Nonlinear flows and optimality of entropy - entropy production methods

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Outline

▷ **Interpolation inequalities on the sphere**

- spectral methods and fractional operators
- bifurcations and flow methods

▷ **Symmetry breaking and linearization**

- The sub-critical and critical Caffarelli-Kohn-Nirenberg inequalities
- Linearization and spectrum
- Diffusions without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows: Rényi entropy powers, self-similar variables and relative entropies, the role of the spectral gap

▷ **Diffusions with weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**

- Towards a parabolic proof
- Large time asymptotics and spectral gaps
- A discussion of optimality cases

Interpolation inequalities on the sphere and eigenvalues of the (fractional) Laplace operator

- ▷ A spectral point of view on the inequalities
- ▷ The *bifurcation* point of view
- ▷ Flows on the sphere
  - *Carré du champ*
  - Can one prove Sobolev's inequalities with a heat flow ?
  - Some open problems: constraints and improved inequalities

[Beckner, 1993], [J.D., Zhang, 2016]
[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008][J.D., Esteban, Loss, 2014 & 2015]

Non-fractional interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ induced by the Lebesgue measure on $\mathbb{R}^{d+1}$

$$1 \le p < 2 \quad \text{or} \quad 2 < p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

Optimal interpolation inequalities for fractional operators

- The sharp *Hardy-Littlewood-Sobolev inequality on* $\mathbb{S}^n$ [Lieb, 1983]

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} F(\zeta)\,|\zeta-\eta|^{-\lambda}\,F(\eta)\,d\mu(\zeta)\,d\mu(\eta) \le \frac{\Gamma(n)\,\Gamma\big(\frac{n-\lambda}{2}\big)}{2^\lambda\,\Gamma\big(\frac{n}{2}\big)\,\Gamma\big(\frac{n}{p}\big)}\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^n)}$$

$\lambda\in(0,n)$, $p=\frac{2\,n}{2\,n-\lambda}\in(1,2)$,$\lambda=\frac{2\,n}{q_\star}$ where $\frac{1}{p}+\frac{1}{q_\star}=1$

- sharp GNS inequalities on $\mathbb{S}^d$: [Backner 1993], [Bidaut-Véron, Véron, 1991]
- A *subcritical interpolation inequality*

$d\mu$ is the uniform probability measure on $\mathbb{S}^n$
$\mathcal{L}_s$ is the fractional Laplace operator of order $s\in(0,n)$
$q\in[1,2)\cup(2,q_\star]$, $q_\star=\frac{2\,n}{n-s}$

$$\frac{\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)}-\|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}}{q-2}\le \mathsf{C}_{q,s}\int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu \quad \forall\,F\in\mathrm{H}^{s/2}(\mathbb{S}^n)$$

The sharp constants

Theorem

[J.D., Zhang] *Let* $n \geq 1$. *If either* $s \in (0,n]$, $q \in [1,2) \cup (2, q_\star]$, *or* $s = n$ *and* $q \in [1,2) \cup (2,\infty)$, *then*

$$\mathsf{C}_{q,s} = \frac{n-s}{2\,s}\,\frac{\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(\frac{n+s}{2}\right)}$$

$$\mathsf{C}_{q,s}^{-1} = \lambda_1(\mathcal{L}_s) = \inf_{F \in \mathrm{H}^{s/2}(\mathbb{S}^n)\setminus\mathbb{R}} \mathcal{Q}[F]\,, \quad \mathcal{Q}[F] := \frac{(q-2)\int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu}{\|F\|^2_{\mathrm{L}^q(\mathbb{S}^n)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^n)}}$$

- *Sharp subcritical fractional logarithmic Sobolev inequalities*

Corollary

[J.D., Zhang] *Let* $s \in (0,n]$

$$\int_{\mathbb{S}^n} |F|^2 \log\left(\frac{|F|}{\|F\|_{\mathrm{L}^2(\mathbb{S}^n)}}\right) d\mu \leq \mathsf{C}_{2,s} \int_{\mathbb{S}^n} F\,\mathcal{L}_s F\,d\mu \quad \forall\, F \in \mathrm{H}^{s/2}(\mathbb{S}^n)$$

From HLS to Sobolev and subcritical inequalities

- Lieb's approach: $F=\sum_{k=0}^\infty F_{(k)}$ (spherical harmonics), HLS and Funk-Hecke formula

$$\iint_{\mathbb{S}^n\times\mathbb{S}^n} F(\zeta)\,|\zeta-\eta|^{-\lambda}\,F(\eta)\,d\mu(\zeta)\,d\mu(\eta)$$
$$=\frac{\Gamma(n)\,\Gamma\big(\frac{n-\lambda}{2}\big)}{2^\lambda\,\Gamma\big(\frac{n}{2}\big)\,\Gamma\big(\frac{n}{p}\big)}\sum_{k=0}^\infty\frac{\Gamma\big(\frac{n}{p}\big)\,\Gamma\big(\frac{n}{p'}+k\big)}{\Gamma\big(\frac{n}{p'}\big)\,\Gamma\big(\frac{n}{p}+k\big)}\int_{\mathbb{S}^n}|F_{(k)}|^2\,d\mu$$

- Duality: the *fractional Sobolev inequality*

$$\|F\|^2_{\mathrm{L}^{q_\star}(\mathbb{S}^n)}\le\int_{\mathbb{S}^n}F\,\mathcal{K}_s\,F\,d\mu:=\sum_{k=0}^\infty\gamma_k\big(\tfrac{n}{q_\star}\big)\int_{\mathbb{S}^n}|F_{(k)}|^2\,d\mu$$

is dual of HLS, where $q_\star=\frac{2\,n}{n-s}$ is the critical exponent and $q\mapsto\gamma_k\big(\frac{n}{q}\big)$ is convex, with $\gamma_k(x):=\frac{\Gamma(x)\,\Gamma(n-x+k)}{\Gamma(n-x)\,\Gamma(x+k)}$ is enough to establish the result in the subcritical range

Fractional flows and related functional inequalities

▷ Sphere: generalized fractional heat flow

$$\frac{\partial u}{\partial t} - q\,\nabla\cdot\left(u^{1-\frac{1}{q}}\,\nabla(-\Delta)^{-1}\,\mathcal{L}_s\,u^{\frac{1}{q}}\right) = 0$$

The entropy decays exponentially because of

$$\frac{1}{q-2}\,\frac{d}{dt}\left[\left(\int_{\mathbb{S}^n} u\,d\mu\right)^{\frac{2}{q}} - \int_{\mathbb{S}^n} u^{\frac{2}{q}}\,d\mu\right] = -\,2\int_{\mathbb{S}^n} u^{\frac{1}{q}}\,\mathcal{L}_s\,u^{\frac{1}{q}}\,d\mu$$

▷ Euclidean space: any smooth nonnegative solution u of

$$\frac{\partial u}{\partial t} = \nabla\cdot\left(\sqrt{u}\,\nabla(-\Delta)^{-s}\,u^{m-\frac{1}{2}}\right)$$

is such that

$$\frac{d}{dt}\int_{\mathbb{R}^d} u^m\,dx = \frac{2\,m\,(1-m)}{2\,m-1}\int_{\mathbb{R}^d}\left|\nabla^{(1-s)}\,u^{m-\frac{1}{2}}\right|^2\,dx$$

where $\nabla^{(1-s)}w := \nabla(-\Delta)^{-s/2}w$. Rates ?

The Bakry-Emery method on the sphere (non-fractional case)

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

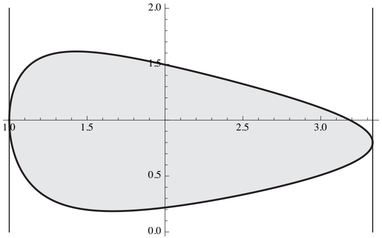
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

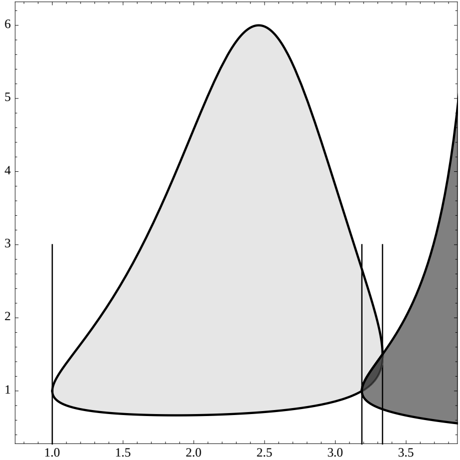
[Demange], [J.D., Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Can one prove Sobolev's inequalities with a heat flow ?

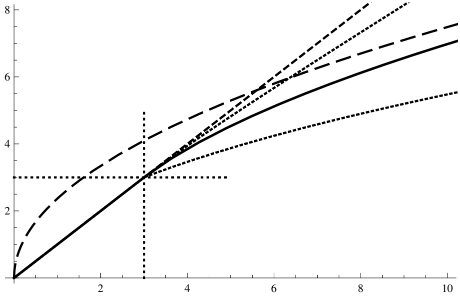


(p, β) *representation,* $d = 5$*. In the dark grey area, the functional is not monotone under the action of the heat flow [J.D., Esteban, Loss]*

The bifurcation point of view

$\mu(\lambda)$ is the optimal constant in the functional inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda) \, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall \, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$



Here $d = 3$ *and* $p = 4$

- A critical point of $u \mapsto \mathcal{Q}_\lambda[u] := \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$ solves

$$-\Delta u + \lambda\,u = |u|^{p-2}\,u \tag{EL}$$

up to a multiplication by a constant (and a conformal transformation if $p = 2^*$)

- The best constant $\mu(\lambda) = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}} \mathcal{Q}_\lambda[u]$ is such that $\mu(\lambda) < \lambda$ if $\lambda > \frac{d}{p-2}$, and $\mu(\lambda) = \lambda$ if $\lambda \le \frac{d}{p-2}$ so that

$$\frac{d}{p-2} = \min\{\lambda > 0\,:\,\mu(\lambda) < \lambda\}$$

- *Rigidity* : the unique positive solution of (EL) is $u = \lambda^{1/(p-2)}$ if $\lambda \le \frac{d}{p-2}$

Constraints and improvements

- Taylor expansion:

$$d=\inf_{u\in \mathrm{H}^1(\mathbb{S}^d,d\mu)\setminus\{0\}}\frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

is achieved in the limit as $\varepsilon\to 0$ with $u=1+\varepsilon\,\varphi_1$ such that

$$-\Delta\varphi_1=d\,\varphi_1$$

▷ This suggest that improved inequalities can be obtained under appropriate orthogonality constraints...

Integral constraints

With the heat flow...

Proposition

For any $p \in (2, 2^{\#})$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \;\; s.t. \;\; \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^{\#} - p) \, (\lambda^\star - d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

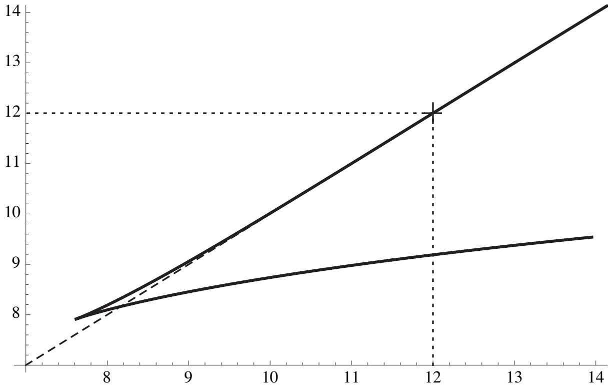
If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

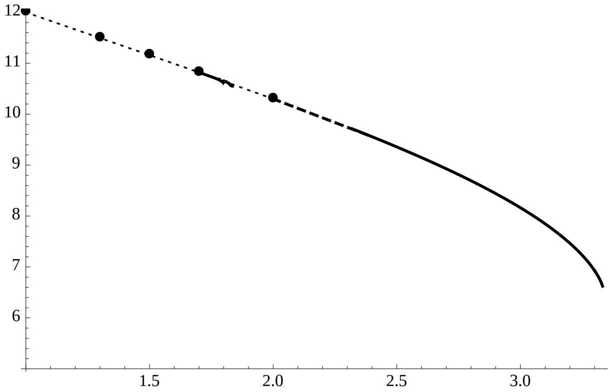
$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

The larger picture: branches of antipodal solutions



Case $d=5$, $p=3$: values of the shooting parameter a as a function of λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d=5$ *and* $1\le p\le 10/3\approx 3.33$. *The limiting value of the constant is numerically found to be equal to* $\lambda_\star=2^{1-2/p}\,d\approx 6.59754$ *with* $d=5$ *and* $p=10/3$

Symmetry breaking results and tools for proving symmetry

- ▷ The critical Caffarelli-Kohn-Nirenberg inequality
- ▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
- ▷ Linearization and spectrum
- ▷ Rényi entropy powers and fast diffusion: [Savaré, Toscani]
- ▷ Faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: [B. Gidas, J. Spruck, 1981], [M.-F. Bidaut-Véron, L. Véron, 1991]
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): [D. Bakry, M. Emery, 1984], [Bakry, Ledoux, 1996], [Demange, 2008], [JD, Esteban, Loss, 2014 & 2015] → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs
  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., [del Pino, JD, 2001], [Blanchet, Bonforte, JD, Grillo, Vázquez] → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*
  ▷ Mass transportation: [Otto] → *C. Villani, Optimal transport. Old and new (2009)*
  ▷ Rényi entropy powers (information theory) [Savaré, Toscani, 2014], [Dolbeault, Toscani]

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

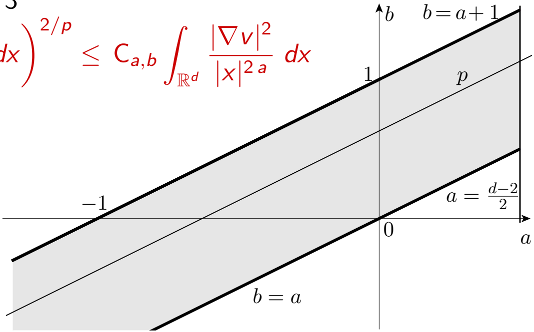
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \leq \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \leq b \leq a+1$ if $d \geq 3$

$a < b \leq a+1$ if $d=2$, $a+1/2 < b \leq a+1$ if $d=1$

and $a < a_c := (d-2)/2$

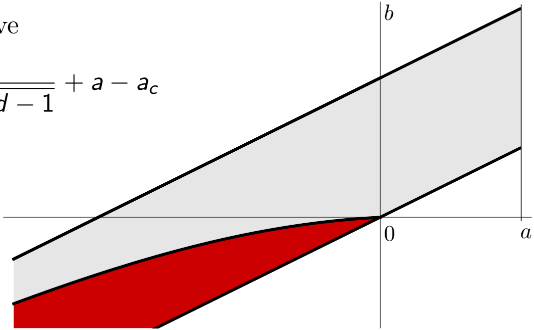
$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



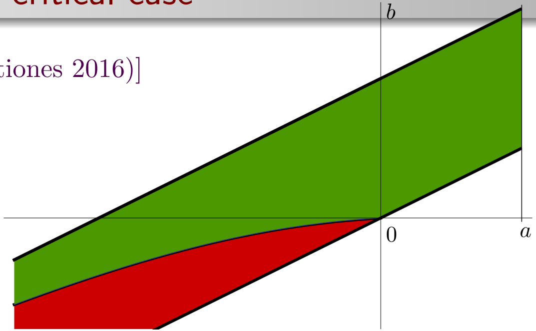
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]
The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (Inventiones 2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r = |x|\,, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})} \ge \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}^{\vartheta}\,\|w\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d\ge 2\,,\quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

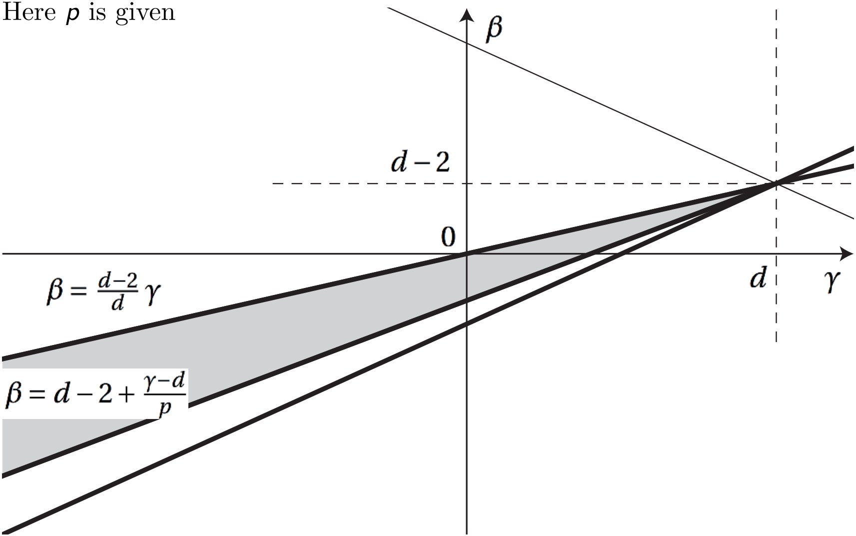
- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\,x\in\mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters

Here p is given



Symmetry and symmetry breaking

[JD, Esteban, Loss, Muratori, 2016]

Let us define $\beta_{\rm FS}(\gamma) := d-2-\sqrt{(d-\gamma)^2-4\,(d-1)}$

Theorem

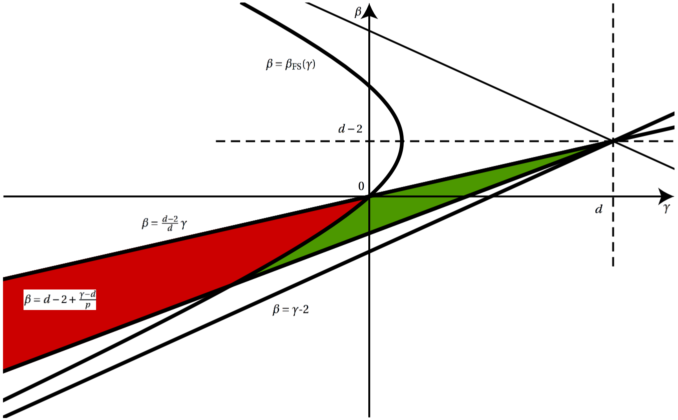
Symmetry breaking holds in (CKN) if

$$\gamma<0 \quad \textit{and} \quad \beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$$

In the range $\beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$

$$w_\star(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$$

is not optimal



The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha > \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta \log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star + \varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \dfrac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{\mathrm{FS}}$$

Symmetry in one slide: 3 steps

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathfrak{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac1s\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^\vartheta\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $\mathcal{L}_\alpha=-\,\mathcal{D}_\alpha^*\,\mathfrak{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal{L}_\alpha u^m$

$$-\,\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}$$
$$\ge(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathfrak{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac1n\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu$$
$$+\,2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

Inequalities without weights and fast diffusion equations

- Rényi entropy powers, the entropy approach without rescaling: [Savaré, Toscani]: scalings, nonlinearity and a concavity property inspired by information theory
  ▷ Faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

- Self-similar variables and relative entropies: the issue of the boundary terms [Carrillo, Toscani], [Carrillo, Vázquez], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter]

- Equivalence of the methods ?

- The role of the spectral gap

Rényi entropy powers: FDE in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \ge 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \ge 0$ such that $\int_{\mathbb{R}^d} v_0\, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2\, v_0\, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa\, t^{1/\mu}\right)^d}\, \mathcal{B}_\star\left(\frac{x}{\kappa\, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\, m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

▷ *Bakry-Emery method:* Compute I'

The Rényi entropy power

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m}\left(\frac{1}{d} + m - 1\right) = \frac{2}{d}\,\frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The *pressure variable* is $\mathsf{P} = \frac{m}{1-m}\,v^{m-1}$

$$\frac{\partial \mathsf{P}}{\partial t} = (m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla\mathsf{P}|^2$$

Then F'' is proportional to

$$(\sigma-1)\,(1-m)\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m \left|\Delta\mathsf{P} - \frac{\int_{\mathbb{R}^d} v\,|\nabla\mathsf{P}|^2\,dx}{\int_{\mathbb{R}^d} v^m\,dx}\right|^2 dx$$
$$+\,2\,\mathsf{E}^{\sigma-1}\int_{\mathbb{R}^d} v^m\,\Big\|\,\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\,\Big\|^2\,dx$$

The concavity property

Theorem

[Toscani-Savaré] *Assume that $m \ge 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \le 0$ and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|w\|^{1-\theta}_{\mathrm{L}^{q+1}(\mathbb{R}^d)} \ge \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1-\frac{1}{d} \le m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

Self-similar variables and relative entropies

A time-dependent rescaling: self-similar variables

$$v(t,x)=\frac{1}{\kappa^d\,R^d}\,u\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\,\log R(t)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial\tau}+\nabla\cdot\Big[u\,\big(\nabla u^{m-1}-2\,x\big)\Big]=0$$

- $\mathcal E[u]:=\int_{\mathbb R^d}\big(\mathfrak B^m-u^m-m\,\mathfrak B^{m-1}\,(\mathfrak B-u)\big)\,dx/m$ is such that

$$\frac{d}{dt}\mathcal E[u]=-\mathcal I[u]\,,\quad\mathcal I[u]:=\int_{\mathbb R^d}u\,\big|\nabla u^{m-1}+2\,x\big|^2\,dx$$

[del Pino, J.D.], [Carrillo, Toscani] $d\ge3$, $m\in[\frac{d-1}{d},+\infty)$, $m>\frac12$, $m\neq1$

$$\mathcal I[u]\ge4\,\mathcal E[u]$$

If $u_0\in L^1_+(\mathbb R^d)$ is such that $|x|^2\,u_0\in L^1(\mathbb R^d)$, $u_0^m\in L^1(\mathbb R^d)$, then

$$\mathcal E[u(t,\cdot)]\le\mathcal E[u_0]\,e^{-4\,t}$$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0 \quad \tau > 0\,, \quad x \in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R > 0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0\,, \quad x \in \partial B_R\,.$$

With $z(\tau, x) := \nabla Q(\tau, x) := \nabla u^{m-1} - 2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau} \int_{B_R} u\,|z|^2\,dx + 4 \int_{B_R} u\,|z|^2\,dx$$
$$+\; 2\,\tfrac{1-m}{m} \int_{B_R} u^m \left(\left\| \mathrm{D}^2 Q \right\|^2 - (1-m)\,(\Delta Q)^2 \right) dx$$
$$= \int_{\partial B_R} u^m \left(\omega \cdot \nabla |z|^2 \right) d\sigma \le 0 \text{ (by Grisvard's lemma)}$$

Reintroducing Rényi entropy powers

- the *relative entropy*

$$\mathcal{E}[u] := -\frac{1}{m}\int_{\mathbb{R}^d}\left(u^m - \mathcal{B}_\star^m - m\,\mathcal{B}_\star^{m-1}\,(u-\mathcal{B}_\star)\right)dx$$

- *relative Fisher information*

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|z|^2\,dx = \int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - 2\,x\right|^2\,dx$$

$$\mathcal{R}_\star[u] = 2\,\tfrac{1-m}{m}\int_{\mathbb{R}^d} u^m\,\left\|\mathrm{D}^2 u^{m-1} - \tfrac{1}{d}\,\Delta u^{m-1}\,\mathrm{Id}\right\|^2\,dx$$

$$+\,2\,(m-m_1)\,\tfrac{1-m}{m}\int_{\mathbb{R}^d} u^m\,\left|\Delta u^{m-1} - 2\,d\right|^2\,dx$$

Proposition

If $1-1/d \le m < 1$ *and* $d \ge 2$, *then*

$$\mathcal{I}[u_0] - 4\,\mathcal{E}[u_0] \ge \int_0^\infty \mathcal{R}[u(\tau,\cdot)]\,d\tau$$

Sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

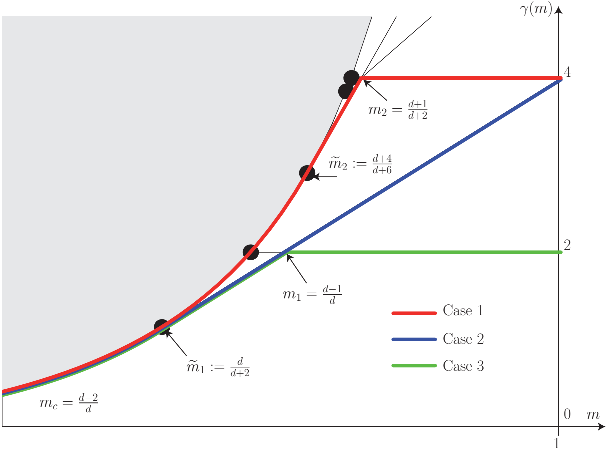
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gaps, asymptotic rates



[Bonforte, J.D., Grillo, Vázquez], [Bonforte, J.D., Grillo, Vázquez], [J.D., Toscani], [Denzler, Koch, McCann]

Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[\,|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0 \qquad \text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{E}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m=\frac{p+1}{2\,p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0\in\mathrm L^{1,\gamma}(\mathbb R^d)$ such that $\|u_0^m\|_{\mathrm L^{1,\gamma}(\mathbb R^d)}$ and $\int_{\mathbb R^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ are finite. Then

$$\mathcal E[v(t,\cdot)]\le\mathcal E[u_0]\,e^{-(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

Towards a parabolic proof

Let $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathfrak{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$ and define the diffusion operator L_α by

$$\mathsf{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha=\alpha^2\left(\partial_r^2+\frac{n-1}{r}\,\partial_r\right)+\frac{\Delta_\omega}{r^2}$$

where Δ_ω denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ and consider the equation

$$\frac{\partial u}{\partial\tau}=\mathsf{D}_\alpha^*\,(u\,z)$$

where

$$z:=\mathsf{D}_\alpha\,\mathsf{q}\,,\quad \mathsf{q}:=u^{m-1}-\mathcal{B}_\alpha^{m-1}\,,\quad \mathcal{B}_\alpha(x):=\left(1+\frac{|x|^2}{\alpha^2}\right)^{\frac{1}{m-1}}$$

If the weight does not introduce any singularity at $x=0$...

$$\frac{m}{1-m}\,\frac{d}{d\tau}\int_{B_R} u\,|z|^2\,d\mu_n$$

$$=\int_{\partial B_R} u^m\left(\omega\cdot\mathrm D_\alpha\,|z|^2\right)|x|^{n-d}\,d\sigma \quad (\le 0 \text{ by Grisvard's lemma})$$

$$-\,2\,\tfrac{1-m}{m}\left(m-1+\tfrac1n\right)\int_{B_R} u^m\,|\mathrm L_\alpha\,q|^2\,d\mu_n$$

$$-\int_{B_R} u^m\left(\alpha^4\,m_1\left|\mathsf q''-\tfrac{\mathsf q'}{r}-\tfrac{\Delta_\omega\,\mathsf q}{\alpha^2\,(n-1)\,r^2}\right|^2+\tfrac{2\,\alpha^2}{r^2}\left|\nabla_\omega\mathsf q'-\tfrac{\nabla_\omega\mathsf q}{r}\right|^2\right)d\mu_n$$

$$-\,(n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)\int_{B_R}\frac{|\nabla_\omega\mathsf q|^2}{r^4}\,d\mu_n$$

A formal computation that still needs to be justified ($x=0$?)

- Other potential application: the computation of Bakry, Gentil and Ledoux (chapter 6) for non-integer dimensions; weights on manifolds

[...]

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \left[|x|^{-\beta} v \nabla \left(v^{m-1} - |x|^{2+\beta-\gamma} \right) \right] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

Theorem

For "good" initial data, there exist positive constants $\mathcal{K}$, ζ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

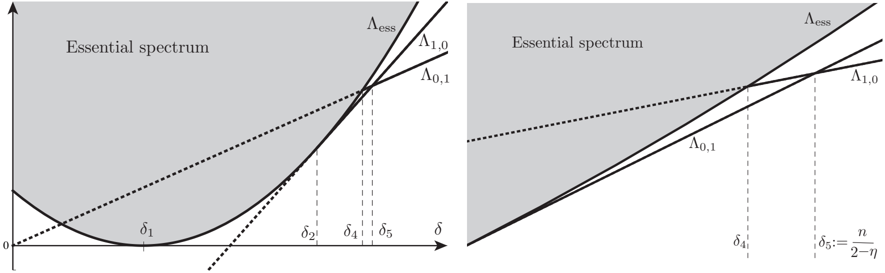
$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0, d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$

Λ is a spectral gap



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\mathrm{ess}}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\, w_t = -\frac{1}{\mathfrak{B}}\, \nabla \cdot \Big(|x|^{-\beta}\, \mathfrak{B}\, w\, \nabla \big((w^{m-1} - 1)\, \mathfrak{B}^{m-1} \big) \Big) & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*: [Chiarenza, Serapioni], Harnack inequalities; relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_* := \frac{n-4}{n-2}$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{E}[v(t)] \leq \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"

From asymptotic to global estimates

- When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{E}[v] \le \frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)] \le \mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t} \quad \forall\, t \ge 0 \quad \text{with} \quad \Lambda_\star := \frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

- Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{E}[v] \le \mathcal{I}[v] \quad \forall\, v \in \mathrm{L}^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)} = M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{E}[v(t)] \le \mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t} \quad \forall\, t \ge 0$$

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any $M>0$, we have*
$$0<\mathcal{K}(M)\le\frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M)\ge\frac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m\in[m_1,1)$

(i) For any $M>0$*, if* $\Lambda(M)=\Lambda_\star$ *then* $\beta=\beta_{\mathrm{FS}}(\gamma)$

(ii) If $\beta>\beta_{\mathrm{FS}}(\gamma)$ *then* $\Lambda_{0,1}<\Lambda_\star$ *and* $\Lambda(M)\in(0,\Lambda_{0,1}]$ *for any* $M>0$

(iii) For any $M>0$*, if* $\beta<\beta_{\mathrm{FS}}(\gamma)$ *and if symmetry holds in (CKN), then*
$$\Lambda(M)>\Lambda_\star$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon \, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathrm{D}^\star \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathrm{D} f \right)$$

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathrm{D} f_1 \cdot \mathrm{D} f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

$$\frac{1}{2}\,\frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx$$

$$= - \int_{\mathbb{R}^d} |\mathrm{D} f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = - \langle\!\langle f, f \rangle\!\rangle$$

for any f smooth enough, and

$$\frac{1}{2}\,\frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathrm{D} f \cdot \mathrm{D}(\mathcal{L}\, f)\, u\, |x|^{-\beta}\, dx = - \langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\,f_1 = \lambda_1\,f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g, g \rangle\!\rangle = -\,\langle g, \mathcal{L}\,g \rangle \geq \lambda_1\,\|g - \bar{g}\|^2\,, \quad \bar{g} := \langle g, 1 \rangle \,/\, \langle 1, 1 \rangle$$

$$-\,\langle\!\langle g, \mathcal{L}\,g \rangle\!\rangle \geq \lambda_1\,\langle\!\langle g, g \rangle\!\rangle$$

Proof: expansion of the square :
$-\,\langle\!\langle (g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle\!\rangle = \langle \mathcal{L}\,(g - \bar{g}), \mathcal{L}\,(g - \bar{g}) \rangle = \|\mathcal{L}\,(g - \bar{g})\|^2$

- Key observation:

$$\lambda_1 \geq 4 \quad \Longleftrightarrow \quad \alpha \leq \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta \log\left(\|\mathsf{D}_\alpha w\|_{\mathrm{L}^{2,\delta}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|w\|_{\mathrm{L}^{p+1,\delta}(\mathbb{R}^d)}\right) - \log\left(\|w\|_{\mathrm{L}^{2p,\delta}(\mathbb{R}^d)}\right)$$

with $\delta := d - n$ and

$$\mathcal{J}[w_\star + \varepsilon\, g] = \varepsilon^2\, \mathcal{Q}[g] + o(\varepsilon^2)$$

where

$$\frac{2}{\vartheta}\, \|\mathsf{D}_\alpha w_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\, \mathcal{Q}[g]$$
$$= \|\mathsf{D}_\alpha g\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)} + \frac{p\,(2+\beta-\gamma)}{(p-1)^2} \left[d-\gamma-p\,(d-2-\beta)\right] \int_{\mathbb{R}^d} |g|^2\, \frac{|x|^{n-d}}{1+|x|^2}\, dx$$
$$-p\,(2\,p-1)\, \frac{(2+\beta-\gamma)^2}{(p-1)^2} \int_{\mathbb{R}^d} |g|^2\, \frac{|x|^{n-d}}{(1+|x|^2)^2}\, dx$$

is a nonnegative quadratic form if and only if $\alpha \le \alpha_{\mathrm{FS}}$

- Symmetry breaking holds if $\alpha > \alpha_{\mathrm{FS}}$

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\,\mathcal{K}[u(\tau,\cdot)] = -\;\text{(sum of squares)}$$

If $\alpha \le \alpha_{\rm FS}$, then $\lambda_1 \ge 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional
With $u_\varepsilon = \mathcal{B}_\star\,\left(1+\varepsilon\,f\,\mathcal{B}_\star^{1-m}\right)$, we observe that

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \liminf_{\varepsilon\to0}\,\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\!\langle f\,,\mathcal{L}\,f\rangle\!\rangle}{\langle\!\langle f\,,f\rangle\!\rangle} = \frac{\langle\!\langle f_1\,,\mathcal{L}\,f_1\rangle\!\rangle}{\langle\!\langle f_1\,,f_1\rangle\!\rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\rm FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\rm FS}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \le \alpha_{\rm FS}$, the fact that $\mathcal{K}/\mathcal{I} \ge 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$$

so that

$$\mathcal{I}[u] - 4\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - 4\,\mathcal{E}[\mathcal{B}_\star] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \le \alpha_{\rm FS}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]$$

is monotone decreasing

- This explains why the method based on nonlinear flows provides the *optimal range for symmetry*

Entropy – production of entropy inequality

Using $\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - \mathcal{C}_2\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$, we know that

$$\mathcal{I}[u] - \mathcal{C}_2\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - \mathcal{C}_2\,\mathcal{E}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$\mathcal{C}_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \ge \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star\,\left(1 + \varepsilon\,f\,\mathcal{B}_\star^{1-m}\right)$, we observe that

$$\mathcal{C}_1 \le \lim_{\varepsilon\to0}\inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f,\mathcal{L}\,f\rangle}{\langle f,f\rangle} = \frac{\langle f_1,\mathcal{L}\,f_1\rangle}{\langle f_1,f\rangle_1} = \lambda_1 = \lim_{\varepsilon\to0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

- If $\lim_{\varepsilon\to0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2 = \lambda_1$

This happens if $\alpha = \alpha_{\rm FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)

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Thank you for your attention !