Interpolation inequalities (Gagliardo-Nirenberg, Sobolev and Onofri inequalities): rigidity results, nonlinear flows and applications

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

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Scope (1/3): rigidity results

Rigidity results for semilinear elliptic PDEs on manifolds...

Let $(\mathfrak{M}, g)$ be a smooth compact connected Riemannian manifold of dimension $d \ge 2$, no boundary, Δ_g is the Laplace-Beltrami operator the Ricci tensor $\mathfrak{R}$ has good properties (which ones ?)

Let $p \in (2, 2^*)$, with $2^* = \frac{2\,d}{d-2}$ if $d \ge 3$, $2^* = \infty$ if $d = 2$

For which values of $\lambda > 0$ the equation

$$-\,\Delta_g v + \lambda\, v = v^{p-1}$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv \lambda^{\frac{1}{p-2}}$?

A typical *rigidity result* is: there exists $\lambda_0 > 0$ such that $v \equiv \lambda^{\frac{2}{p-2}}$ if $\lambda \in (0, \lambda_0]$

Assumptions ?
Optimal λ_0 *?*

Scope (2/3): interpolation inequalities

Still on a smooth compact connected Riemannian manifold $(\mathfrak{M}, g)$ we assume that $\mathrm{vol}_g(\mathfrak{M}) = 1$

For any $p \in (1,2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$, consider the *interpolation inequality*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

What is the largest possible value of λ ?

- using $u = 1 + \varepsilon\, \varphi$ as a test function proves that $\lambda \leq \lambda_1$
- the minimum of $v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right]$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if λ is above the rigidity threshold

- the threshold case $p = 2$ is the *logarithmic Sobolev inequality*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \lambda \int_{\mathfrak{M}} u^2 \, \log \left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathfrak{M})}} \right) dv_g \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

Scope (3/3): flows

We shall consider a flow of porous media / fast diffusion type

$$u_t = u^{2-2\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,, \quad \kappa = 1 + \beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying as long as λ is not too big. Hence, if the limit as $t \to \infty$ is 0 (convergence to the constants), we know that $\mathcal{F}[u] \geq 0$

Structure ? Link with computations in the rigidity approach

Some references (1/2)

Some references (incomplete) and *goals*

1. rigidity results and elliptic PDEs: [Gidas-Spruck 1981], [Bidaut-Véron & Véron 1991], [Licois & Véron 1995]
   $\longrightarrow$ *systematize and clarify the strategy*
2. semi-group approach and Γ_2 or *carré du champ* method: [Bakry-Emery 1985], [Bakry & Ledoux 1996], [Bentaleb et al., 1993-2010], [Fontenas 1997], [Brouttelande 2003], [Demange, 2005 & 2008]
   $\longrightarrow$ *emphasize the role of the flow, get various improvements*
   $\longrightarrow$ *get rid of pointwise constraints on the curvature, discuss optimality*
3. harmonic analysis, duality and spectral theory: [Lieb 1983], [Beckner 1993]
   $\longrightarrow$ *apply results to get new spectral estimates*

Outline

Joint work with:

M.J. Esteban, G. Jankowiak, M. Kowalczyk, A. Laptev and M. Loss

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

The sphere

- The case of the sphere as a simple example

Inequalities on the sphere

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere:

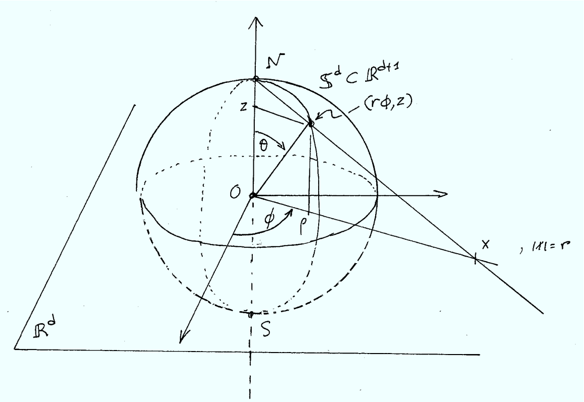
$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\,d\,v_g+\int_{\mathbb{S}^d}|u|^2\,d\,v_g\geq\left(\int_{\mathbb{S}^d}|u|^p\,d\,v_g\right)^{2/p}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,dv_g)$$

- for any $p\in(2,2^*]$ with $2^*=\frac{2\,d}{d-2}$ if $d\geq 3$
- for any $p\in(2,\infty)$ if $d=2$

Here dv_g is the uniform probability measure: $v_g(\mathbb{S}^d)=1$

- 1 is the optimal constant, equality achieved by constants
- $p=2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \geq \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Extended inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \ge \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p \, d\,v_g \right)^{2/p} - \int_{\mathbb{S}^d} |u|^2 \, d\,v_g \right] \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

is valid

- for any $p \in (1,2) \cup (2,\infty)$ if $d = 1,\, 2$
- for any $p \in (1,2) \cup (2, 2^*]$ if $d \ge 3$

- Logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \ge \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \, d\,v_g} \right) d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

  - case $p = 2$

- Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \ge d \int_{\mathbb{S}^d} |u - \bar{u}|^2 \, d\,v_g \quad \text{with} \quad \bar{u} := \int_{\mathbb{S}^d} u \, d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

  - case $p = 1$

A spectral approach when $p \in (1,2)$ – 1st step

[Dolbeault-Esteban-Kowalczyk-Loss] adapted from [Beckner] (case of Gaussian measures).

Nelson's hypercontractivity result. Consider the heat equation

$$\frac{\partial f}{\partial t} = \Delta_g f$$

with initial datum $f(t=0,\cdot) = u \in L^{2/p}(\mathbb{S}^d)$, for some $p \in (1,2]$, and let $F(t) := \|f(t,\cdot)\|_{\mathrm{L}^{p(t)}(\mathbb{S}^d)}$. The key computation goes as follows.

$$\frac{F'}{F} = \frac{p'}{p^2\,F^p}\left[\int_{\mathbb{S}^d} v^2 \log\left(\frac{v^2}{\int_{\mathbb{S}^d} v^2\;d\,v_g}\right)d\,v_g + 4\,\frac{p-1}{p'}\int_{\mathbb{S}^d}|\nabla v|^2\;d\,v_g\right]$$

with $v := |f|^{p(t)/2}$. With $4\,\frac{p-1}{p'} = \frac{2}{d}$ and $t_* > 0$ e such that $p(t_*) = 2$, we have

$$\|f(t_*,\cdot)\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^{2/p}(\mathbb{S}^d)} \quad \text{if} \quad \frac{1}{p-1} = e^{2\,d\,t_*}$$

A spectral approach when $p \in (1,2)$ – $2^{\rm nd}$ step

Spectral decomposition. Let $u = \sum_{k\in\mathbb{N}} u_k$ be a spherical harmonics decomposition, $\lambda_k = k\,(d+k-1)$, $a_k = \|u_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ so that

$$\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k \text{ and } \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} \lambda_k\, a_k$$

$$\|f(t_*,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k\in\mathbb{N}} a_k\, e^{-2\,\lambda_k\,t_*}$$

$$\begin{aligned}
\frac{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}{2-p} &\le \frac{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f(t_*,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{2-p}\\
&= \frac{1}{2-p}\sum_{k\in\mathbb{N}^*} \lambda_k\, a_k\,\frac{1-e^{-2\,\lambda_k\,t_*}}{\lambda_k}\\
&\le \frac{1-e^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\sum_{k\in\mathbb{N}^*}\lambda_k\,a_k = \frac{1-e^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}
\end{aligned}$$

The conclusion easily follows if we notice that $\lambda_1 = d$, and $e^{-2\,\lambda_1\,t_*} = p-1$ so that $\frac{1-e^{-2\,\lambda_1\,t_*}}{(2-p)\,\lambda_1} = \frac{1}{d}$

Optimality: a perturbation argument

- The optimality of the constant can be checked by a Taylor expansion of $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$, small
- For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or 2, it is remarkable that

$$\mathcal{Q}[u] := \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \geq \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)} \mathcal{Q}[u] = \frac{1}{d}$$

is achieved by $\mathcal{Q}[1 + \varepsilon\, v]$ as $\varepsilon \to 0$ and v is an eigenfunction associated with the first nonzero eigenvalue of Δ_g
- $p > 2$ no simple proof based on spectral analysis: [Beckner], an approach based on Lieb's duality, the Funk-Hecke formula and some (non-trivial) computations
- elliptic methods / Γ_2 formalism of Bakry-Emery / flow... they are the same (main contribution) and can be simplified (!) As a side result, you can go beyond these approaches and discuss optimality

Schwarz symmetry and the ultraspherical setting

$(\xi_0,\, \xi_1, \ldots \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on ξ_d

- Let $d\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\, v \in \mathrm{H}^1([0,\pi], d\sigma)$

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2\,d\sigma + \int_0^\pi |v(\theta)|^2\,d\sigma \geq \left(\int_0^\pi |v(\theta)|^p\,d\sigma\right)^{\frac{2}{p}}$$

- Change of variables $z = \cos\theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\,\nu\,d\nu_d + \int_{-1}^1 |f|^2\,d\nu_d \geq \left(\int_{-1}^1 |f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $\nu_d(z)\,dz = d\nu_d(z) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1\, f_2\; d\nu_d\,, \quad \|f\|_p = \left(\int_{-1}^{1} f^p\; d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\, f := (1 - z^2)\, f'' - d\, z\, f' = \nu\, f'' + \frac{d}{2}\,\nu'\, f'$$

which satisfies $\langle f_1, \mathcal{L}\, f_2 \rangle = -\int_{-1}^{1} f_1'\, f_2'\; \nu\; d\nu_d$

Proposition

Let $p \in [1,2) \cup (2, 2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L}\, f \rangle = \int_{-1}^{1} |f'|^2\; \nu\; d\nu_d \geq d\, \frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \quad \forall\, f \in \mathrm{H}^1([-1,1], d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Heat flow and the Bakry-Emery method

With $g = f^p$, *i.e.* $f = g^\alpha$ with $\alpha = 1/p$

$$\text{(Ineq.)} \qquad -\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L} g^\alpha \rangle =: \mathcal{I}[g] \geq d\, \frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\, \|g\|_1 = 0\,, \quad \frac{d}{dt}\, \|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\, f \rangle = 2\,(p-2) \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d$$

which finally gives

$$\frac{d}{dt} \mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\, \frac{d}{dt}\, \|g^{2\,\alpha}\|_1 = -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

$$\text{Ineq.} \iff \frac{d}{dt} \mathcal{F}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{F}[g(t,\cdot)] \Longleftarrow \frac{d}{dt} \mathcal{I}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\,\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} < \frac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof on two slides

$$\left[\frac{d}{dz},\mathcal{L}\right]u=(\mathcal{L}\,u)'-\mathcal{L}\,u'=-2\,z\,u''-d\,u'$$

$$\int_{-1}^{1}(\mathcal{L}\,u)^2\,d\nu_d=\int_{-1}^{1}|u''|^2\,\nu^2\,d\nu_d+d\int_{-1}^{1}|u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^{1}(\mathcal{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d=\frac{d}{d+2}\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\int_{-1}^{1}\frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t=u^{2-2\beta}\left(\mathcal{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right)$$

If $\kappa=\beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^{1}u^{\beta p}\,d\nu_d=\beta\,p\,(\kappa-\beta\,(p-2)-1)\int_{-1}^{1}u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d=0$$

$f=u^\beta$, $\|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)\geq 0$?

$$\mathcal{A}:=-\frac{1}{2\,\beta^2}\,\frac{d}{dt}\int_{-1}^1\left(|(u^\beta)'|^2\,\nu+\frac{d}{p-2}\,\left(u^{2\beta}-\overline{u}^{2\beta}\right)\right)d\nu_d$$

$$=\int_{-1}^1\left(\mathcal{L}\,u+(\beta-1)\,\frac{|u'|^2}{u}\,\nu\right)\left(\mathcal{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right)d\nu_d$$

$$+\frac{d}{p-2}\,\frac{\kappa-1}{\beta}\int_{-1}^1|u'|^2\,\nu\;d\nu_d$$

$$=\int_{-1}^1|u''|^2\,\nu^2\;d\nu_d-2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^1u''\,\frac{|u'|^2}{u}\,\nu^2\;d\nu_d$$

$$+\left[\kappa\,(\beta-1)+\frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\;d\nu_d$$

$$=\int_{-1}^1\left|u''-\frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d\geq 0\quad\text{if } p=2^* \text{ and } \beta=\frac{4}{6-p}$$

$\mathcal{A}$ is nonnegative for some β if $\displaystyle\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p)\geq 0$

the rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\mathcal{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\,d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms.

Spectral consequences

- A quantitative deviation with respect to the semi-classical regime

Some references (2/2)

Consider the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$ and denote by $(\lambda_k)_{k\geq 1}$ its eigenvalues

- Euclidean case [Keller, 1961]

$$|\lambda_1|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

[Lieb-Thirring, 1976]

$$\sum_{k\geq 1} |\lambda_k|^\gamma \leq \mathrm{L}_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

$\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

- Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

An interpolation inequality (I)

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (2, 2^*)$. *Then there exists a concave increasing function* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following properties*

$$\mu(\alpha) = \alpha \quad \forall\, \alpha \in \left[0, \tfrac{d}{q-2}\right] \quad \textit{and} \quad \mu(\alpha) < \alpha \quad \forall\, \alpha \in \left(\tfrac{d}{q-2}, +\infty\right)$$

$$\mu(\alpha) = \mu_{\rm asymp}(\alpha)\,(1 + o(1)) \quad \textit{as} \quad \alpha \to +\infty\,, \quad \mu_{\rm asymp}(\alpha) := \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\alpha)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

If $d \ge 3$ *and* $q = 2^*$, *the inequality holds with* $\mu(\alpha) = \min\{\alpha, \alpha_*\}$, $\alpha_* := \frac{1}{4}\,d\,(d-2)$

- $\mu_{\rm asymp}(\alpha) := \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}$, $\vartheta := d\,\frac{q-2}{2\,q}$ corresponds to the *semi-classical regime* and $\mathsf{K}_{q,d}$ is the optimal constant in the *Euclidean* Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}_{q,d}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

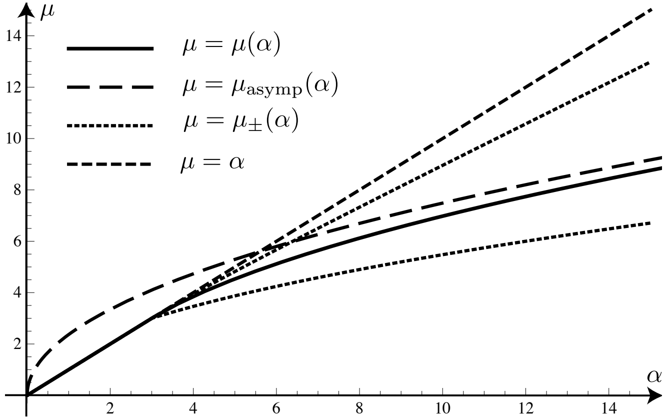
- Let φ be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

$$-\,\Delta\varphi = d\,\varphi$$

Consider $u = 1 + \varepsilon\,\varphi$ as $\varepsilon \to 0$ Taylor expand $\mathcal{Q}_\alpha$ around $u = 1$

$$\mu(\alpha) \le \mathcal{Q}_\alpha[1+\varepsilon\,\varphi] = \alpha + \big[d + \alpha\,(2-q)\big]\,\varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2\,d\,v_g + o(\varepsilon^2)$$

By taking ε small enough, we get $\mu(\alpha) < \alpha$ for all $\alpha > d/(q-2)$
Optimizing on the value of $\varepsilon > 0$ (not necessarily small) provides an interesting test function...


μ
α
14
12
10
8
6
4
2
2
4
6
8
10
12
14
$\mu = \mu(\alpha)$
$\mu = \mu_{\text{asymp}}(\alpha)$
$\mu = \mu_{\pm}(\alpha)$
$\mu = \alpha$

Consider the Schrödinger operator $-\Delta - V$ and the energy

$$\mathcal{E}[u] := \int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V\,|u|^2$$
$$\geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq -\,\alpha(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{if } \mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

Theorem (Dolbeault-Esteban-Laptev)

Let $d \geq 1$, $p \in \big(\max\{1, d/2\}, +\infty\big)$. Then there exists a convex increasing function α s.t. $\alpha(\mu) = \mu$ if $\mu \in \big[0, \frac{d}{2}\,(p-1)\big]$ and $\alpha(\mu) > \mu$ if $\mu \in \big(\frac{d}{2}\,(p-1), +\infty\big)$

$$|\lambda_1(-\Delta - V)| \leq \alpha\big(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}\big) \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)$$

For large values of μ, we have $\alpha(\mu)^{p-\frac{d}{2}} = \mathrm{L}^1_{p-\frac{d}{2},d}\,(\kappa_{q,d}\,\mu)^p\,(1+o(1))$ and the above estimate is optimal
If $p = d/2$ and $d \geq 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \alpha_*]$

A Keller-Lieb-Thirring inequality

Corollary (Dolbeault-Esteban-Laptev)

Let $d \geq 1, \gamma = p - d/2$

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim \mathrm{L}^1_{\gamma,d} \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}} \quad as \quad \mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ *or* $\gamma = 1/2$ *and* $d = 1$

However, if $\mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4}\, d\,(2\,\gamma + d - 2)$, *then we have*

$$|\lambda_1(-\Delta - V)|^{\gamma+\frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ *and this estimate is optimal*

$\mathrm{L}^1_{\gamma,d}$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} \phi_+^{\gamma+\frac{d}{2}}\, dx$$

Another interpolation inequality (II)

Let $d \geq 1$ and $\gamma > d/2$ and assume that $\mathrm{L}^1_{-\gamma,d}$ is the optimal constant in

$$\lambda_1(-\Delta+\phi)^{-\gamma} \leq \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2}-\gamma}\,dx$$

$$q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}$$

Theorem (Dolbeault-Esteban-Laptev)

$$\big(\lambda_1(-\Delta+W)\big)^{-\gamma} \lesssim \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma} \quad as \quad \beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

However, if $\gamma \geq \frac{d}{2}+1$ *and* $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4}\,d\,(2\,\gamma-d+2)$

$$\big(\lambda_1(-\Delta+W)\big)^{\frac{d}{2}-\gamma} \leq \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}$$

and this estimate is optimal

$\mathsf{K}^*_{q,d}$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}^*_{q,d}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

and $\mathcal{L}^{\,1}_{-\gamma,d} := \left(\mathsf{K}^*_{q,d}\right)^{-\gamma}$ with $q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2}$, $\delta := \frac{2\,q}{2\,d-q\,(d-2)}$

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (0,2)$ and $d \ge 1$. There exists a concave increasing function ν

$$\nu(\beta) \le \beta \quad \forall\,\beta > 0 \quad \textit{and} \quad \nu(\beta) < \beta \quad \forall\,\beta \in \left(\tfrac{d}{2-q}, +\infty\right)$$

$$\nu(\beta) = \beta \quad \forall\,\beta \in \left[0, \tfrac{d}{2-q}\right] \quad \textit{if} \quad q \in [1,2)$$

$$\nu(\beta) = \mathsf{K}^*_{q,d}\,\left(\kappa_{q,d}\,\beta\right)^\delta\,\left(1 + o(1)\right) \quad \textit{as} \quad \beta \to +\infty$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \ge \nu(\beta)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

The threshold case: $q = 2$

Lemma (Dolbeault-Esteban-Laptev)

Let $p > \max\{1, d/2\}$. *There exists a concave nondecreasing function* ξ

$$\xi(\alpha) = \alpha \quad \forall\, \alpha \in (0, \alpha_0) \quad \textit{and} \quad \xi(\alpha) < \alpha \quad \forall\, \alpha > \alpha_0$$

for some $\alpha_0 \in \left[\frac{d}{2}\,(p-1), \frac{d}{2}\,p\right]$, *and* $\xi(\alpha) \sim \alpha^{1-\frac{d}{2\,p}}$ *as* $\alpha \to +\infty$ *such that, for any* $u \in \mathrm{H}^1(\mathbb{S}^d)$ *with* $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$

$$\int_{\mathbb{S}^d} |u|^2 \log |u|^2 \, d\,v_g + p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) \le p\,\log\left(1 + \tfrac{1}{\alpha}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Corollary (Dolbeault-Esteban-Laptev)

$$e^{-\,\lambda_1(-\Delta - W)/\alpha} \le \frac{\alpha}{\xi(\alpha)}\left(\int_{\mathbb{S}^d} e^{-\,p\,W/\alpha}\, d\,v_g\right)^{1/p}$$

Improvements of the inequalities (subcritical range)

- as long as the exponent is either in the range $(1,2)$ or in the range $(2,2^*)$, on can establish *improved inequalities*

[Dolbeault-Esteban-Kowalczyk-Loss]

What does “improvement” mean ?

An *improved* inequality is

$$d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\Phi\left(\frac{\mathsf{e}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\le \mathsf{i}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

for some function Φ such that $\Phi(0)=0$, $\Phi'(0)=1$, $\Phi'>0$ and $\Phi(s)>s$ for any s. With $\Psi(s):=s-\Phi^{-1}(s)$

$$\mathsf{i}-d\,\mathsf{e}\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi\circ\Phi)\left(\frac{\mathsf{e}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{d}{p-2}\left[\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right]$$

$$\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi\circ\Phi)\left(\mathsf{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,\|u^r-\bar{u}^r\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

$s(p):=\max\{2,p\}$ and $p\in(1,2)$: $q(p):=2/p$, $r(p):=p$; $p\in(2,4)$: $q=p/2$, $r=2$; $p\ge4$: $q=p/(p-2)$, $r=p-2$

Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

$$w_t = \mathcal{L}\, w + \kappa\, \frac{|w'|^2}{w}\, \nu$$

With $2^\sharp := \frac{2\,d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d>1\,, \qquad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d=1$$

If $p \in [1,2) \cup (2,2^\sharp]$ and w is a solution, then

$$\frac{d}{dt}\,(\mathsf{i} - \, d\,\mathsf{e}) \le -\,\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2}\, d\nu_d \le -\,\gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \, d\,\mathsf{e}' \ge \gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

which after integration implies an inequality of the form

$$d\,\Phi(\mathsf{e}(0)) \le \mathsf{i}(0)$$

$$w_t = w^{2-2\beta}\left(\mathcal{L}\,w + \kappa\,\frac{|w'|^2}{w}\right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta\,(p-2)+1$, $\frac{d}{dt}\int_{-1}^{1} w^{\beta p}\,d\nu_d = 0$

$$-\frac{1}{2\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(w^{\beta})'|^2\,\nu + \frac{d}{p-2}\,\left(w^{2\beta} - \overline{w}^{2\beta}\right)\right)d\nu_d \geq \gamma\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d$$

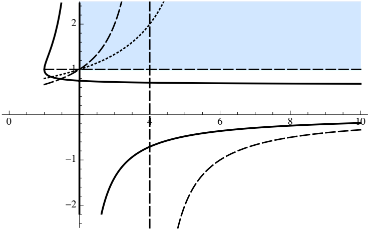
Lemma

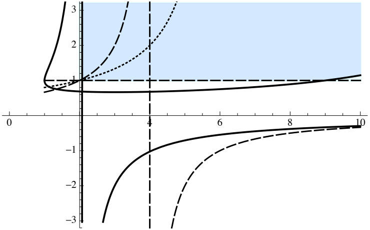
For all $w \in \mathrm{H}^1\big((-1,1), d\nu_d\big)$, such that $\int_{-1}^{1} w^{\beta p}\,d\nu_d = 1$

$$\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d \geq \frac{1}{\beta^2}\,\frac{\int_{-1}^{1}|(w^{\beta})'|^2\,\nu\,d\nu_d\int_{-1}^{1}|w'|^2\,\nu\,d\nu_d}{\left(\int_{-1}^{1} w^{2\beta}\,d\nu_d\right)^{\delta}}$$

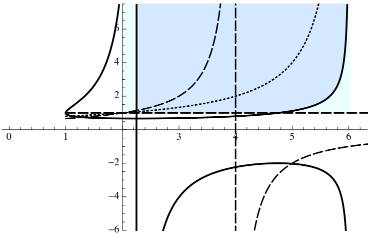
.... but there are conditions on β

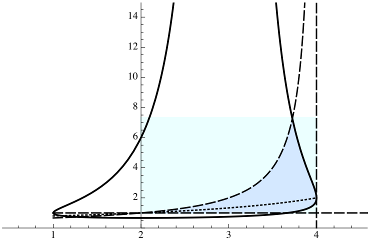
Admissible (p, β) for $d = 1, 2$



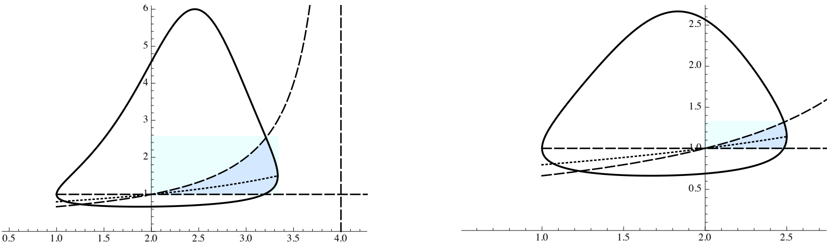


Admissible (p, β) for $d = 3, 4$





Admissible (p, β) for $d = 5, 10$



Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established
- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold dimension d, no boundary, Δ_g is the Laplace-Beltrami operator $\mathrm{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi\,,\xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad \textit{where} \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\,\Delta_g v + \frac{\lambda}{p-2}\,\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1,2) \cup (2,2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\,(\Delta_g u)^2 + \frac{\theta\,d}{d-1}\,\mathfrak{R}(\nabla u, \nabla u)\right] d\,v_g}{\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$*:* $u \equiv 1$

$\lim_{p \to 1_+} \theta(p) = 0 \Longrightarrow \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$ if ρ is bounded
$\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \le \lambda_\star \le \lambda_1$$

Riemannian manifolds: second improvement

$\mathrm{H}_g u$ denotes Hessian of u and $\theta = \dfrac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}$

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

$$\Lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{(1-\theta)\displaystyle\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g + \frac{\theta\,d}{d-1}\displaystyle\int_{\mathfrak{M}} \left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right]}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1,2) \cup (2, 2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0, \Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ we get $\lambda \leq \lambda_1$
A minimum of

$$v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right]$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tools the flow

$$u_t = u^{2-2\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1+\beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying

J. Demange, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611. Also see C. Villani, *Optimal Transport, Old and New*

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2\, d\,v_g = \frac{d}{d-1} \int_{\mathfrak{M}} \|\,\mathrm{L}_g u\,\|^2\, d\,v_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u)\, d\,v_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2}\,\Delta\,|\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, d\,v_g$$
$$= \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, d\,v_g - \frac{2\,d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] : \left[\frac{\nabla u \otimes \nabla u}{u} \right] d\,v_g$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d\,v_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, d\,v_g \quad \forall\, u \in \mathrm{H}^2(\mathfrak{M})$$

and λ_1 is the optimal constant in the above inequality

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right] d\,v_g$$

Lemma

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g - \mathcal{G}[u] + \lambda\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g$$

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \frac{1}{\theta}\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,d\,v_g\right] - \mu\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g$$

with $\mu := \frac{1}{\theta}\left(\frac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\frac{d}{d+2}$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall\, p \in (1, 2^*)$$

where $\beta_\pm := \frac{\mathsf{b} \pm \sqrt{\mathsf{b}^2 - \mathsf{a}}}{2\,\mathsf{a}}$ with $\mathsf{a} = 2 - p + \left[\frac{(d-1)\,(p-1)}{d+2}\right]^2$ and $\mathsf{b} = \frac{d+3-p}{d+2}$
Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2) + p - 1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1,2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \leq (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g$$

The line

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\theta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R})} \quad \text{if} \quad p\in(2,\infty)$$

$$\|f\|_{\mathrm{L}^2(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\eta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\eta}_{\mathrm{L}^p(\mathbb{R})} \quad \text{if} \quad p\in(1,2)$$

with $\theta=\frac{p-2}{2\,p}$ and $\eta=\frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p\to2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2\,\log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right)dx \le \frac12\,\|u\|^2_{\mathrm{L}^2(\mathbb{R})}\,\log\left(\frac{2}{\pi\,e}\,\frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{R})}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right)$$

If $p>2$, $u_\star(x)=(\cosh x)^{-\frac{2}{p-2}}$ solves

$$-\,(p-2)^2\,u''+4\,u-\,2\,p\,|u|^{p-2}\,u=0$$

If $p\in(1,2)$ consider $u_*(x)=(\cos x)^{\frac{2}{2-p}}$, $x\in(-\pi/2,\pi/2)$

Mass transportation

Theorem (Dolbeault-Esteban-Laptev-Loss)

If $p \in (2,\infty)$, we have

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{p+2}{3p-2}}\,dy}{\left(\int_{\mathbb{R}} G\,|y|^2\,dy\right)^{\frac{p-2}{3p-2}}\left(\int_{\mathbb{R}} G\,dy\right)^{\frac{4}{3p-2}}} = \mathsf{c}_p \inf_f \frac{\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2(p-2)}{3p-2}}\,\|f\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2(p+2)}{3p-2}}}{\|f\|_{\mathrm{L}^p(\mathbb{R})}^{\frac{4p}{3p-2}}}$$

and if $p \in (1,2)$, we obtain

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{2}{4-p}}\,dy}{\left(\int_{\mathbb{R}} G\,|y|^2\,dy\right)^{\frac{2-p}{2(4-p)}}\left(\int_{\mathbb{R}} G\,dy\right)^{\frac{p+2}{2(4-p)}}} = \mathsf{c}_p \inf_f \frac{\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2-p}{4-p}}\,\|f\|_{\mathrm{L}^p(\mathbb{R})}^{\frac{2p}{4-p}}}{\|f\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{p+2}{4-p}}}$$

for some explicit numerical constant c_p

Flow

Let us define on $\mathrm{H}^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{\mathrm{L}^2(\mathbb{R})} + \frac{4}{(p-2)^2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R})} - \mathsf{C}\,\|v\|^2_{\mathrm{L}^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}}\left[v'' + \frac{2\,p}{p-2}\,z\,v' + \frac{p}{2}\,\frac{|v'|^2}{v} + \frac{2}{p-2}\,v\right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2,\infty)$. Then

$$\frac{d}{dt}\mathcal{F}[v(t)] \le 0 \quad \textit{and} \quad \lim_{t\to\infty}\mathcal{F}[v(t)] = 0$$

$\frac{d}{dt}\mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_\star(x - x_0)$

Similar result for $p \in (1,2)$

The inequality ($p>2$) and the ultraspherical operator

- *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}}|v'|^2\,dx+\frac{4}{(p-2)^2}\int_{\mathbb{R}}|v|^2\,dx\ge \mathsf{C}\left(\int_{\mathbb{R}}|v|^p\,dx\right)^{\frac{2}{p}}$$

With

$$z(x)=\tanh x\,,\quad v_\star=(1-z^2)^{\frac{1}{p-2}}\quad\text{and}\quad v(x)=v_\star(x)\,f(z(x))$$

equality is achieved for $f=1$ and, if we let $\nu(z):=1-z^2$, then

$$\int_{-1}^{1}|f'|^2\,\nu\,d\nu_p+\frac{2\,p}{(p-2)^2}\int_{-1}^{1}|f|^2\,d\nu_p\ge\frac{2\,p}{(p-2)^2}\left(\int_{-1}^{1}|f|^p\,d\nu_p\right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z):=\frac{1}{\zeta_p}\,\nu^{\frac{2}{p-2}}\,dz$

$$d=\tfrac{2\,p}{p-2}\quad\Longleftrightarrow\quad p=\tfrac{2\,d}{d-2}$$

The change of variables amounts to the stereographic projection composed with the Emden-Fowler transformation

The Moser-Trudinger-Onofri inequality

Joint work with Maria J. Esteban and G. Jankowiak

Three equivalent forms

- The Euclidean (Moser-Trudinger-)Onofri inequality:

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) - \int_{\mathbb{R}^2} u\,d\mu$$

$d\mu = \mu(x)\,dx$, $\mu(x) = \frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x \in \mathbb{R}^2$

- The Onofri inequality on the two-dimensional sphere $\mathbb{S}^2$:

$$\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,d\sigma \ge \log\left(\int_{\mathbb{S}^2} e^v\,d\sigma\right) - \int_{\mathbb{S}^2} v\,d\sigma$$

$d\sigma$ is the uniform probability measure

- The Onofri inequality on the two-dimensional cylinder $\mathcal{C} = \mathbb{S}^1 \times \mathbb{R}$:

$$\frac{1}{16\,\pi}\int_{\mathcal{C}}|\nabla w|^2\,dy \ge \log\left(\int_{\mathcal{C}} e^w\,\nu\,dy\right) - \int_{\mathcal{C}} w\,\nu\,dy$$

$y = (\theta, s) \in \mathcal{C} = \mathbb{S}^1 \times \mathbb{R}$, $\nu(y) = \frac{1}{4\pi}\,(\cosh s)^{-2}$

[Moser (1971)], [Onofri (1982)]

The inequality seen as a limit case of the Gagliardo-Nirenberg inequalities

Proposition

[JD] *Assume that* $u \in \mathcal{D}(\mathbb{R}^2)$ *is such that* $\int_{\mathbb{R}^2} u \, d\mu = 0$ *and let*

$$f_p := F_p \left(1 + \frac{u}{2\,p}\right) , \quad F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^2$$

Then we have

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2} \frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)} \, \|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx}}{\int_{\mathbb{R}^2} e^{u} \, d\mu}$$

Rigidity method in the symmetric case

Under an appropriate normalization, a critical point of

$$\mathsf{G}_\lambda[f] := \frac{1}{8}\int_{-1}^{1} |f'|^2\,\nu\,dz + \frac{\lambda}{2}\int_{-1}^{1} f\,dz \geq \log\left(\frac{1}{2}\int_{-1}^{1} e^f\,dz\right)$$

solves the Euler-Lagrange equation

$$-\frac{1}{2}\,\mathcal{L}f + \lambda = e^f$$

Theorem

For any $\lambda \in (0,1)$, the EL equation has a unique smooth solution $f = \log\lambda$. If $\lambda = 1$, f has to satisfy the differential equation $f'' = \frac{1}{2}\,|f'|^2$ and is either a constant or

$$f(z) = C_1 - 2\,\log(C_2 - z)$$

$$\frac{1}{8}\int_{-1}^{1} \nu^2\,\left|f'' - \tfrac{1}{2}\,|f'|^2\right|^2 e^{-f/2}\,d\nu_p + \frac{1-\lambda}{4}\int_{-1}^{1} \nu\,|f'|^2\,e^{-f/2}\,d\nu_p = 0$$

Rigidity method in the symmetric case: proof

Multiply by $\mathcal{L}\big(e^{-f/2}\big)$ and integrate by parts

$$0=\int_{-1}^{1}\left(-\tfrac{1}{2}\,\mathcal{L}f+\lambda-e^{f}\right)\,\mathcal{L}\big(e^{-f/2}\big)\,d\nu_p$$

$$=\frac{1}{4}\int_{-1}^{1}\nu^2\,|f''|^2\,e^{-f/2}\,d\nu_p-\frac{1}{8}\int_{-1}^{1}\nu^2\,|f'|^2\,f''\,e^{-f/2}\,d\nu_p$$

$$+\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{-f/2}\,d\nu_p-\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{f/2}\,d\nu_p$$

Multiply by $\frac{\nu}{2}\,|f'|^2\,e^{-f/2}$ and integrate by parts

$$0=\int_{-1}^{1}\left(-\tfrac{1}{2}\,\mathcal{L}f+\lambda-e^{f}\right)\left(\tfrac{\nu}{2}\,|f'|^2\,e^{-f/2}\right)\,d\nu_p$$

$$=\frac{1}{8}\int_{-1}^{1}\nu^2\,|f'|^2\,f''\,e^{-f/2}\,d\nu_p-\frac{1}{16}\int_{-1}^{1}\nu^2\,|f'|^4\,e^{-f/2}\,d\nu_p$$

$$+\frac{\lambda}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{-f/2}\,d\nu_p-\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{f/2}\,d\nu_p$$

A nonlinear flow method in the general case

On $\mathbb{S}^2$ let us consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2}\left(e^{-f/2}\right) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator. Let us define

$$\mathcal{R}_\lambda[f] := \frac{1}{2}\int_{\mathbb{S}^2}\|\mathrm{L}_{\mathbb{S}^2} f - \frac{1}{2}\,\mathrm{M}_{\mathbb{S}^2} f\|^2\,e^{-f/2}\,d\sigma + \frac{1}{2}\,(1-\lambda)\int_{\mathbb{S}^2}|\nabla f|^2\,e^{-f/2}\,d\sigma$$

where

$$\mathrm{L}_{\mathbb{S}^2} f := \mathrm{Hess}_{\mathbb{S}^2} f - \frac{1}{2}\,\Delta_{\mathbb{S}^2} f\,\mathrm{Id} \quad \text{and} \quad \mathrm{M}_{\mathbb{S}^2} f := \nabla f\otimes\nabla f - \frac{1}{2}\,|\nabla f|^2\,\mathrm{Id}$$

Theorem

Assume that f is a solution to with initial datum $v - \log\left(\int_{\mathbb{S}^2} e^v\,d\sigma\right)$, where $v\in\mathcal{L}^1(\mathbb{S}^2)$ is such that $\nabla v\in\mathcal{L}^2(\mathbb{S}^2)$. Then for any $\lambda\in(0,1]$ we have

$$\mathcal{G}_\lambda[v] \ge \int_0^\infty \mathcal{R}_\lambda[f(t,\cdot)]\,dt$$

A summary

- the sphere: the flow tells us what to do, and provides a simple proof (*choice of the exponents / of the nonlinearity*) once the problem is reduced to the ultraspherical setting

- the spectral point of view on the inequality: how to measure the deviation with respect to the *semi-classical* estimates, a nice example of bifurcation (and *symmetry breaking*)

- *Riemannian manifolds:* no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The flow explores the energy landscape... and generically shows the non-optimality of the improved criterion

- the flow is a nice way of exploring an energy space. *Rigidity* result tell you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal

http://www.ceremade.dauphine.fr/∼dolbeaul
▷ Preprints (or arxiv, or HAL)

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Thank you for your attention !