A review on L^2 Hypocoercivity methods

based on lower order perturbations, diffusion limits and interpolation inequalities

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Outline

Hypocoercivity: global overview

H^1 hypocoercivity: an example

$$\frac{\partial f}{\partial t} + \mathsf{T} f = \Delta_v f + \nabla_v \cdot (v\, f)\,, \quad \mathsf{T} f := v \cdot \nabla_x f - x \cdot \nabla_v f \qquad \text{(VFP)}$$

(JD, X. Li) take $h = (f/f_*)^{2/p}$, $p \in (1,2)$

$$\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\,\frac{|\nabla_v h|^2}{h}\,, \qquad \mathsf{L}\, h := \Delta_v h - v \cdot \nabla_v h$$

Twisted Fisher information

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2\, d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2\, d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\, d\mu$$

Theorem

For some $t \mapsto \lambda(t)$, there is a function $t \mapsto \rho(t) > 1$ a.e.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le -\rho(t)\, \mathcal{J}_{\lambda(t)}[h(t,\cdot)] \quad \forall\, t \ge 0$$

and $\mathcal{J}_{\lambda(t)}[h(t,\cdot)] \le \mathcal{J}_{1/2}[h_0] \exp\left(-\int_0^t \rho(s)\, ds\right)$

(C. Villani) (JD, X. Li) + coupling with a Beckner type inequality

L^2 hypocoercivity: the strategy

(JD, C. Mouhot, C. Schmeiser) Π is the orthogonal projection on $\mathrm{Ker}(\mathsf{L})$

$$\varepsilon \frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\mathsf{L}F$$

$F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$, $u = F_0 = \Pi F_0$

$$\partial_t u + (\mathsf{T}\Pi)^* (\mathsf{T}\Pi) u = 0$$

▷ The *microscopic coercivity* assumption

$$\frac{1}{2}\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \le -\lambda_m \|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ *decays exponentially* and a *macroscopic coercivity* assumption (Poincaré inequality) is needed

$$\|\mathsf{T}\Pi F\|^2 \ge \lambda_M \|\Pi F\|^2$$

$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that $\langle \mathsf{A}\mathsf{T}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\|\Pi F\|^2$ and we can use the L^2 entropy / Lyapunov functional

$$\mathsf{H}[F] := \frac{1}{2}\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

H^{-1} hypocoercivity: a brief summary

▷ (S. Armstrong, J.-C. Mourrat, 2019), (D. Albritton, S. Armstrong, J.-C. Mourrat, M. Novack, 2024), (Y. Cao, J. Lu, L. Wang, 2020-23). Variational methods for the kinetic Fokker-Planck equation.

▷ (G. Brigati, 2021), (G. Brigati & *al.*). Time averages for kinetic Fokker-Planck equations

Consider the *kinetic-Ornstein-Uhlenbeck equation*

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h := \Delta_v h - \alpha\, v\, \langle v \rangle^{\alpha-2} \cdot \nabla_v h\,, \quad h(0,\cdot,\cdot) = h_0 \qquad \text{(OU)}$$

on $\mathbb{R}^+ \mathbb{T}^d \times \mathbb{R}^d$ with local equilibrium $\gamma_\alpha(v) = Z_\alpha^{-1}\, e^{-\langle v \rangle^\alpha}$

Theorem (G. Brigati)

Let $\alpha \ge 1$ and $\tau > 0$. There exists a constant $\lambda > 0$ such that, for all $h_0 \in L^2(dx\, d\gamma_\alpha)$ with zero-average,

$$\fint_t^{t+\tau} \|h(s,\cdot,\cdot)\|^2_{L^2(dx\, d\gamma_\alpha)}\, ds \le \|h_0\|^2_{L^2(dx\, d\gamma_\alpha)}\, e^{-\lambda\, t} \quad \forall\, t \ge 0$$

Averaging lemma + Poincaré inequality (based on JL Lions' lemma)

Vlasov-Fokker-Planck equation: history, methods

Vlasov-Fokker-Planck equation with external potential ψ

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f = \Delta_v f + \nabla_v (v\, f) \qquad \text{(VFP)}$$

- Decomposition on Hermite functions ($\psi(x) = |x|^2/2$) and spectral results
- Green's function as in Kolmogorov's computation

$$G(t,x,v) = \frac{\exp\left(-\frac{\gamma(t)\,|x|^2+\alpha(t)\,|v|^2+\beta(t)\,x\cdot v}{4\,\alpha(t)\,\gamma(t)-\beta^2(t)}\right)}{(2\,\pi)^d\left(4\,\alpha(t)\,\gamma(t)-\beta^2(t)\right)^{d/2}}$$

- Hypoelliptic methods (Hörmander, 1965)
- H^1 hypocoercivity (Villani, 2001 & 2005)
- L^2 hypocoercivity (Mouhot, Neumann, 2006), (Hérau, 2006), (JD, Mouhot, Schmeiser 2009 & 2015) + Lyapunov functions method (Arnold, Erb, 2014)
- H^{-1} hypocoercivity (Armstrong, Mourrat, 2019), (Brigati, 2021), (Cao, Lu, Wang, 2020), (Albritton, Armstrong, Mourrat-Novack, 2021), (Brigati, Stoltz, 2023), (Brigati, Lörler, L. Wang, 2024), (Brigati, Stoltz, A.Q. Wang, L. Wang, 2024)

L^2 Hypocoercivity

▷ Abstract statement, diffusion limit (the compact case)

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$
* denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$

Π is the orthogonal projection onto the null space of L

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F \rangle \le -\lambda_m \,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

$\Leftarrow$ *microscopic coercivity*

Formal macroscopic / diffusion limit

$F = F(t,x,v)$, $\mathsf{T} = v\cdot\nabla_x$, L good collision operator. Scaled evolution equation

$$\varepsilon\,\frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. With $F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned} \varepsilon^{-1}: &\quad \mathsf{L}F_0 = 0\,,\\ \varepsilon^{0}: &\quad \mathsf{T}F_0 = \mathsf{L}F_1\,,\\ \varepsilon^{1}: &\quad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2 \end{aligned}$$

- ▷ the first equation reads as $u = F_0 = \Pi F_0$
- ▷ the second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
- ▷ after projection, the third equation is $\frac{d}{dt}\,(\Pi F_0) - \Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$

$\Leftarrow$ *macroscopic coercivity*

The assumptions in the compact case

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \,\|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \,\|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi\, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \,\|(1-\Pi)F\| \tag{H4}$$

Equivalence and entropy decay

For some $\delta > 0$ to be chosen, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \frac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Entropy decay: $\frac{d}{dt}\mathsf{H}[F] = -\,\mathsf{D}[F]$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\mathsf{D}[F] \ge \lambda\,\mathsf{H}[F]$$

(JD, C. Mouhot, C. Schmeiser, 2015)

Theorem

Under (H1)–(H4), *for any* $t \ge 0$,

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda\,t}$$

$$\|F(t,\cdot)\|^2 \le \mathcal{C}\,\|F_0\|^2\,e^{-\lambda\,t} \quad \textit{with} \quad \mathcal{C} = \frac{2+\delta}{2-\delta}$$

Details of the proof

$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that $\langle \mathsf{AT}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$
With $X := \|(\mathrm{Id}-\Pi)F\|$ and $Y := \|\Pi F\|$, we can prove that

$$\frac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle =: \mathsf{H}[F] \le \frac{1}{2}\,(X^2 + Y^2) + \frac{\delta}{2}\,X\,Y$$

$$\mathsf{D}[F] \ge (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y$$

The largest value of $\lambda = \lambda(\delta)$ for which $\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \ge 0$ can be estimated by the largest value of λ for which

$$\begin{aligned}\mathcal{Q}(X,Y) :=& (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\,(X^2+Y^2) - \frac{\lambda}{2}\,\delta\,X\,Y\\ =& \left(\lambda_m - \delta - \frac{\lambda}{2}\right)X^2 - \delta\left(C_M + \frac{\lambda}{2}\right)X\,Y + \left(\frac{\delta\,\lambda_M}{1+\lambda_M} - \frac{\lambda}{2}\right)Y^2\end{aligned}$$

is a nonnegative quadratic form

A classical example

- Vlasov-Fokker-Planck equation (harmonic potential)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v\, f) \qquad \text{(VFP)}$$

With $\rho := \int_{\mathbb{R}^d} F(v)\, dv$, $\Pi F := \rho\, \mathcal{M}(v)$, $\phi(x) = \frac{1}{2}\,|x|^2$

▷ *microscopic coercivity:* Gaussian Poincaré inequality in v

$$\int_{\mathbb{R}^d} |F(v) - \rho\, \mathcal{M}(v)|^2 \, \frac{dv}{\mathcal{M}(v)} \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_v F(v)|^2 \, \frac{dv}{\mathcal{M}(v)} \qquad \text{(H1)}$$

▷ *macroscopic coercivity:* Gaussian Poincaré inequality in x

$$\int_{\mathbb{R}^d} \left| \rho(x) - \frac{M\, e^{-|x|^2/2}}{(2\pi)^{d/2}} \right|^2 e^{\frac{|x|^2}{2}}\, dx \le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_x \rho(x)|^2\, e^{\frac{|x|^2}{2}}\, dx \qquad \text{(H2)}$$

▷ *parabolic macroscopic dynamics* (H3) and *bounded auxiliary operators* (H4) are consequences of elliptic estimates

Without confinement

▷ Nash's inequality and consequences

Basic examples without confinement

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f_0(x,v)$$

L is the *Fokker-Planck operator* L_1 or the *linear BGK operator* L_2

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\,f) \quad \text{and} \quad \mathsf{L}_2 f := \rho_f\,\mu - f$$

$\mu(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\,\pi)^{d/2}}$ is the normalized Gaussian function

$\rho_f := \int_{\mathbb{R}^d} f\,dv$ is the spatial density

$$d\gamma := \gamma(v)\,dv \quad \text{where} \quad \gamma := \frac{1}{\mu}$$

Weighted norm

$$\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} |f(x,v)|^2\,dx\,d\gamma$$

Results (whole space, no external potential)

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \delta\,\langle \mathsf{A}f, f\rangle_{dx\,d\gamma}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

(Bouin, JD, Mischler, Mouhot, Schmeiser, 2020)

Theorem

There exists a constant $C > 0$ such that, for any $t \ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

By the *Nash inequality* (1958)

$$\|u\|^2_{L^2(dx)} \le \mathcal{C}_{\rm Nash}\, \|u\|^{\frac{4}{d+2}}_{L^1(dx)}\, \|\nabla u\|^{\frac{2\,d}{d+2}}_{L^2(dx)} \quad \forall\, u \in L^1 \cap H^1(\mathbb{R}^d)$$

$$\|\Pi f\|^2 \le \Phi^{-1}\big(2\,\langle \mathsf{AT\Pi}f, f\rangle\big) \quad \text{with} \quad \Phi^{-1}(y) := y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}} \quad \forall\, y \ge 0$$

where $\mathsf{c} = 2\,\Theta\,\mathcal{C}_{\rm Nash}^{-1-\frac{2}{d}}\, \|f_0\|^{-\frac{4}{d}}_{L^1(dx\,dv)}$

We deduce the entropy decay inequality

$$\mathsf{D}[F] = -\frac{d}{dt}\mathsf{H}[f] \gtrsim \left(\|f_0\|^{\frac{4}{d+2}}_{L^2(dx\,d\gamma)} + \|f_0\|^{\frac{4}{d+2}}_{L^1(dx\,dv)}\right)^{-\frac{d+2}{d}} \mathsf{H}[f]^{1+\frac{2}{d}}$$

Fokker-Planck equations with various external potentials, moments and functional inequalities

▷ Adapted functional inequalities

Strong confinement case: Poincaré inequality

If $\phi(x) = \alpha^{-1} \langle x \rangle^\alpha$ with $\alpha \geq 1$, $\langle x \rangle := \sqrt{1+|x|^2}$, and $d\mu = e^{-\phi}\,dx$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2\,d\mu \geq \lambda_M \int_{\mathbb{R}^d} |u|^2\,d\mu \quad \forall\, u \in \mathscr{D}(\mathbb{R}^d) \quad \text{s.t.} \int_{\mathbb{R}^d} u\,d\mu = 0$$

with $\lambda_M > 0$ (false if $\alpha < 1$). A solution $\rho = e^\phi\,u$ of

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho\,\nabla \phi) \tag{FP}$$

satisfies

$$\frac{d}{dt}\|\rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d,d\mu)} = -\,2\,\|\nabla \rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d,d\mu)} \leq -\,2\,\lambda_M\,\|\rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d,d\mu)}$$

which yields the estimate

$$\|\rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d,d\mu)} \leq \|\rho_0\|^2_{L^2(\mathbb{R}^d,d\mu)}\,e^{-2\,\lambda_M\,t} \quad \forall\, t \geq 0$$

Weak confinement case: weighted Poincaré inequality

Let $\phi(x)=\alpha^{-1}\langle x\rangle^\alpha$ with $\alpha\in(0,1)$
If $u=\rho\,e^\phi$ solves the *Ornstein-Uhlenbeck* (*backward Kolmogorov*) equation

$$\frac{\partial u}{\partial t}=\Delta u-\nabla\phi\cdot\nabla u \qquad \text{(OU)}$$

▷ *Moments.*

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2\,\langle x\rangle^k\,e^{-\phi}\,dx+2\int_{\mathbb{R}^d}|\nabla_x u(t,x)|^2\,\langle x\rangle^k\,e^{-\phi}\,dx \le \int_{\mathbb{R}^d}\left(a_k-b_k\,\langle x\rangle^{\alpha-2}\right)|u(t,x)|^2\,\langle x\rangle^k\,e^{-\phi}\,dx$$

There exists a constant $\mathcal{K}(k)>0$ such that for any $t\ge0$

$$\int_{\mathbb{R}^d}\langle x\rangle^k\,|\rho(t,x)|^2\,e^\phi\,dx\le\mathcal{K}(k)\int_{\mathbb{R}^d}\langle x\rangle^k\,|\rho_0|^2\,e^\phi\,dx$$

▷ *Weighted Poincaré inequality*. With $\bar u=\int_{\mathbb{R}^d}u\,d\mu/\int_{\mathbb{R}^d}d\mu$

$$\int_{\mathbb{R}^d}|\nabla_x u|^2\,d\mu\ge\mathscr{C}_\alpha^{\mathrm{wP}}\int_{\mathbb{R}^d}|u-\bar u|^2\,\frac{e^{-\phi}}{\langle x\rangle^{2\,(1-\alpha)}}\,dx$$

(E. Bouin, JD, L. Lafleche, C. Schmeiser, 2020)

Theorem

For any $t \ge 0$

$$\int_{\mathbb{R}^d} |\rho(t,x) - \rho_\star(x)|^2 \, e^{\phi} \, dx \le \left(\|\rho_0 - \rho_\star\|_{L^2(\mathbb{R}^d, d\mu)}^{-4\,(1-\alpha)/k} + \frac{4\,(1-\alpha)\,\mathscr{C}_\alpha^{\mathrm{wP}}}{k\,\mathcal{K}_*^{2\,(1-\alpha)/k}} \, t \right)^{-\frac{k}{2\,(1-\alpha)}}$$

Hölder's inequality with $k \ge 2\,(1-\alpha)$ and $\theta = k/\big(k + 2\,(1-\alpha)\big)$

$$\int_{\mathbb{R}^d} |u - \bar{u}|^2 \, e^{-\phi} \, dx \le \left(\int_{\mathbb{R}^d} |u - \bar{u}|^2 \, \frac{e^{-\phi}}{\langle x \rangle^{2\,(1-\alpha)}} \, dx \right)^{\theta} \left(\int_{\mathbb{R}^d} |u - \bar{u}|^2 \, \langle x \rangle^k \, e^{-\phi} \, dx \right)^{1-\theta}$$

$$\mathcal{K}_* := \mathcal{K}(k)^2 \int_{\mathbb{R}^d} \langle x \rangle^k \, |\rho_0|^2 \, e^{\phi} \, dx + \int_{\mathbb{R}^d} \langle x \rangle^k \, e^{-\phi} \, dx \, \|\rho_0\|_{L^1(\mathbb{R}^d)}^2$$

Weak confinement, a limit case

In the limit as $\alpha \to 0_+$, take

$$\phi(x) = \gamma \log\langle x\rangle$$

with $\gamma > d$: there is a $\mathrm{L}^1(\mathbb{R}^d)$ stationary solution of (FP)

▷ *Hardy-Poincaré* inequality

$$\int_{\mathbb{R}^d} |\nabla_x u|^2 \langle x\rangle^k \, d\mu \ge \mathscr{C}^{\mathrm{HP}}_{\gamma-k} \int_{\mathbb{R}^d} |u-\bar{u}|^2 \langle x\rangle^{k-2} \, d\mu$$

▷ *Moments*

Theorem

For any $t \ge 0$

$$\int_{\mathbb{R}^d} |\rho(t,x)-\rho_\star(x)|^2 \, d\mu \le \int_{\mathbb{R}^d} |\rho_0-\rho_\star|^2 \, d\mu \, (1+c\,t)^{-\frac{k}{2}} \quad \forall\, t \ge 0$$

where c depends on d, γ, k, $\int_{\mathbb{R}^d} |\rho_0|^2 \langle x\rangle^{k-\gamma} \, dx$ and $\|\rho_0\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$

(E. Bouin, JD, L. Ziviani, 2024)

Very weak confinement case: CKN inequality

Let $1 \le \gamma < d$, $k = \max\{2, \gamma/2\}$, $a = (d+2k-\gamma)/(d+2+2k-\gamma)$ and

$$\phi(x) = \gamma \log\langle x\rangle$$

With $e^{-\phi} = \langle x\rangle^{-\gamma}$, if $u = \rho\,\langle x\rangle^{\gamma}$ solves (OU)

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2\,\langle x\rangle^{-\gamma}\,dx = -\,2\int_{\mathbb{R}^d} |\nabla u(t,x)|^2\,\langle x\rangle^{-\gamma}\,dx$$

$$M_k(t) := \int_{\mathbb{R}^d} \langle x\rangle^k\,\rho(t,x)\,dx$$
$$\le 2^{\frac{k-2}{2}}\left(M_0 + \left(\big(M_k(0)-M_0\big)^{2/k} + 2\,(d+k-2-\gamma)\,M_0^{2/k}\,t\right)^{k/2}\right)$$

▷ *Inhomogeneous Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^d} |u|^2\,\langle x\rangle^{-\gamma}\,dx \le \mathscr{C}_{k,\gamma}^{\rm CKN}\left(\int_{\mathbb{R}^d} |\nabla u|^2\,\langle x\rangle^{-\gamma}\,dx\right)^{a}\left(\int_{\mathbb{R}^d} u\,\langle x\rangle^{k-\gamma}\,dx\right)^{2(1-a)}$$

Theorem

For any $t \ge 0$

$$\|\rho(t,\cdot)\|^2_{L^2(\mathbb{R}^d,d\mu)} \le \|\rho_0\|^2_{L^2(\mathbb{R}^d,d\mu)}\,(1+c\,t)^{-\frac{d-\gamma}{2}}$$

(E. Bouin, JD, C. Schmeiser, 2020)

No potential case: Nash's inequality

We assume that $\phi = 0$ so that (FP) is the heat equation

$$\frac{d}{dt}\,\|\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = -2\,\|\nabla\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ *Nash's inequality*

$$\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} \le \mathscr{C}_{\mathrm{Nash}}\,\|\nabla u\|^{\frac{d}{d+2}}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|u\|^{\frac{2}{d+2}}_{\mathrm{L}^1(\mathbb{R}^d)}$$

Hence $y(t) := \|\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ solves $y' \le -2\,\mathscr{C}_{\mathrm{Nash}}^{-1}\,\|\rho_0\|^{-\frac{4}{d}}_{\mathrm{L}^1(\mathbb{R}^d)}\,y^{1+\frac{2}{d}}$

Theorem

For any $t \ge 0$

$$\|\rho(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \left(\|\rho_0\|^{-4/d}_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{4}{d\,\mathscr{C}_{\mathrm{Nash}}}\,\|\rho_0\|^{-4/d}_{\mathrm{L}^1(\mathbb{R}^d)}\,t\right)^{-d/2}$$

(E. Bouin, JD, S. Mischler, C. Mouhot, C. Schmeiser, 2020)

Potential	$\phi = 0$	$\phi(x) = \gamma \log\langle x\rangle$ $\gamma < d$	$\phi(x) = \gamma \log\langle x\rangle$ $\gamma > d$	$\phi(x) = \frac{1}{\alpha}\langle x\rangle^\alpha$ $\alpha \in (0,1)$	$\phi(x) = \frac{1}{\alpha}\langle x\rangle^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn-Nirenberg	Hardy-Poincaré	Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-k/2}$ convergence	$t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence
References	[7]	[4]	[5]	[3]	$(*)$

Table 1 Alternative: *weak Poincaré inequalities*: see [8, Theorem 2.1], [1, Theorem 1.4] and [6]. $(*)$ *Poincaré inequality*: [7], [2]

References [1] D. Bakry, P. Cattiaux, A. Guillin, 2008; [2] D.Bakry, I. Gentil, M. Ledoux, 2014; [3] E. Bouin, JD, L. Lafleche, C. Schmeiser, 2020; [4] E. Bouin, JD, C. Schmeiser, 2020; [5] E. Bouin, JD, L. Ziviani, 2024; [6] O. Kavian, S.Mischler, M. Ndao, 2021; [7] J.Nash, 1958; [8] M. Röckner, F.Y. Wang, 2001

Generalized kinetic Fokker-Planck equation

Generalized kinetic Fokker-Planck equation

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi\cdot\nabla_v f = e^{-\psi(v)}\,\nabla_v\left(e^{\psi(v)}\,\nabla_v f\right), \quad f_\star(x,v) = Z^{-1}\,\exp\left(-\phi(x)-\psi(v)\right) \qquad \text{(gKFP)}$$

Potential	$\phi = 0$	$\phi(x) = \frac{1}{\alpha}\langle x\rangle^\alpha$ $\alpha\in(0,1)$	$\phi(x) = \frac{1}{\alpha}\langle x\rangle^\alpha$ $\alpha\ge 1$, or $\mathbb{T}^d$ Macro Poincaré
$\psi(v) = \frac{1}{\beta}\langle v\rangle^\beta$ $\beta\ge 1$ Micro Poincaré	$t^{-d/2}$ decay [4]	e^{-t^b}, $b<1$ $\beta = 2$ convergence [7]	$e^{-\lambda t}$ convergence [11, 13, 9, 10, 12, 1, 8, 5, 6]
$\psi(v) = \frac{1}{\beta}\langle v\rangle^\beta$ $\beta\in(0,1)$	$t^{-\zeta}$ $\zeta = \min\{\frac{d}{2}, \frac{\ell}{2(1-\beta)}\}$ decay, [3]	$t^{-\zeta}$ convergence (*)	$t^{-\zeta}$ convergence (*)
Limit as $\beta\to 0_+$ $\psi(v) = -(d+\varepsilon)\,\log\langle v\rangle$	$\varepsilon\in(0,2)$ fractional diffusion limit, [2]	(*)	$t^{-\zeta}$ if $\varepsilon > 2$ convergence (*)

Table 2 Additional assumptions are needed: if $(\alpha,\beta)\in[1+\infty)\times(0,1)$, $f_0\in L^2(\mathbb{R}^d\times\mathbb{R}^d,\langle v\rangle^{(1-\beta)\sigma}\,d\mu)$; if $\phi = 0$ and $\beta\in(0,1)$, $f_0\in L^2(\mathbb{R}^d\times\mathbb{R}^d,\langle v\rangle^{\ell/2}\,d\mu)$; if $(\alpha,\beta)\in(0,1)\times(0,1)$, uniform bounds (weak Poincaré inequality)

References [1] D. Albritton, S. Armstrong, J.C. Mourrat, M. Novack, 2024; [2] E. Bouin, JD, L. Lafleche, 2022; [3] E. Bouin, JD, L. Lafleche, C. Schmeiser, 2020; [4] E. Bouin, JD, S. Mischler, C. C. Mouhot Schmeiser, 2020; [5] G. Brigati, 2023; [6] G. Brigati, G. Stoltz, 2023; [7] C. Cao, 2019; [8] Y. Cao, J. Lu, L. Wang, 2023; [9] JD, C.C. Mouhot Schmeiser, 2009; [10] JD, C.C. Mouhot Schmeiser, 2015; [11] F. Hérau, 2006; [12] S. Mischler, C. Mouhot, 2016; [13] C. Mouhot, L. Neumann, 2006

With the transport operator defined by the Poisson bracket as

$$\mathsf{T} f := \nabla_v \mathscr{E} \cdot \nabla_x f - \nabla_x \mathscr{E} \cdot \nabla_v f$$

corresponding to the Hamiltonian energy $\mathscr{E}(x,v) := \frac{1}{\beta} \langle v \rangle^\beta + \phi(x)$
(E. Bouin, JD, L. Ziviani, 2024) Under the condition

$$0 \le f_0 \le C\, f_\star$$

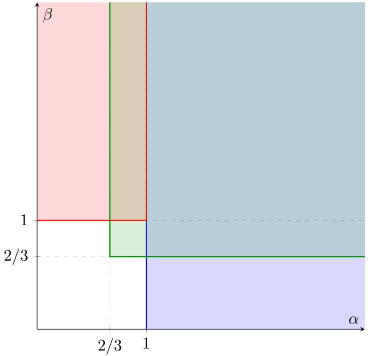
Potential	$\phi = 0$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \in (0,1)$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \ge 1$
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \ge 1$ Micro Poincaré	$t^{-d/2}$ decay	$t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \in (0,1)$	$t^{-\min\left\{\frac{d}{2}, \frac{\ell}{2(1-\beta)}\right\}}$ convergence	$t^{-\min\left\{\frac{k}{2(1-\alpha)}, \frac{\ell}{2(1-\beta)}\right\}}$ convergence	$t^{-\frac{\ell}{2(1-\beta)}}$ convergence

... Without the assumption

$$0 \le f_0 \le C\, f_\star$$

Regions in the (α, β) plane:
Green: H and moments
Red: Lyapunov function method
Blue: modified DMS method

(L. Ziviani, in progress)



With

$$\mathsf{T} f := v \cdot \nabla_x f - \nabla_x \mathscr{E} \cdot \nabla_v f$$

instead of $\mathsf{T} f := \nabla_v \mathscr{E} \cdot \nabla_x f - \nabla_x \mathscr{E} \cdot \nabla_v f$... ?
(E. Bouin, L. Ziviani, 2025) (for very weak potentials)

Hypocoercivity: further results

- Mode-by-mode methods
- Fractional diffusion limits
- Macroscopic modes

Mode-by-mode methods

Fourier variables

On $\mathbb{R}^d$ or on $\mathbb{T}^d$, the transport operator $\mathsf{T} = v \cdot \nabla_x$ is rewritten as a simple multiplication operator $i\, v \cdot \xi$ and

$$(\mathsf{A}F)(v) = -\frac{i\,\xi}{1+|\xi|^2} \cdot \int_{\mathbb{R}^d} w\, F(w)\, dw\, F(v)$$

With $X := \|(\mathrm{Id} - \Pi)F\|$ and $Y := \|\Pi F\|$, we obtain

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle \ge X^2 + \frac{\delta\,|\xi|^2}{1+|\xi|^2}\, Y^2$$

Proving that $\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \ge 0$ amounts to choose λ such that the quadratic form $\mathcal{Q}(X,Y)$ is nonnegative: a discriminant condition (Bouin, JD, Mischler, Mouhot, Schmeiser, 2020), (A. Arnold, JD, C. Schmeiser, T. Wöhrer, 2021)

Optimisation on the parameters

$$\lambda_\star(\delta) := \sup\Big\{\lambda \in (0, 2\,\lambda_m) \,:\, h_\star(\delta,\lambda) \le 0\Big\} \quad \text{and} \quad C_\star(\delta) := \tfrac{2+\delta}{2-\delta}$$

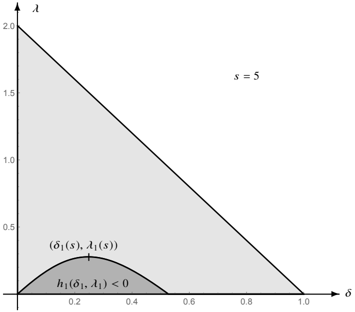
$$h_\star(\delta,\lambda) := \delta^2 \left(C_M + \tfrac{\lambda}{2}\right)^2 - 4\left(\lambda_m - \delta - \tfrac{\lambda}{2}\right)\left(\tfrac{\delta\,\lambda_M}{1+\lambda_M} - \tfrac{\lambda}{2}\right)$$

With $s := |\xi|$, $\lambda_m = 1$, $\lambda_M = s^2$ and $C_M = s\left(1 + \sqrt{3}\,s\right)/(1+s^2)$

Mode-by-mode optimisation

$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda\,t}$

$\|F(t,\cdot)\|^2 \le C_\star(\delta)\,e^{-\lambda_\star(\delta)\,t}\,\|F_0\|^2$



Fractional diffusion limits

Fractional diffusion limits and hypocoercivity

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f$$

with *fat tail local equilibrium* μ

$$\forall\, v \in \mathbb{R}^d, \quad \mu(v) = \frac{c_\gamma}{\langle v \rangle^{d+\gamma}} \quad \text{where} \quad \langle v \rangle := \sqrt{1 + |v|^2}\,.$$

▷ *Fokker-Planck* type operator $(\beta = 2)$

$$\mathsf{L}_1 f := \nabla_v \cdot \big(\mu\, \nabla_v \big(\mu^{-1} f\big)\big)$$

▷ *Linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f := \int_{\mathbb{R}^d} \mathrm{b}(\cdot, v') \Big(f(v')\, \mu(\cdot) - f(\cdot)\, \mu(v') \Big)\, \mathrm{d}v'$$

with *collision frequency* $\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v, v')\, \mu(v')\, \mathrm{d}v' \underset{|v|\to+\infty}{\sim} |v|^{-\beta}$

▷ the *fractional Fokker-Planck* operator $(0 < \sigma < 2,\ \beta = \sigma - \gamma)$

$$\mathsf{L}_3 f := \Delta_v^{\sigma/2} f + \nabla_v \cdot (E\, f)$$

\+ technical conditions

Decay result

(Bouin, JD, Lafleche, 2022)

Theorem

Let $d \geq 2$, $\beta \in \mathbb{R}$, $\gamma > \max\{0, -\beta\}$ *and* $k \in [0, \gamma)$ *such that* $\gamma \neq 2 + \beta$ *or if* $\gamma = 2 + \beta$ *and* $\frac{k}{\beta_+} > \frac{d}{2}$. *If* f *is a solution with initial condition* $f^{\rm in} \in \mathrm{L}^1(\,\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2\big(\langle v\rangle^k\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v\big)$, *then for any* $t \geq 0$,

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v)} \lesssim \frac{\left\|f^{\rm in}\right\|^2_{\mathrm{L}^1(\,\mathrm{d}x\,\mathrm{d}v)} + \left\|f^{\rm in}\right\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mu^{-1}\,\mathrm{d}v)}}{(1+t)^\tau}$$

with $\tau = \min\left\{\frac{d}{\alpha}, \frac{k}{\beta_+}\right\}$ *and* $\alpha = \min\left\{\frac{\gamma+\beta}{1+\beta}, 2\right\}$

Threshold case $\gamma = 2 + \beta$, and with either $k = 0$ if $\beta < 0$ or $k > 0$ if $\beta \geq 0$, $O(t\log t)^{-d/2}$ + specific statement if $d = 1$

Tools for the proof

▷ The *Lyapunov function* property: there exist three positive constant a, b and R, a real parameter β, and a (smooth) positive *Lyapunov function* $F = F(v)$ on $\mathbb{R}^d$ such that

$$-\mathsf{L}F \le \left(a\,\mathbb{1}_{B_R} - b\,\langle v\rangle^{-\beta}\right) F$$

▷ Moment estimates + stationary states are not factorized + fractional Nash inequality

▷ The L^2-hypocoercivity with $\mathsf{A}_\xi := \frac{1}{\langle v\rangle^2}\,\Pi\,\frac{(-i\,v\cdot\xi)\langle v\rangle^\beta}{1+\langle v\rangle^{2\,|1+\beta|}\,|\xi|^2}$ (in Fourier variables) and the *entropy functional*

$$\mathsf{H}_\xi[f] := \left\|\widehat{f}\right\|^2 + \delta\,\mathrm{Re}\langle\mathsf{A}_\xi\widehat{f},\widehat{f}\rangle$$

▷ Fractional diffusion: the exponent α arises from

$$\partial_t u + (-\Delta)^{\alpha/2} u = 0$$

(A. Mellet, S. Mischler, C. Mouhot, 2011), (M. Jara, T. Komorowski, S. Olla, 2011), (Cattiaux, Noureddine, Puel, 2019), (Fournier, Tardif, 2021), (E. Bouin, L. Kanzler, C. Mouhot),...

Fractional diffusion limit

The scaled equation

$$\varepsilon^\alpha\,\partial_t f+\varepsilon\,v\cdot\nabla_x f=\mathsf{L} f$$

written in Fourier variables is $\varepsilon^\alpha\,\partial_t\widehat{f}+i\,\varepsilon\,v\cdot\xi\,\widehat{f}=\mathsf{L}\widehat{f}$ where $\mathsf{L}=\mathsf{L}_2$ is

$$\mathsf{L}\widehat{f}=\langle v\rangle^{-\beta}\left(\mathsf{r}_f\,\frac{F}{Z}-\widehat{f}\right)\quad\text{with}\quad\mathsf{r}_f(t,\xi):=\int_{\mathbb{R}^d}\langle v'\rangle^{-\beta}\,\widehat{f}(t,\xi,v')\,dv'$$

with flux $i\,\varepsilon^{1-\alpha}\int_{\mathbb{R}^d}v\cdot\xi\,\widehat{f}\,dv$ given by

$$\frac{\mathsf{r}_f}{Z}\,\varepsilon^{2-\alpha}\int_{\mathbb{R}^d}\frac{\langle v\rangle^{-\beta}\,(v\cdot\xi)^2\,F}{\langle v\rangle^{-2\,\beta}+\varepsilon^2\,(v\cdot\xi)^2}\,dv-i\,\varepsilon\int_{\mathbb{R}^d}\frac{v\cdot\xi\,\partial_t\widehat{f}}{\langle v\rangle^{-\beta}+i\,\varepsilon\,v\cdot\xi}\,dv$$

With β and γ such that $0<\alpha<2$, the macroscopic limit of the continuity equation as $\varepsilon\to0$ is (formally) the *fractional heat equation*

$$\partial_t\widehat{\rho_0}+\kappa\,|\xi|^\alpha\,\widehat{\rho_0}=0$$

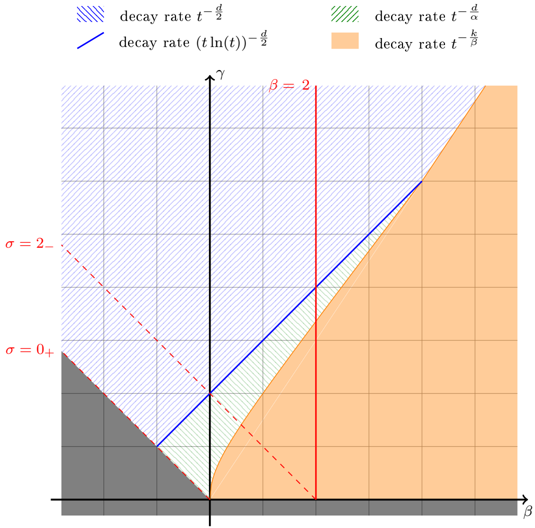


Figure: $d = 3$. If $\mathsf{L} = \mathsf{L}_3$, γ is limited to the strip enclosed between the two dashed red lines

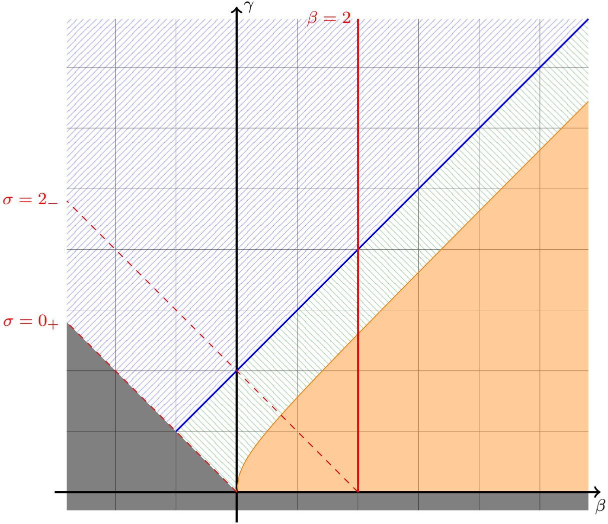


Figure: $d = 2$

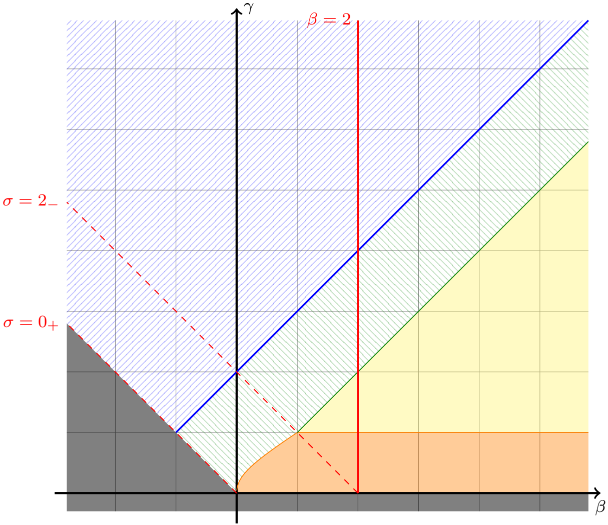


Figure: $d = 1$

Special macroscopic modes and hypocoercivity

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Joint work with with Kleber Carrapatoso, Frédéric Hérau, Stéphane Mischler, Clément Mouhot, Christian Schmeiser

The equation

Consider the kinetic equation

$$\partial_t f = \mathscr{L} f := \mathscr{T} f + \mathscr{C} f \,, \quad f_{|t=0} = f_0$$

with *transport operator* $\mathscr{T}$ given by

$$\mathscr{T} f := -v \cdot \nabla_x f + \nabla_x \phi \cdot \nabla_v f$$

where $\phi \in C^2(\mathbb{R}^d, \mathbb{R})$. Let $\rho(x) := e^{-\phi(x)}$ and $\langle \varphi \rangle := \int_{\mathbb{R}^d} \varphi \, \rho \, \mathrm{d}x$

Linear collision operator

$\mathscr{C}$ acts only on $v \in \mathbb{R}^d$, is self-adjoint in $\mathrm{L}^2(\mu^{-1})$, with $\mu(v) := \frac{e^{-|v|^2/2}}{(2\pi)^{d/2}}$ and has the $(d+2)$-dimensional kernel of *collision invariants* given by

$$\operatorname{Ker}\mathscr{C} = \operatorname{Span}\left\{\mu, v_1\,\mu, \ldots, v_d\,\mu, |v|^2\,\mu\right\}$$

▷ *Spectral gap property*

$$-\int_{\mathbb{R}^d} f(v)\,\mathscr{C}f(v)\,\frac{dv}{\mu(v)} \ge \mathrm{c}_{\mathscr{C}}\,\|f-\Pi f\|^2_{\mathrm{L}^2(\mu^{-1})}$$

where Π denotes the $\mathrm{L}^2(\mu^{-1})$-orthogonal projection onto $\operatorname{Ker}\mathscr{C}$

▷ For any polynomial function $p(v):\mathbb{R}^d\to\mathbb{R}$ of degree at most 4, the function $p\,\mu$ is in the domain of $\mathscr{C}$ and

$$C(p) := \|\mathscr{C}(p\,\mu)\|_{\mathrm{L}^2(\mu^{-1})} < \infty$$

Other assumptions (1/2)

▷ *Normalization conditions*:

$$\int_{\mathbb{R}^d} \rho(x)\,\mathrm{d}x = 1\,, \quad \int_{\mathbb{R}^d} x\,\rho(x)\,\mathrm{d}x = 0\,, \quad \left\langle \nabla_x^2 \phi \right\rangle = \int_{\mathbb{R}^d} \nabla_x^2 \phi\,\rho\,\mathrm{d}x = \mathrm{Id}_{d\times d}$$

▷ *Growth/regularity assumption*

$$|\nabla_x^2 \phi| \le \varepsilon\,|\nabla_x \phi|^2 + C_\varepsilon$$

▷ *Poincaré inequality*

$$c_{\mathrm{P}} \int_{\mathbb{R}^d} |u - \langle u \rangle|^2\,\rho\,\mathrm{d}x \le \int_{\mathbb{R}^d} |\nabla_x u|^2\,\rho\,\mathrm{d}x$$

▷ *Moment bounds on* ρ

$$\int_{\mathbb{R}^d} \left(|x|^4 + |\phi|^2 + |\nabla_x \phi|^4\right) \rho\,\mathrm{d}x \le \mathrm{C}_\phi$$

Other assumptions (2/2)

▷ *Semi-group property*

$$t \mapsto e^{t\mathscr{L}} \text{ is a strongly continuous semi-group on } \mathrm{L}^2(\mathscr{M}^{-1})$$

where $\mathscr{M}$ is the *global Maxwellian equilibrium*

$$\mathscr{M}(x,v) := \rho(x)\,\mu(v) = \frac{e^{-\frac{1}{2}|v|^2-\phi(x)}}{(2\,\pi)^{d/2}}$$

Special macroscopic modes (1/2)

Special macroscopic modes $\mathscr{C} F = 0\,,\quad \partial_t F = \mathscr{T} F$

$$F = \big(r(t,x) + m(t,x)\cdot v + e(t,x)\,\mathfrak{E}(v)\big)\,\mathscr{M}\,,\quad \mathfrak{E}(v) := \frac{|v|^2 - d}{\sqrt{2\,d}}$$

▷ *Energy mode* $F = \mathcal{H}\,\mathscr{M}$ with

$$\mathcal{H}(x,v) := \tfrac{1}{2}\,\big(|v|^2 - d\big) + \phi(x) - \langle\phi\rangle$$

▷ The set of *infinitesimal rotations compatible with* ϕ defined as

$$\mathcal{R}_\phi := \Big\{x \mapsto A\,x\,:\,A \in \mathfrak{M}^{\rm skew}_{d\times d}(\mathbb{R})\text{ s.t. }\forall\,x\in\mathbb{R}^d\,,\;\nabla_x\phi(x)\cdot A\,x = 0\Big\}$$

gives rise *rotation modes compatible with* ϕ

$$(A\,x\cdot v)\,\mathscr{M}(x,v)\,,\quad A\in\mathcal{R}_\phi$$

Special macroscopic modes (2/2): harmonic modes

Harmonic directions $E_\phi := \underset{\mathbb{R}^d}{\mathrm{Span}} \{\nabla_x \phi(x) - x\}_{x \in \mathbb{R}^d}$, $d_\phi := \dim E_\phi$

▷ the potential is *partially harmonic* if $1 \le d_\phi \le d-1$
harmonic directional modes are defined by

$$(x_i \cos t - v_i \sin t)\,\mathscr{M}\,, \quad (x_i \sin t + v_i \cos t)\,\mathscr{M}\,, \quad i \in I_\phi := \{d_\phi + 1, \ldots, d\}$$

▷ If $d_\phi = 0$, the potential $\phi(x) = \frac{1}{2}|x|^2 + \frac{d}{2}\log(2\pi)$ is *fully harmonic*
In addition to the harmonic directional modes, there are *harmonic pulsating modes*

$$\left(\tfrac{1}{2}\left(|x|^2 - |v|^2\right)\cos(2t) - x \cdot v \sin(2t)\right)\mathscr{M}$$
$$\left(\tfrac{1}{2}\left(|x|^2 - |v|^2\right)\sin(2t) + x \cdot v \cos(2t)\right)\mathscr{M}$$

(Boltzmann, 1876) (Uhlenbeck, Ford, 1963) (Cercignani, 1983)

Theorem (Special macroscopic modes and hypocoercivity)

(1) *All special macroscopic modes are given by*

$$F = \alpha\,\mathscr{M} + \beta\,\mathcal{H}\,\mathscr{M} + A\,x\cdot v\,\mathscr{M} + F_{\rm dir} + F_{\rm pul}$$

(2) *There are explicit constants* $C>0$ *and* $\lambda>0$ *such that, for any solution* $f\in\mathcal{C}\big(\mathbb{R}^+;\mathrm{L}^2(\mathscr{M}^{-1})\big)$ *with initial datum* f_0*, there exists a unique special macroscopic mode* F *such that*

$$\forall\,t\ge0\,,\quad \|f(t)-F(t)\|_{\mathrm{L}^2(\mathscr{M}^{-1})}\le C\,e^{-\lambda\,t}\,\|f_0-F(0)\|_{\mathrm{L}^2(\mathscr{M}^{-1})}$$

A micro-macro decomposition

$$\partial_t h = \mathcal{L}\, h := \mathcal{T}\, h + \mathcal{C} h\,, \quad \mathcal{C} h := \mu^{-1}\, \mathscr{C}\,(\mu\, h)$$

with $\mathrm{Ker}\,\mathcal{C} = \mathrm{Span}\left\{1, v_1, \ldots, v_d, |v|^2\right\}$ and

$$h := \frac{f - \alpha\,\mathscr{M} - \beta\,\mathcal{H}\,\mathscr{M} - F_{\mathrm{rot}} - F_{\mathrm{dir}} - F_{\mathrm{pul}}}{\mathscr{M}}$$

Micro-macro decomposition

$$h = h^{\parallel} + h^{\perp}\,, \quad h^{\parallel} := r + m \cdot v + e\,\mathfrak{E}(v)$$

$$(r, m, e)(t, x) := \int_{\mathbb{R}^d} \big(1, v, \mathfrak{E}(v)\,\big)\, h(t, x, v)\, \mu(v)\, \mathrm{d}v$$

- f is a special macroscopic modes iff $h^{\perp} = 0$
- all steady states are special macroscopic modes: factorization (use entropy-dissipation arguments)

Sketch of the proof

The function $h = h^\parallel + h^\perp = r + m\cdot v + e\,\mathfrak{E}(v) + h^\perp$ is such that

$$\frac{\mathrm d}{\mathrm dt}\|h\|^2 \le -\,2\,\mathrm c_{\mathscr C}\,\|h^\perp\|^2$$

With the *Witten-Laplace operator* $\Omega := -\,\Delta_x + \nabla_x\phi\cdot\nabla_x + 1$ and

$$E[h] := \int_{\mathbb R^d} (v\otimes v - \mathrm{Id}_{d\times d})\,h\,\mu\,\mathrm dv\,,\quad \Theta[h] := \int_{\mathbb R^d} v\left(\mathfrak{E}(v) - \sqrt{\tfrac2d}\right)h\,\mu\,\mathrm dv$$

we build a *Lyapunov functional*

$$\begin{aligned}\mathcal F[h] := \|h\|^2 &+ \varepsilon\left\langle\Omega^{-1}\,\nabla_x e,\Theta[h]\right\rangle + \varepsilon^{\frac32}\left\langle\Omega^{-1}\,\nabla_x^{\rm sym}\,m_s, E[h] - \sqrt{\tfrac2d}\,\langle e\rangle\,\mathrm{Id}_{d\times d}\right\rangle\\ &+ \varepsilon^{\frac74}\left\langle\Omega^{-1}\,\nabla_x w_s, m_s\right\rangle + \varepsilon^{\frac{15}8}\left\langle-\Omega^{-1}\,\partial_t w_s, w_s\right\rangle\\ &- \varepsilon^{\frac{61}{32}}\left\langle(X - Y\cdot\nabla_x\phi),\nabla_x\phi\cdot A\,x\right\rangle - \varepsilon^{\frac{62}{32}}\left\langle b,b'\right\rangle - \varepsilon_6\left\langle c',c''\right\rangle\end{aligned}$$

such that, for some $\lambda\ge0$,

$$\frac{\mathrm d}{\mathrm dt}\mathcal F[h] \le -\,\lambda\,\mathcal F[h] \quad\text{and}\quad \|h\|^2 \lesssim \mathcal F[h] \lesssim \|h\|^2$$

These slides can be found at

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Thank you for your attention !