d\sigma \quad \square
\]
The large time limit

\[
\Sigma_\gamma[f(t)|M] + \int_0^t \Sigma_\gamma^+ [f(s)|M] ds \leq \Sigma_\gamma[f_0|M]
\]

\((f^n(x,v,t), \phi^n(x,t)) = (f(x,v,t+t_n), \phi(x,t+t_n))\)

\[
\lim_{n \to +\infty} \int_{\mathbb{R}^+} \Sigma_\gamma^+ [f^n(s)|M] \, ds = 0
\]

\[
\sup_{t>0} \Sigma_\gamma[f^n(t)|M] \leq C.
\]

\(\) Dunford-Pettis criterion: \((f^n, \phi^n) \rightharpoonup (f^\infty, \phi^\infty)\) weakly in \(L^1_{\text{loc}}(dt, L^1(\Omega)) \times L^1_{\text{loc}}(dt, H^1_0(\omega)).\)

\[
f^\infty \equiv M \text{ on } \Gamma
\]
Is $f^\infty$ stationary? A partial answer for $d = 1$.

**Theorem 8** Assume that $\gamma$ satisfies Property ($P$) and consider a solution $(f^\infty, \phi^\infty)$ such that $f^\infty \equiv M$ on $\Gamma$ on the interval $\omega = (0, 1)$ ($d = 1$). If $\phi_0$ is analytic in $x$ with $C^\infty$ (in time) coefficients and if $\phi_0$ is analytic with $-\frac{d^2\phi_0}{dx^2} \geq 0$ on $\omega$, then $(f, \phi)$ is the unique stationary solution, given by: $f = M$, $\phi = U[M]$.

**Proof.** Let $\phi_0(0) = 0$, $\phi_0(1) \geq 0$: $\phi'_0(0) \geq 0$. Characteristics

$$\frac{\partial X}{\partial t} = V, \quad \frac{\partial V}{\partial t} = -\frac{\partial \phi}{\partial x}(X, t) - \frac{d\phi_0}{dx}(X),$$

$$X(s; x, v, s) = x, \quad V(s; x, v, s) = v$$

are defined on $(T_{in}(s; x, v), T_e(s; x, v))$: either $T_{in}(s; x, v) = -\infty$ or $(X_{in}, V_{in})(s; x, v) \in \Gamma^-$; either $T_e(s; x, v) = +\infty$ or $(X_e, V_e)(s; x, v) \in \Gamma^+$. **Step 1:** the electric field is repulsive at $x = 0$. 

Step 2: Analysis of the characteristics in a neighborhood of $(0,0,t)$.

\[ f(X_{in}, V_{in}, T_{in}) = \gamma \left( \frac{1}{2} |V_{in}|^2 \right) \]
\[ f(X_e, V_e, T_e) = \gamma \left( \frac{1}{2} |V_e|^2 \right) . \]

The function $\gamma$ is strictly decreasing:

\[ |V_{in}(t_0, x_0, 0)| = |V_e(t_0, x_0, 0)| \]

Characteristics are parametrized by

\[ \frac{dt^\pm}{dX} = \frac{1}{V}, \quad \frac{dV}{dX} = -\frac{1}{V} \frac{\partial \Phi}{\partial x}(X, t^\pm) . \]

Let $e_\pm(X) = \frac{1}{2}V^2(X)$.

\[ t^\pm(X) = t_0 \mp \int_{x_0}^X \frac{dY}{\sqrt{2e_\pm(Y)}} \quad \forall \ X \in [0, x_0] , \]

\[ \frac{de_\pm}{dX} = -\frac{\partial \Phi}{\partial x}(X, t^\pm(X)) , \quad e_\pm(x_0) = 0 . \]

Rescaling: $x_0 = \varepsilon^2$, $X = \varepsilon^2 (1 - x)$ and $e_\pm(X) := \frac{\varepsilon^2}{2} e^\varepsilon(x)$
\[
\frac{de^\varepsilon}{dx} = 2 \frac{\partial \Phi}{\partial x} \left( \varepsilon^2 (1 - x), t_0 \pm \varepsilon \int_0^x \frac{dy}{\sqrt{e^\varepsilon_\pm(y)}} \right)
\]

with the condition \( e^\varepsilon_+ (1) = e^\varepsilon_- (1) \) for any \( \varepsilon > 0 \) small enough.

\[
e^\varepsilon_\pm = \sum_{n=0}^{+\infty} \varepsilon^n e^\varepsilon_n.
\]

**Lemma 9** With the above notations, for all \( n \in \mathbb{N} \), we have the following identities:

(i) \( \frac{d e^\varepsilon_{2n}}{dx}(x) = \frac{2(1-x)^n}{n!} \frac{\partial^{n+1} \Phi(0, t_0)}{\partial x^{n+1}} \)

(ii) \( \frac{d e^\varepsilon_{2n+1}}{dx}(x) = \pm \frac{2(1-x)^n}{(n+1)!} \left( \int_0^x \frac{dy}{\sqrt{e_0(y)}} \right) \frac{\partial_t \partial_x^{n+1} \Phi(0, t_0)}{\partial x^{n+1}} \)

(iii) \( \partial_t \partial_x^{n+1} \Phi(0, t_0) = 0 \).
3. With collision kernels

Vlasov-Poisson-Boltzmann system with injection boundary conditions

\[
\begin{aligned}
&\partial_t f + v \cdot \nabla_x f - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla v f = Q(f) \\
f|_{t=0} = f_0, & \quad f|_{\Gamma^{-} \times \mathbb{R}^+}(x, v, t) = \gamma(\frac{1}{2}|v|^2 + \phi_0(x)), \\
-\Delta \phi = \rho = & \int_{\mathbb{R}^d} f \, dv, (x, t) \in \omega \times \mathbb{R}^+, \\
\text{and} & \quad \phi(x, t) = 0, (x, t) \in \partial \omega \times \mathbb{R}^+.
\end{aligned}
\]

Additional assumptions:

(H8) The collision operator \( Q \) preserves the mass \( \int_{\mathbb{R}^d} Q(g) \, dv = 0 \), and satisfies the H-theorem

\[
D[g] = -\int_{\mathbb{R}^d} Q(g) \left[ \frac{1}{2}|v|^2 - \gamma^{-1}(g) \right] \, dv \geq 0,
\]

for any nonnegative function \( g \) in \( L^1(\mathbb{R}^d) \).

(H9) \( D[g] = 0 \iff Q(g) = 0 \)
Examples

1. **Pure Vlasov-Poisson system.**
The inflow function $\gamma$ is arbitrary.

2. **The Vlasov-Poisson-Fokker-Planck system.**

   \[
   Q_{FP}(f) = \text{div}_v(vf + \theta \nabla_v f)
   \]

   \[
   Q_{FP,\alpha}(f) = \text{div}_v(vf(1 - \alpha f) + \theta \nabla_v f)
   \]

   Take $M_\theta = (2\pi \theta)^{-d/2} e^{-|v|^2/(2\theta)}$

   \[
   \gamma(u) = \frac{1}{\alpha + e^{(u - \mu)/\theta}}
   \]

**Lemma 10** **[H-theorem]** $\alpha \geq 0$, $0 \leq f \leq \alpha^{-1}$.

\[
\mathcal{H}(f) = \int_{\mathbb{R}^d} Q_{FP,\alpha}(f) \log \left( \frac{f}{(1 - \alpha f)M_\theta} \right) \, dv \leq 0
\]

and $\mathcal{H}(f) = 0 \iff Q_{FP,\alpha}(f) = 0 \iff f(v) = (\alpha + e^{(|v|^2/2 - \mu)/\theta})^{-1}$. 

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3. **BGK approx. of the Boltzmann operator.**

\[ Q_\alpha(f) = \int_{\mathbb{R}^d} \sigma(v, v') [M_\theta(v) f(v')(1 - \alpha f(v)) - M_\theta(v') f(v)(1 - \alpha f(v'))] \, dv' \]

*H*-theorem: same as for the Vlasov-Poisson-Fokker-Planck system.

4. **Linear elastic collision operator.**

\[ Q_E(f) = \int_{\mathbb{R}^d} \chi(f(v') - f(v)) \delta(|v'|^2 - |v|^2) \, dv' , \]

\( \chi(v, v') \) is a symmetric positive cross-section. Let \( \lambda(v) = \int_{\mathbb{R}^d} \chi(v, v') \delta(|v|^2 - |v'|^2) \, dv' \in L^\infty. \)

**Lemma 11** \( Q_E \) is bounded on \( L^1 \cap L^\infty(\mathbb{R}^d). \)

*For any function \( \psi \) and \( H \) (increasing)*

\[ \int_{\mathbb{R}^d} Q_E(f) \cdot \psi(|v|^2) \, dv = 0 \]

\[ H(f) = \int_{\mathbb{R}^d} Q_E(f) \cdot H(f) \, dv \leq 0 . \]

*If \( H \) is strictly increasing, \( H(f) = 0 \iff Q_E(f) = 0 \iff f(v) = \psi(|v|^2). \gamma \) is arbitrary.*

5. **Electron-Electron collision operator.**

Boltzmann collision operator \( Q_{ee,\alpha}^B \) or Fokker-Planck-Landau collision operator \( Q_{ee,\alpha}^L. \)
**Relative entropy and irreversibility**

**Theorem 12** Assume that \( f_0 \in L^1 \cap L^\infty \) is a nonnegative function such that \( \Sigma \gamma [f_0|M] < +\infty \). Under Assumptions (H5)-(H7),

\[
\frac{d}{dt} \Sigma \gamma [f(t)|M] = -\Sigma^+ \gamma [f(t)|M] - \int_\omega D[f](x, t) \, dx
\]

\( \Sigma^+ \gamma \) is the boundary relative entropy flux

\[
\Sigma^+ \gamma [g|h] = \int_{\Gamma_+} \left( \beta_\gamma (g) - \beta_\gamma (h) - (g - h) \beta'_\gamma (h) \right) \, d\sigma.
\]

**Proof.**

\[
\frac{d}{dt} \int_\Omega \beta(f) \, dxdv = \sum_{\pm} \int_{\Gamma_\mp} \beta(f) \, d\sigma + \int_\Omega \beta'(f) Q(f) \, dxdv
\]

\[
\frac{d}{dt} \int_\Omega f \left( \frac{1}{2} |v|^2 + \frac{1}{2} U[f] + \phi_0 \right) \, dxdv = \sum_{\pm} \int_{\Gamma_\mp} f \left( \frac{1}{2} |v|^2 + \phi_0 \right) \, d\sigma + \int_\Omega \frac{1}{2} |v|^2 Q(f) \, dxdv
\]

\[
\int_{\Gamma_\pm} f \left( \frac{1}{2} |v|^2 + \phi_0(x) \right) \, d\sigma = -\int_{\Gamma_\pm} f \beta'_\gamma (M) \, d\sigma \quad \square
\]
The large time limit

\[
\sum_{\gamma}[f(t)|M] + \int_0^t \sum_{\gamma}^+[f(s)|M]ds \\
+ \int_0^t \int_{\omega} D[f(x, \cdot, s)] dx ds \leq \sum_{\gamma}[f_0|M]
\]

\[(f^n(x, v, t), \phi^n(x, t)) = (f(x, v, t+t_n), \phi(x, t+t_n))\]

\[\lim_{n \to +\infty} \int_{\mathbb{R}^+} \sum_{\gamma}^+[f^n(s)|M] \, ds = 0\]

\[\lim_{n \to +\infty} \int_{\mathbb{R}^+} \int_{\omega} D[f^n(x, \cdot, s)] dx ds = 0\]

\[\sup_{t>0} \sum_{\gamma}[f^n(t)|M] \leq C .\]

+ Dunford-Pettis criterion: \((f^n, \phi^n) \rightharpoonup (f^\infty, \phi^\infty)\) weakly in \(L^1_{\text{loc}}(dt, L^1(\Omega)) \times L^1_{\text{loc}}(dt, H^1_0(\omega)).\)

\[f^\infty \equiv M \text{ on } \Gamma\]
Ex. 2, 3, 4: $f^\infty \in \text{Ker } Q$ depends only on $|v|^2$

**Lemma 13** Let $f \in L^1_{loc}$ be a solution. If $f$ is even (or odd) with respect to the $v$ variable, then it does not depend on $t$.

**Proof.** The operator $\partial_t$ conserves the $v$ parity while $v \cdot \nabla_x - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v$ transforms the $v$ parity into its opposite. \hfill $\square$

**Corollary 14** Let $f$ be a solution s.t. $f \equiv M$ on $\Gamma$, with $Q = Q_E, Q_{FP,\alpha}, Q_{\alpha, Q_{ee,\alpha} + Q_E, Q_{ee,\alpha} + Q_{\alpha, Q_{ee,\alpha} + Q_{FP,\alpha}}}$ or a combination of these operators. Then on $\Omega$, $f \equiv M$ is stationary if there are no closed characteristics.

**Theorem 15** Assume that $\alpha \geq 0$. Let $0 \leq f \leq F_D(x,v) = \left(\alpha + e^{(\frac{1}{2}|v|^2 + \phi_0(x) - \mu)/\theta}\right)^{-1}$ on $\Gamma^-$.\[\begin{align*}
\partial_t f + v \cdot \nabla_x f - \nabla_x \phi_0 \cdot \nabla_v f &= Q_{\alpha}(f) \\
f(x,v) &= g(x,v), \quad (x,v) \in \Gamma^-
\end{align*}\]

Any solution such that $0 \leq f_0 \leq F_D$ converges to the unique stationary solution.
4. Other boundary conditions: entropies

**Diffuse reflection boundary conditions for $f$.**

For any $(x, t) \in \partial \omega \times \mathbb{R}^+$, let

$$\rho_+(x, t) := \int_{\Sigma^+(x)} f(x, v, t) \cdot \nu(x) \, dv.$$ 

There exists a unique function $\mu : \partial \omega \times \mathbb{R}^+ \to \mathbb{R}$ s.t.

$$\rho_+ = \int_{\Sigma^-(x)} \gamma \left( \frac{1}{2} |v|^2 + \phi_0(x) - \mu(x, t) \right) |v \cdot \nu(x)| \, dv$$

With the notation

$$m_f(x, v, t) := \gamma \left( \frac{1}{2} |v|^2 + \phi_0(x) - \mu(x, t) \right),$$

we shall say that $f$ is subject to **diffuse reflection boundary conditions** (DRBC) iff

$$f(x, v, t) = m_f(x, v, t), \quad \forall (x, v, t) \in \Gamma^- \times \mathbb{R}^+.$$ 

Note that under this condition, the total mass is preserved:

$$\frac{d}{dt} \int_{\Omega} f(x, v, t) \, dx \, dv = 0.$$
Lemma 16 For any $M > 0$, the system with diffuse reflection boundary conditions has at least one nonnegative stationary solution
\[
M(x,v) = \gamma \left( \frac{1}{2} |v|^2 + \phi_0(x) + U[M] - \mu M \right)
\]
such that $\|M\|_{L^1(\Omega)} = M$.

\[
\Sigma_{\gamma}[g|h] = \int_{\Omega} (\beta_{\gamma}(g) - \beta_{\gamma}(h) - (g-h)\beta'_{\gamma}(h)) dx dv + \frac{1}{2} \int_\omega |\nabla U[g-h]|^2 dx
\]
\[
\Sigma^+_\gamma[g|h] = \int_{\Gamma^+} (\beta_{\gamma}(g) - \beta_{\gamma}(h) - (g-h)\beta'_{\gamma}(h)) d\sigma
\]

Theorem 17 Let $(f,\phi)$ be a solution with diffuse reflection boundary conditions, $M = \|f_0\|_{L^1}$. Under (H5)-(H8) hold, if $\lim_{s \to -\infty} \gamma(s) = +\infty$. Then the relative entropy satisfies
\[
\frac{d}{dt} \Sigma_{\gamma}[f(t)|M] = -\Sigma^+_\gamma[f|m_f] + \int_\omega D[f(x,\cdot,s)] dx
\]
If there are no closed trajectories and if $\omega$ is connected, then there exists a unique continuous nonnegative stationary solution.
5. Relation with nonlinear diffusion equations

Consider a solution in \( C^0(\mathbb{R}^+, L_1^1(\mathbb{R}^d)) \) of

\[
\rho_t = \Delta \nu(\rho) + \nabla (\rho \nabla \phi_0) + \nabla \cdot (\rho \nabla \phi)
\]

\[
\phi = V[\rho] = |S^{d-1}|^{-1} |x|^{-(d-2)} \ast \rho
\]

\( \Gamma^{-1}(u) = -\nu'(u)/u \) is nonnegative decreasing

\[
\beta'(u) = -\int_0^u \Gamma^{-1}(z) \, dz = \int_0^u \frac{\nu'(z)}{z} \, dz
\]

\[
\rho(t, \cdot) \to \rho_\infty(x) = \Gamma(\phi_0 + V[\rho_\infty] - \mu)
\]

\[
\Sigma[\rho|\rho_\infty] = \int_{\mathbb{R}^d} (\beta(\rho) - \beta(\rho_\infty) - \beta'(\rho_\infty)(\rho - \rho_\infty)) \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^d} |\nabla V[\rho - \rho_\infty]|^2 \, dx
\]

\[
\frac{d}{dt} \Sigma[\rho|\rho_\infty] = -I[\rho]
\]

\[
I[\rho] = \int_{\mathbb{R}^d} \rho \left| \nabla \phi_0 + \nabla V[\rho] - \nabla (\Gamma^{-1}(\rho)) \right|^2 \, dx
\]

Csiszár-Kullback inequality:

\[
\Sigma[\rho|\rho_\infty] \geq C \|\rho - \rho_\infty\|^2_{L_1^1(\mathbb{R}^d)} + \frac{1}{2} \|\nabla U[\rho - \rho_\infty]\|^2_{L_2^2(\mathbb{R}^d)}
\]
The nonlinear diffusion equation can be obtained from a singular perturbation of a kinetic equation. Let $\gamma$ satisfy Property $\mathcal{P}$ and consider a function $g(v)$:

$$m_g(v) = \gamma \left( \frac{1}{2} |v|^2 + a_g \right)$$

such that $\int_{\mathbb{R}^d} m_g(v) \, dv = \int_{\mathbb{R}^d} g(v) \, dv$. Consider the Vlasov-Poisson-Boltzmann system

$$\partial_t f^\eta + \frac{1}{\eta} [v \cdot \nabla_x f^\eta - (\nabla_x \phi_0 + \varepsilon \nabla_x \phi) \cdot \nabla_v f^\eta] = \frac{1}{\eta^2} Q(f^\eta)$$

where $Q(g) = m_g - g$. $H$-Theorem

$$\frac{d}{dt} \sum_\gamma [f^\eta(\cdot, \cdot, t)|M] = -\frac{1}{\eta^2} \int_{\mathbb{R}^d} D[f^\eta(x, \cdot, t)] \, dx.$$ 

$$-D[g] = \int_{\mathbb{R}^d} Q(g) \left[ \frac{1}{2} |v|^2 - \gamma^{-1}(g) \right] \, dv = \int_{\mathbb{R}^d} (m_g - g) \left( \gamma^{-1}(m_g) - \gamma^{-1}(g) \right) \, dv \leq 0$$

Formal limit as $\eta \to 0$: $f^0 = \gamma \left( \frac{1}{2} |v|^2 + a_{f_0}(x, t) \right)$
\[ \rho(x,t) = \int_{\mathbb{R}^d} f^0(x,v,t) \, dv \text{ satisfies the nonlinear diffusion equation with} \]
\[ \nu(u) = -\int_0^u s (\Gamma^{-1})'(s) \, ds \]
\[ \Gamma(u) = \int_{\mathbb{R}^d} \gamma \left( \frac{1}{2} |v|^2 + u \right) \, dv \]
\[ \rho(x,t) = \Gamma(a_{f0}(x,t)). \]

**Proof.**

\[ \partial_t \rho^n + \text{div}_x j^n = 0 \]
with \( \rho^n = \int_{\mathbb{R}^d} f^n \, dv \) and \( j^n = \frac{1}{\eta} \int_{\mathbb{R}^d} vf^n \, dv \)

\[-j^n = \eta^2 \partial_t j^n + \text{div}_x \left[ \int v \otimes v f^n \, dv \right] + (\nabla \phi_0 + \nabla \phi^n) \rho^n \]
\[ \rightarrow \text{div}_x \left[ \int v \otimes \gamma \left( \frac{1}{2} |v|^2 + a_{f0}(x,t) \right) \, dv \right] + (\nabla \phi_0 + \nabla \phi) \rho \]
\[ = \frac{1}{d} \nabla_x \int_{\mathbb{R}^d} |v|^2 \gamma \left( \frac{1}{2} |v|^2 + a_{f0} \right) \, dv \]
\[ = - (\Gamma \circ a_{f0}) \nabla_x a_{f0} = \nabla_x \nu(\rho). \]

This formal limit holds not only at the level of the equations but also for the *relative entropy*. 28
Hilbert expansion for $f^n$:

$$f^n = f^0 + \eta f_1 + O(\eta^2).$$

$$D_{f^0}Q(f_1) = v \cdot \nabla_x f^0 - (\nabla_x \phi_0 + \nabla_x \phi) \cdot \nabla_v f^0 = -f_1$$

$$\frac{1}{\eta^2} \int D[f^n] \, dx = -\frac{1}{\eta^2} \int Q(f^n) \left[ \frac{1}{2} |v|^2 - \gamma^{-1}(f^n) \right] \, dv \, dx$$

$$\to \int D_{f^0}Q(f_1) \cdot (\gamma^{-1})'(f^0) f_1 \, dv \, dx$$

$$= -\int (f_1)^2 (\gamma^{-1})'(f^0) \, dv \, dx$$

$$= -\int \left\{ \int \gamma' \left( \frac{1}{2} |v|^2 + a \right) \right\} \left[ \left( -\nabla_x a f_0 + \nabla_x \phi_0 + \nabla_x \phi \right) \cdot v \right]^2 \, dv \right\} \, dx$$

$$= -\frac{1}{d} \int \left\{ \int |v|^2 \gamma' \left( \frac{1}{2} |v|^2 + a \right) \, dv \right\} \left[ -\nabla_x a f_0 + \nabla_x \phi_0 + \nabla_x \phi \right]^2 \, dx$$

$$= -\int \rho \left[ -\nabla_x a f_0 + \nabla_x \phi_0 + \nabla_x \phi \right]^2 \, dx = I[\rho]$$
Example. Fermi-Dirac statistics

\[ f_\infty(x, v) = \gamma \left( \frac{1}{2} |v|^2 + \phi_0 + \phi - \mu \right) \]

\[ \gamma(u) = (\alpha + e^u)^{-1}, \] where \( \alpha > 0 \) is a parameter related to Planck’s constant.

\[ \rho_\infty(x) = \Gamma(\phi_0 + \phi - \mu) \]

is the unique equilibrium density of

\[ \rho_t = \nabla \cdot (\nabla \nu(\rho) + \rho \nabla \phi), \]

where \( \nu(u) = -\int_0^u s (\Gamma^{-1})'(s) \, ds \) and

\[ \Gamma(u) = |S^{d-1}| \int_0^{+\infty} (2s)^{d/2-1} \gamma(s + u) \, ds. \]
Proof. Taylor development at order two

\[ \Sigma_\sigma [f|g] = \frac{1}{2} \int_{\mathbb{R}^6} \sigma''(\xi)|f - g|^2 \, d(x, v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x (\phi[f]|^2 \, d(x, v) \]

\[ \geq \frac{A}{2} \int_{\mathbb{R}^6} \xi^{p-2} |f - g|^2 \, d(x, v) + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla_x (\phi[f]|^2 \, d(x, v) \]

Hölder's inequality, for any \( h > 0 \) and for any measurable set \( A \subset \mathbb{R}^6 \),

\[ \int_A |f - g| \, h^{-\alpha} h^\alpha \, d(x, v) \leq \left( \int_A |f - g|^2 h^{p-2} \, d(x, v) \right)^{p/2} \]

with \( \alpha = p(2 - p)/2, \ s = 2/(2 - p). \)

\[ \left( \int_A |f - g|^2 h^{p-2} \, d(x, v) \right)^{p/2} \geq \left( \int_A |f - g|^p \, d(x, v) \right) \left( \int_A \right)^{p/2} \]

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[i)] On $A = A_1 = \{(x,v) \in \mathbb{R}^6 : f(x,v) > g(x,v)\}$, use $\xi^{p-2} > f^{p-2}$ and take $h = f$:

$$\left(\int_{A_1} |f - g|^2 \xi^{p-2} \, d(x,v)\right)^{p/2} \geq \left(\int_{A_1} |f - g|^p \, d(x,v)\right)^p \geq \left(\int_{A_1} |f - g|^p \, d(x,v)\right)^{p/2} \geq \left(\int_{A_1} |f - g|^p \, d(x,v)\right)^p \geq \left(\int_{A_1} |f - g|^p \, d(x,v)\right)^p$$

[ii)] On $A = A_2 = \{(x,v) \in \mathbb{R}^6 : f(x,v) \leq g(x,v)\}$, use $\xi^{p-2} \geq g^{p-2}$ and take $h = g$:

$$\left(\int_{A_2} |f - g|^2 \xi^{p-2} \, d(x,v)\right)^{p/2} \geq \left(\int_{A_2} |f - g|^p \, d(x,v)\right)^p \geq \left(\int_{A_2} |f - g|^p \, d(x,v)\right)^{p/2} \geq \left(\int_{A_2} |f - g|^p \, d(x,v)\right)^p \geq \left(\int_{A_2} |f - g|^p \, d(x,v)\right)^p$$

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