Fast diffusion equations: matching large time asymptotics by relative entropy methods

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ON DISSIPATIVE SYSTEMS: ENTROPY METHODS, CLASSICAL AND QUANTUM PROBABILITY

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Fast diffusion equations: outline

- Introduction
  - Fast diffusion equations: entropy methods and Gagliardo-Nirenberg inequalities [del Pino, J.D.]
  - Fast diffusion equations: the finite mass regime
  - Fast diffusion equations: the infinite mass regime
- Relative entropy methods and linearization
  - the linearization of the functionals approach: [Blanchet, Bonforte, J.D., Grillo, Vázquez]
  - sharp rates: [Bonforte, J.D., Grillo,Vázquez]
  - An improvement based on the center of mass: [Bonforte, J.D., Grillo, Vázquez]
- An improvement based on the variance: [J.D., Toscani]
- Quantum mechanics ?

Some references

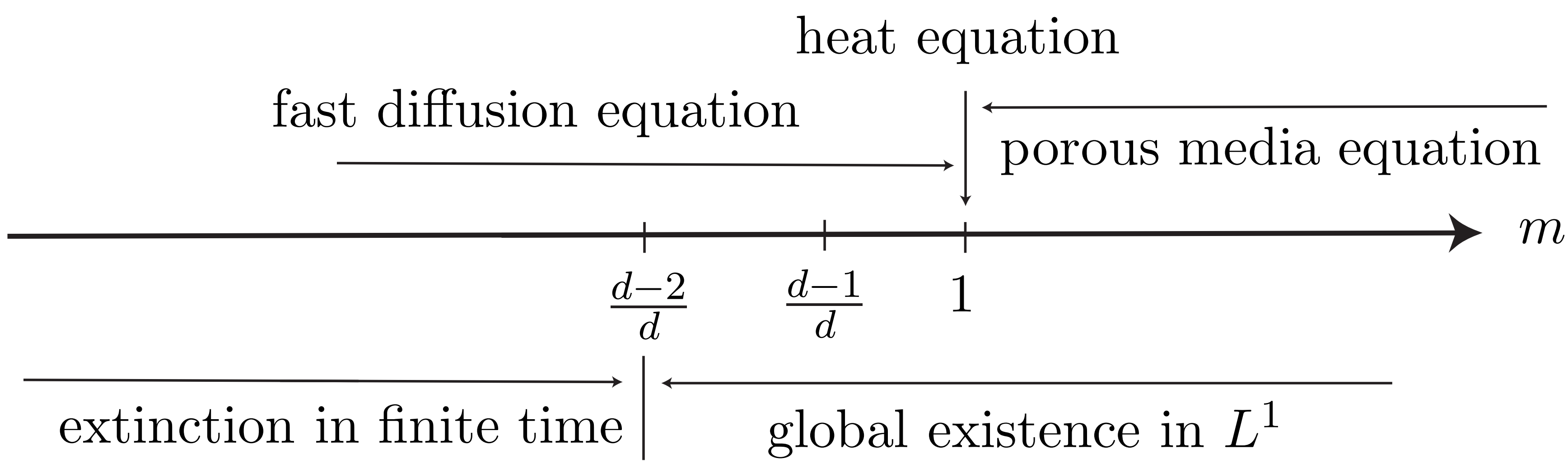
- J.D. and G. Toscani, Fast diffusion equations: matching large time asymptotics by relative entropy methods, Preprint
- M. Bonforte, J.D., G. Grillo, and J.-L. Vázquez. Sharp rates of decay of solutions to the nonlinear fast diffusion equation via functional inequalities, Proc. Nat. Acad. Sciences (2010)
- A. Blanchet, M. Bonforte, J.D., G. Grillo, and J.-L. Vázquez. Asymptotics of the fast diffusion equation via entropy estimates. Archive for Rational Mechanics and Analysis, 191 (2): 347-385, 02, 2009
- A. Blanchet, M. Bonforte, J.D., G. Grillo, and J.-L. Vázquez. Hardy-Poincaré inequalities and applications to nonlinear diffusions. C. R. Math. Acad. Sci. Paris, 344(7): 431-436, 2007
- M. Del Pino and J.D., Best constants for Gagliardo-Nirenberg inequalities and applications to nonlinear diffusions. J. Math. Pures Appl. (9), 81 (9): 847-875, 2002

Fast diffusion equations: entropy methods

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Some references

Generalized entropies and nonlinear diffusions (EDP, uncomplete):

[Toscani], [Arnold, Markowich,Toscani, Unterreiter], [Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],... [del Pino, Sáez], [Daskalopulos, Sesum]... (incomplete, to be continued)

Some methods

1) [J.D., del Pino] relate entropy and **Gagliardo-Nirenberg** inequalities

2) *entropy – entropy-production method:* the **Bakry-Emery** point of view

3) mass transport techniques

4) hypercontractivity for appropriate semi-groups

5) the approach by **linearization** of the entropy

... Fast diffusion equations and Gagliardo-Nirenberg inequalities

Time-dependent rescaling, Free energy

- **Time-dependent rescaling**: Take $u(\tau, y) = R^{-d}(t)\, v(t, y/R(\tau))$ where

$$\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}\ , \quad R(0) = 1\ , \quad t = \log R$$

The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

$$\Sigma[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\, |x|^2 v \right)\, dx - \Sigma_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt}\Sigma[v] = -I[v]\ , \quad I[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Relative entropy and entropy production

- **Stationary solution:** choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2\,m}\,|x|^2 \right)_+^{-1/(1-m)}$$

Relative entropy: Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form: for $m \neq 1$,

$$\Sigma[v] = \int_{\mathbb{R}^d} \psi\left(\frac{v}{v_\infty}\right)\, v_\infty^m\, dx \quad with\ \psi(t) = \frac{t^m - 1 - m\,(t-1)}{m-1}$$

- **Entropy – entropy production inequality**

Theorem 1. $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

Corollary 2. *A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\Sigma[v(t,\cdot)] \leq \Sigma[u_0]\,e^{-\,2\,t} \quad \forall\ t \geq 0$$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\Sigma[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2}\, I[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx + K \geq 0$$

- $1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \iff$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$; $K < 0$
- $0 < p < 1 \iff$ Porous medium case: $m > 1$, $K > 0$
- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal
- $m = m_1 := \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Theorem 3. [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *(fast diffusion case) and* $d \geq 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A \, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1}$$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$ + Csiszár-Kullback inequalities

Undo the change of variables, with

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\,(x/R(t))$$

Theorem 4. [Del Pino, J.D.] *Consider a solution of $u_t = \Delta u^m$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\, u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$*

- **Fast diffusion case:** $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t\to+\infty}\; t^{\frac{1-d(1-m)}{2-d(1-m)}}\; \|u^m - u_\infty^m\|_{L^1} < +\infty$$

- **Porous medium case:** $1 < m < 2$

$$\limsup_{t\to+\infty}\; t^{\frac{1+d(m-1)}{2+d(m-1)}}\; \|\,[u - u_\infty]\, u_\infty^{m-1}\,\|_{L^1} < +\infty$$

Fast diffusion equations: the finite mass regime

Can we consider $m < m_1$?

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- Displacement convexity holds in the same range of exponents, $m \in (m_1, 1)$, as for the Gagliardo-Nirenberg inequalities
- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance if $m > \widetilde{m}_1 := \frac{d}{d+2}$
- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass

...the Bakry-Emery method

Consider the generalized Fisher information

$$I[v] := \int_{\mathbb{R}^d} v\,|Z|^2 \;dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\,I[v(t,\cdot)] + 2\,I[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\mathrm{div} Z)^2\,dx - 2\sum_{i,\,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i Z^j)^2\,dx$$

- the Fisher information decays exponentially: $I[v(t,\cdot)] \le I[u_0]\,e^{-\,2\,t}$
- $\lim_{t\to\infty} I[v(t,\cdot)] = 0$ and $\lim_{t\to\infty} \Sigma[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(I[v(t,\cdot)] - 2\,\Sigma[v(t,\cdot)]\Big) \le 0$ means $I[v] \ge 2\,\Sigma[v]$

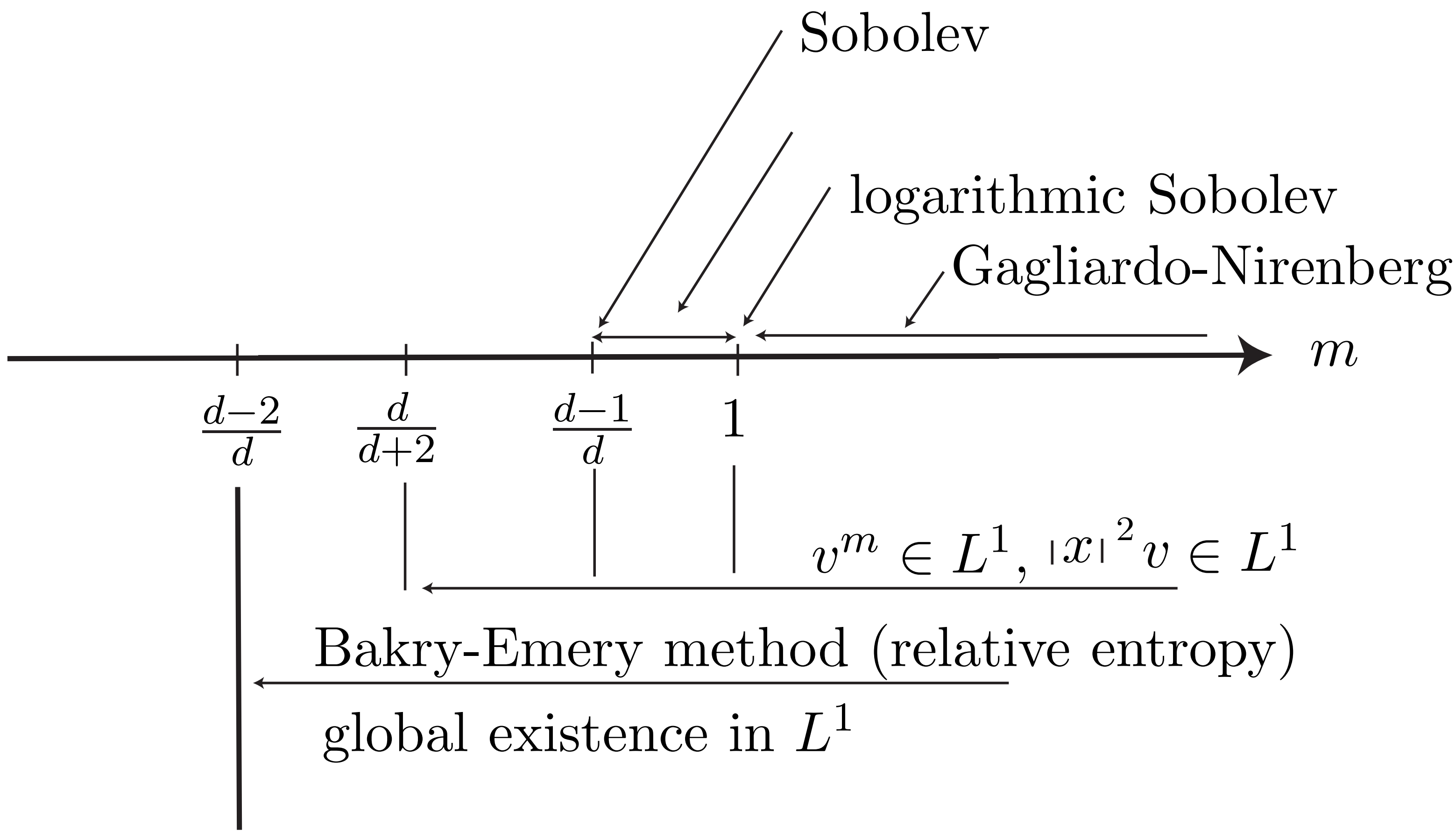
[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

$$I[v] \ge 2\,\Sigma[v]$$

holds for any $m > m_c$

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

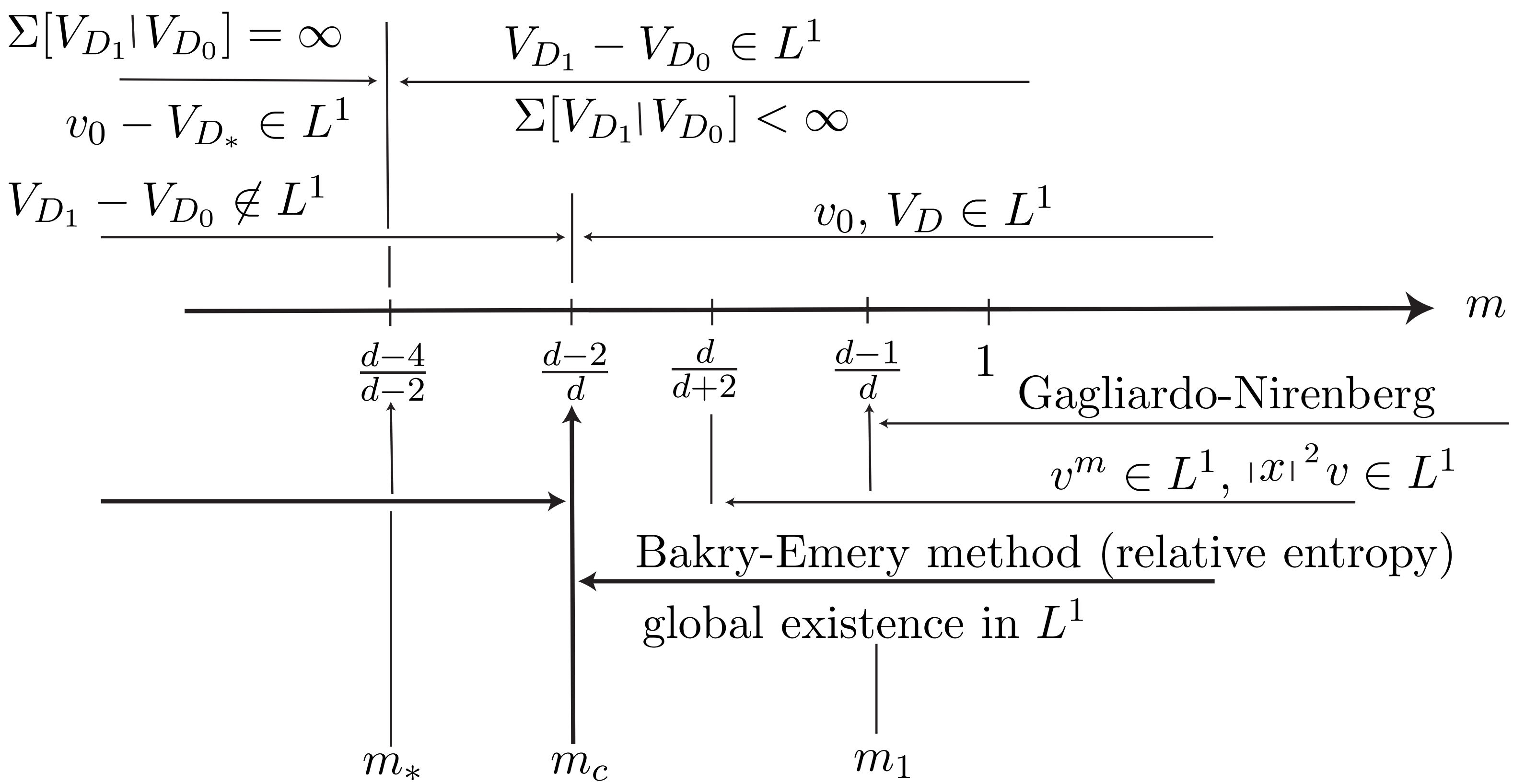
More references: Extensions and related results

- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev], [Kim, McCann], [Koch, McCann, Slepčev]

Fast diffusion equations: the infinite mass regime – Linearization of the entropy

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass.
- For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \leq m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez], [J.D., Toscani]

- work in relative variables
- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ *Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions*

Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where $m_c := \frac{d-2}{d}$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

Alternative approach by comparison techniques: [Daskalopoulos, Sesum] (without rates)

Fast diffusion equation and Barenblatt solutions

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \, \nabla u^{m-1}) = \frac{1-m}{m} \, \Delta u^m \tag{1}$$

with $m < 1$. We look for positive solutions $u(\tau, y)$ for $\tau \geq 0$ and $y \in \mathbb{R}^d$, $d \geq 1$, corresponding to nonnegative initial-value data $u_0 \in L^1_{\mathrm{loc}}(dx)$
In the limit case $m = 0$, u^m/m has to be replaced by $\log u$

Barenblatt type solutions are given by

$$U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d} \left(D + \frac{1-m}{2\,d\,|m-m_c|} \left| \frac{y}{R(\tau)} \right|^2 \right)_+^{-\frac{1}{1-m}}$$

- If $m > m_c := (d-2)/d$, $U_{D,T}$ with $R(\tau) := (T+\tau)^{\frac{1}{d\,(m-m_c)}}$ describes the large time asymptotics of the solutions of equation (1) as $\tau \to \infty$ (mass is conserved)
- If $m < m_c$ the parameter T now denotes the *extinction time* and $R(\tau) := (T-\tau)^{-\frac{1}{d\,(m_c-m)}}$
- If $m = m_c$ take $R(\tau) = e^\tau$, $U_{D,T}(\tau, y) = e^{-d\,\tau} \left(D + e^{-2\tau} \, |y|^2/2 \right)^{-d/2}$

Two crucial values of m: $m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{d} < 1$

Rescaling

A time-dependent change of variables

$$t := \tfrac{1-m}{2} \log\left(\tfrac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\tfrac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

If $m = m_c$, we take $t = \tau/d$ and $x = e^{-\tau}\, y/\sqrt{2}$

The generalized Barenblatt functions $U_{D,T}(\tau, y)$ are transformed into stationary *generalized Barenblatt profiles* $V_D(x)$

$$V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

If u is a solution to (1), the function $v(t,x) := R(\tau)^d\, u(\tau, y)$ solves

$$\frac{\partial v}{\partial t} = -\nabla \cdot \left[v\,\nabla\left(v^{m-1} - V_D^{m-1}\right)\right] \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{2}$$

with initial condition $v(t=0, x) = v_0(x) := R(0)^{-d}\, u_0(y)$

Goal

We are concerned with the *sharp rate* of convergence of a solution v of the rescaled equation to the *generalized Barenblatt profile* V_D in the whole range $m < 1$. Convergence is measured in terms of the **relative entropy**

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - V_D^m - m\, V_D^{m-1} (v - V_D) \right] dx$$

for all $m \neq 0$, $m < 1$

Assumptions on the initial datum v_0

(H1) $V_{D_0} \leq v_0 \leq V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \geq 3$ *and* $m \leq m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

- The case $m = m_* = \frac{d-4}{d-2}$ will be discussed later
- If $m > m_*$, we define D as the unique value in $[D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$

Our goal is to find the best possible rate of decay of $\mathcal{E}[v]$ *if* v *solves (2)*

Sharp rates of convergence

Theorem 5. [Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_*$*, the entropy decays according to*

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0$$

The sharp decay rate Λ *is equal to the best constant* $\Lambda_{\alpha,d} > 0$ *in the Hardy–Poincaré inequality of Theorem* 6 *with* $\alpha := 1/(m-1) < 0$
The constant $C > 0$ *depends only on* m, d, D_0, D_1, D *and* $\mathcal{E}[v_0]$

- Notion of *sharp rate* has to be discussed
- Rates of convergence in more standard norms: $L^q(dx)$ for $q \geq \max\{1, d\,(1-m)/\,[2\,(2-m) + d\,(1-m)]\}$, or C^k by interpolation
- By undoing the time-dependent change of variables, we deduce results on the *intermediate asymptotics* of (1), i.e. rates of decay of $u(\tau, y) - U_{D,T}(\tau, y)$ as $\tau \to +\infty$ if $m \in [m_c, 1)$, or as $\tau \to T$ if $m \in (-\infty, m_c)$

Strategy of proof

Assume that $D = 1$ and consider $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$, with $\alpha = 1/(m-1) < 0$, and $\mathcal{L}_{\alpha,d} := -h_{1-\alpha}\, \mathrm{div}\, [\, h_\alpha\, \nabla \cdot\,]$ on $L^2(d\mu_\alpha)$:

$$\int_{\mathbb{R}^d} f\, (\mathcal{L}_{\alpha,d}\, f)\, d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$$

A first order expansion of $v(t,x) = h_\alpha(x) \left[1 + \varepsilon\, f(t,x)\, h_\alpha^{1-m}(x)\right]$ solves

$$\frac{\partial f}{\partial t} + \mathcal{L}_{\alpha,d}\, f = 0$$

Theorem 6. *Let $d \geq 3$. For any $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

under the additional condition $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$

$$\Lambda_{\alpha,d} = \begin{cases} \frac{1}{4}\,(d - 2 + 2\,\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+2}{2}, \alpha_*\right) \cup (\alpha_*, 0) \\ -\,4\,\alpha - 2\,d & \text{if } \alpha \in \left[-d, -\frac{d+2}{2}\right) \\ -\,2\,\alpha & \text{if } \alpha \in (-\infty, -d) \end{cases}$$

[Denzler, McCann], [Blanchet, Bonforte, J.D., Grillo, Vázquez]

Proof: Relative entropy and relative Fisher information and interpolation

For $m \neq 0, 1$, the *relative entropy* of J. Ralston and W.I. Newmann and the *generalized relative Fisher information* are given by

$$\mathcal{F}[w] := \tfrac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \tfrac{1}{m} \left(w^m - 1 \right) \right] \, V_D^m \, dx$$

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \tfrac{1}{m-1} \, \nabla \left[\left(w^{m-1} - 1 \right) V_D^{m-1} \right] \right|^2 v \, dx$$

where $w = \frac{v}{V_D} \to 1$. If v is a solution of (2), then $\frac{d}{dt} \mathcal{F}[w(t,\cdot)] = -\mathcal{I}[w(t,\cdot)]$

- **Linearization:** $f := (w-1) \, V_D^{m-1}$, $h_1(t) := \inf_{\mathbb{R}^d} w(t,\cdot)$, $h_2(t) := \sup_{\mathbb{R}^d} w(t,\cdot)$ and $h := \max\{h_2, 1/h_1\}$. We notice that $h(t) \to 1$ as $t \to +\infty$

$$h^{m-2} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx \leq \frac{2}{m} \, \mathcal{F}[w] \leq h^{2-m} \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, V_D \, dx \leq [1 + X(h)] \, \mathcal{I}[w] + Y(h) \int_{\mathbb{R}^d} |f|^2 \, V_D^{2-m} \, dx$$

where X and Y are functions such that $\lim_{h \to 1} X(h) = \lim_{h \to 1} Y(h) = 0$

$$h_2^{2(2-m)} / h_1 \leq h^{5-2m} =: 1 + X(h)$$

$$\left[(h_2/h_1)^{2(2-m)} - 1 \right] \leq d \, (1-m) \left[h^{4(2-m)} - 1 \right] =: Y(h)$$

Proof (continued)

- A new **interpolation** inequality: for $h > 0$ small enough

$$\mathcal{F}[w] \leq \frac{h^{2-m}\,[1+X(h)]}{2\,\left[\Lambda_{\alpha,d} - m\,Y(h)\right]}\,m\,\mathcal{I}[w]$$

- Another **interpolation** allows to close the system of estimates: for some C, t large enough,

$$0 \leq h-1 \leq \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

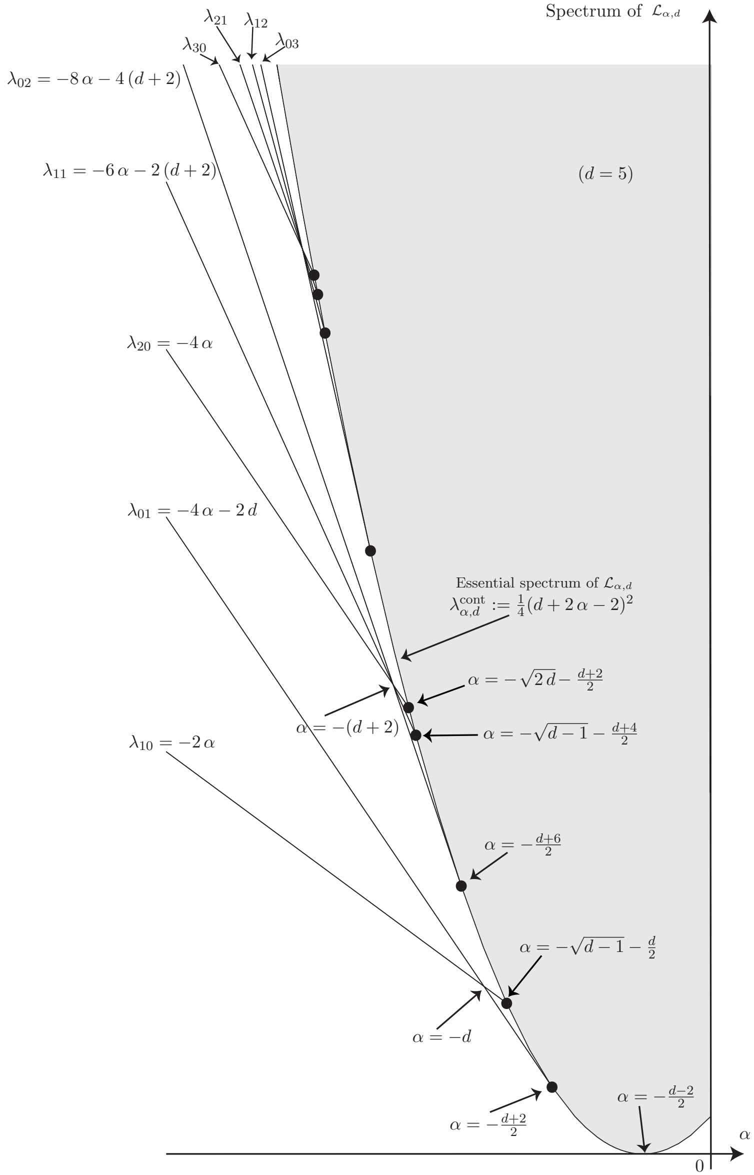
Hence we have a nonlinear differential inequality

$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] \leq -2\,\frac{\Lambda_{\alpha,d} - m\,Y(h)}{\left[1+X(h)\right]\,h^{2-m}}\,\mathcal{F}[w(t,\cdot)]$$

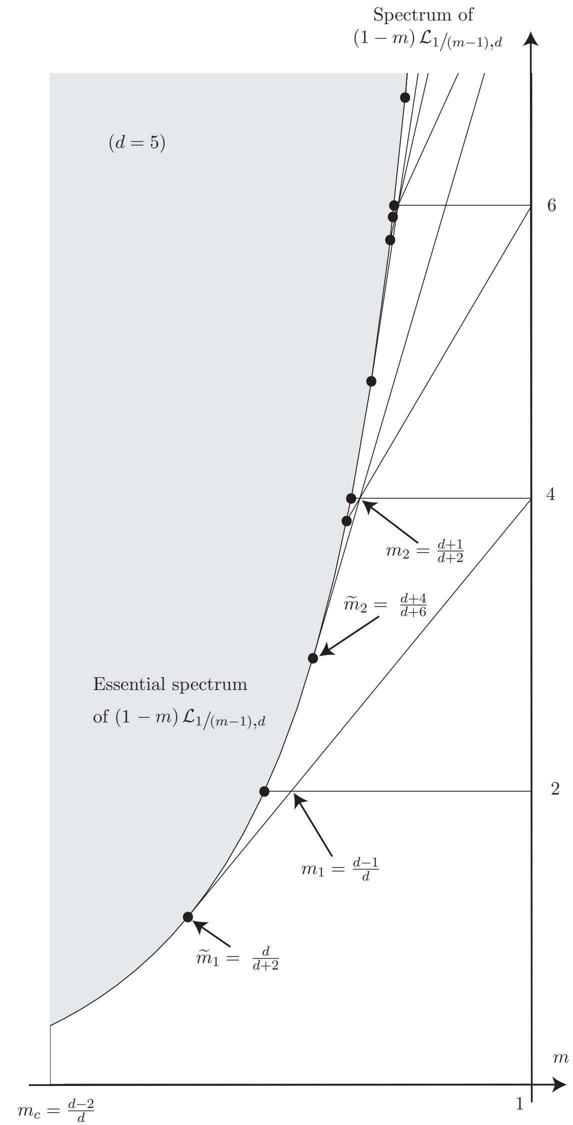
- A **Gronwall** lemma (take $h = 1 + \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$) then shows that

$$\limsup_{t\to\infty}\,\mathrm{e}^{\,2\,\Lambda_{\alpha,d}\,t}\mathcal{F}[w(t,\cdot)] < +\infty$$

Plots $(d = 5)$



Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\mathrm{cont}}_{\alpha,d} := \frac{1}{4}(d + 2\,\alpha - 2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0



Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

Remarks, improvements

- Optimal constants in interpolation inequalities does not mean optimal *asymptotic* rates
- The critical case $(m = m_*,\, d \geq 3)$: **Slow asymptotics** [Bonforte, Grillo, Vázquez] If $|v_0 - V_D|$ is bounded a.e. by a radial $L^1(dx)$ function, then there exists a positive constant C^* such that $\mathcal{E}[v(t,\cdot)] \leq C^*\, t^{-1/2}$ for any $t \geq 0$
- Can we improve the rates of convergence by imposing restrictions on the initial data ?
  - [Carrillo, Lederman, Markowich, Toscani (2002)] Poincaré inequalities for linearizations of very fast diffusion equations (radially symmetric solutions)
  - Formal or partial results: [Denzler, McCann (2005)], [McCann, Slepčev (2006)], [Denzler, Koch, McCann (announcement)],
- Faster convergence ?
  - Improved Hardy-Poincaré inequality: under the conditions $\int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$ and $\int_{\mathbb{R}^d} x\, f\, d\mu_{\alpha-1} = 0$ (center of mas),

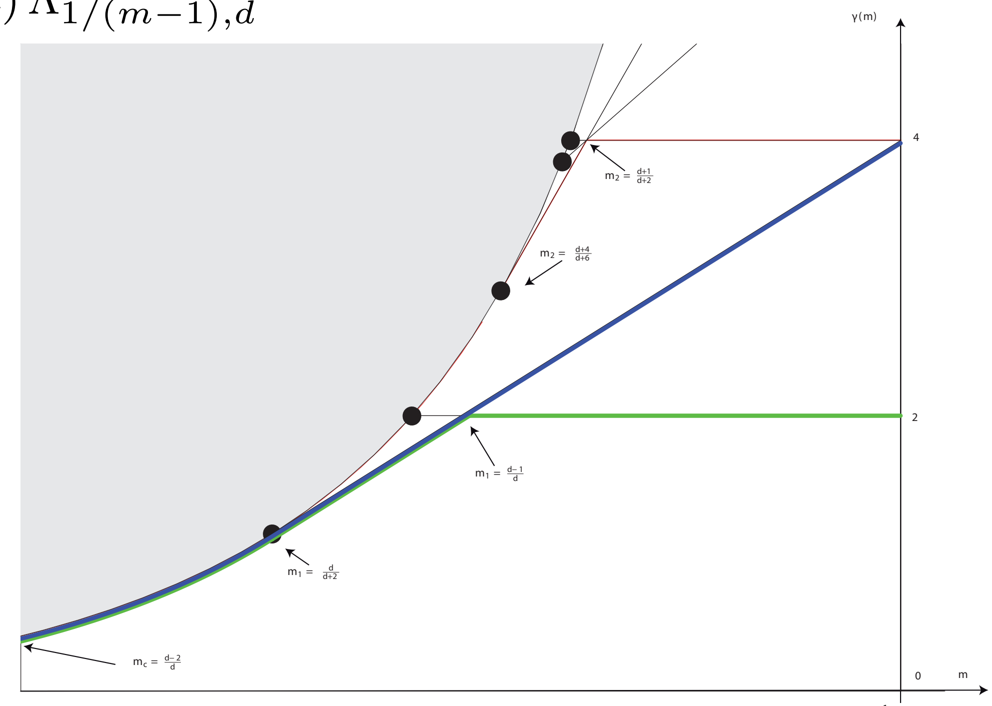
$$\widetilde{\Lambda}_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha$$

  - Next ? Can we kill other linear modes ?

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of (2) with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$ and if D is chosen so that $\int_{\mathbb{R}^d} (v_0 - V_D)\, dx = 0$, then

$$\mathcal{E}[v(t,\cdot)] \leq \widetilde{C}\, \mathrm{e}^{-\gamma(m)\, t} \quad \forall\, t \geq 0$$

with $\gamma(m) = (1-m)\, \widetilde{\Lambda}_{1/(m-1),d}$



Higher order matching asymptotics

For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. We do not use the scaling of self-similar solutions. Define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \, \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \, \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2 \, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2 \, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{3}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1}\,(v - B_\sigma)\right]\, dx$$

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt}\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v]\right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1}\int_{\mathbb{R}^d}\left(v^{m-1} - B_{\sigma(t)}^{m-1}\right)\frac{\partial v}{\partial t}\, dx$$

$$\Longleftrightarrow$$

Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$ (4)

Second step: the entropy / entropy production estimate

According to the definition of B_σ, we know that $2\,x = \sigma^{\frac{d}{2}(m-m_c)}\,\nabla B_\sigma^{m-1}$
Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1} - B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w - 1 - \frac{1}{m}\big(w^m - 1\big)\right] B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\,\nabla\left[(w^{m-1}-1)\,B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)^{\frac{d}{2}(m-m_c)}\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

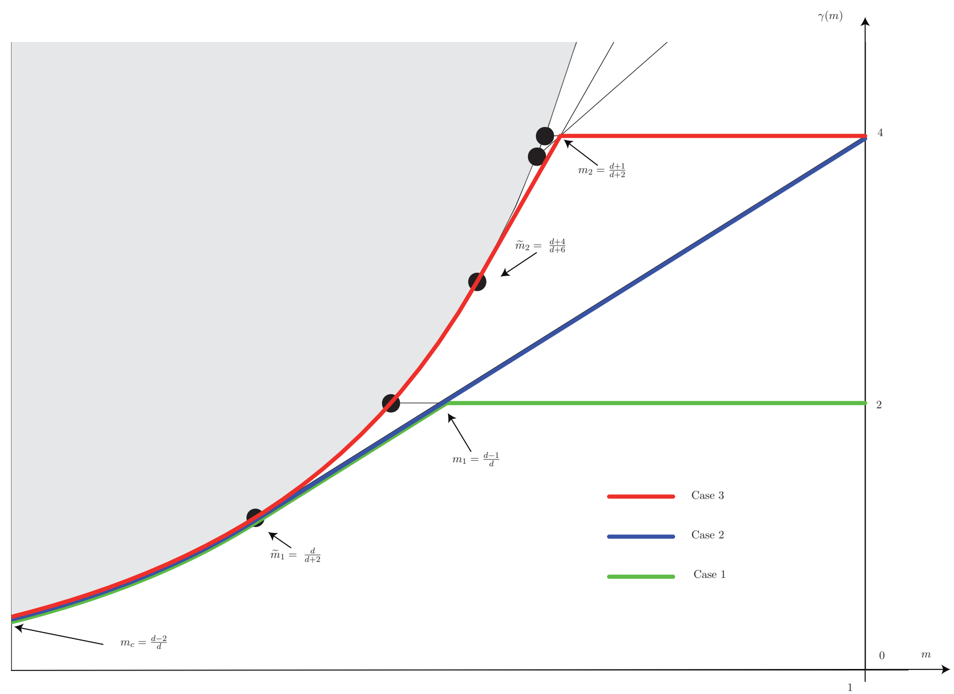
Improved rates of convergence

Theorem 7. *Let* $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$



[Denzler, Koch, McCann], in progress

Quantum mechanics ?

Let V be a smooth bounded nonpositive potential on $\mathbb{R}^d$,

$H_V = -\frac{\hbar^2}{2m}\Delta + V$ with eigenvalues $\lambda_1(V) < \lambda_2(V) \leq \lambda_3(V) \leq \ldots \lambda_N(V) < 0$

$$C^{(1)}_{\mathrm{LT}}(\gamma) := \inf_{\substack{V \in \mathcal{D}(\mathbb{R}^d) \\ V \leq 0}} \frac{|\lambda_1(V)|^\gamma}{\int_{\mathbb{R}^d} |V|^{\gamma+\frac{d}{2}}\, dx}$$

Gagliardo-Nirenberg inequality:

$$C_{\mathrm{GN}}(\gamma) = \inf_{\substack{u \in H^1(\mathbb{R}^d) \\ u \not\equiv 0 \ a.e.}} \frac{\|\nabla u\|^{\frac{d}{2\gamma+d}}_{L^2(\mathbb{R}^d)} \|u\|^{\frac{2\gamma}{2\gamma+d}}_{L^2(\mathbb{R}^d)}}{\|u\|_{L^{2\frac{2\gamma+d}{2\gamma+d-2}}(\mathbb{R}^d)}}$$

Theorem 8. *Let $d \in \mathbb{N}$, $d \geq 1$. For any $\gamma > 1 - \frac{d}{2}$,*

$$C^{(1)}_{\mathrm{LT}}(\gamma) = \kappa_1(\gamma)\left[C_{\mathrm{GN}}(\gamma)\right]^{-\kappa_2(\gamma)}$$

where $\kappa_1(\gamma) = \frac{2}{d}\left(\frac{d}{2\gamma+d}\right)^{1+\frac{d}{2\gamma}}$ and $\kappa_2(\gamma) = 2 + \frac{d}{\gamma}$

Lieb-Thirring inequality and interpolation inequalities

$$\sum_{i=1}^{N} |\lambda_i(V)|^\gamma \le C_{\rm LT}(\gamma) \int_{\mathbb{R}^d} |V|^{\gamma+\frac{d}{2}}\,dx$$

can be seen as an interpolation inequality: for any $m > 1$ (porous medium type), there exists a constant $K > 0$ such that

$$K \int_{\mathbb{R}^d} n_\rho^q\,dx \le Tr(-\Delta\rho) + Tr(\rho^m)$$

if ρ is a trace-class Hilbert-Schmidt operator: $m := \frac{\gamma}{\gamma-1}$ and $q = \frac{2\,\gamma+d}{2\,\gamma+d-2}$ and n_ρ is the spatial density associated to ρ: if $\rho = \sum_i \mu_i\,|\psi_i\rangle\,\langle\psi_i|$, then $n_\rho(x) = \sum_i \mu_i\,|\psi_i(x)|^2$

Other inequalities [J.D., Felmer, Loss, Paturel]

- (fast diffusion type): $m \in (d/(d+2), 1)$

$$K\;Tr(\rho^m) \le Tr(-\Delta\rho) + \int_{\mathbb{R}^d} n_\rho^q\,dx$$

- (logarithmic Sobolev type): $m = 1$

$$\int_{\mathbb{R}^d} n_\rho\,\log n_\rho\,dx + \frac{d}{2}\,\log(4\pi)\int_{\mathbb{R}^d} n_\rho\,dx \le Tr(-\Delta\rho) + Tr(\rho\,\log\rho)$$

Minimizers of free energy functionals and dynamical stability results

- [J.D., P. Felmer, J. Mayorga] Compactness properties for trace-class operators and applications to quantum mechanics
- [J.D., P. Felmer, M. Lewin] Orbitally stable states in generalized Hartree-Fock theory
- [G.L. Aki, J.D., C. Sparber] Thermal effects in gravitational Hartree systems

but...

- which relaxation mechanisms ?
- what about gradient flows ? [Degond, Gallego, Méhats, Ringhofer] [Mayorga]

... Thank you for your attention !