Fast diffusion equations: matching large time asymptotics by relative entropy methods

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Fast diffusion equations: outline

Introduction

Fast diffusion equations: entropy methods and Gagliardo-Nirenberg inequalities
[del Pino, J.D.]

Fast diffusion equations: the finite mass regime

Fast diffusion equations: the infinite mass regime

Relative entropy methods and linearization

the linearization of the functionals approach: [Blanchet, Bonforte, J.D., Grillo, Vázquez]

sharp rates: [Bonforte, J.D., Grillo, Vázquez]

An improvement based on the center of mass: [Bonforte, J.D., Grillo, Vázquez]

An improvement based on the variance: [J.D., Toscani]

Quantum mechanics?
Some references

J.D. and G. Toscani, Fast diffusion equations: matching large time asymptotics by relative entropy methods, Preprint


Fast diffusion equations: entropy methods

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d, \ t > 0$$

Self-similar (Barenblatt) function: $U(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t, \cdot) - U(t, \cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$

Existence theory, critical values of the parameter $m$
Intermediate asymptotics for fast diffusion & porous media

Some references
Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Del Pino, J.D.], [Carrillo, Toscani], [Otto],
[Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler,
J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo,
Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],... [del Pino, Sáez],
[Daskalopoulos, Sesum]... (incomplete, to be continued)

Some methods
1) [J.D., del Pino] relate entropy and Gagliardo-Nirenberg inequalities
2) entropy – entropy-production method: the Bakry-Emery point of view
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups
5) the approach by linearization of the entropy

... Fast diffusion equations and Gagliardo-Nirenberg inequalities
**Time-dependent rescaling, Free energy**

- **Time-dependent rescaling:** Take \( u(\tau, y) = R^{-d}(t) v(t, y/R(\tau)) \) where

\[
\frac{\partial R}{\partial \tau} = R^{d(1-m)-1}, \quad R(0) = 1, \quad t = \log R
\]

The function \( v \) solves a Fokker-Planck type equation

\[
\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x v), \quad v|_{\tau=0} = u_0
\]

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

\[
\Sigma[v] := \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \Sigma_0
\]

Entropy production is measured by the **Generalized Fisher information**

\[
\frac{d}{dt} \Sigma[v] = -I[v], \quad I[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx
\]
Relative entropy and entropy production

**Stationary solution:** choose $C$ such that $\|v_{\infty}\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_{\infty}(x) := \left(C + \frac{1-m}{2m} |x|^2\right)^{-1/(1-m)}_+$$

**Relative entropy:** Fix $\Sigma_0$ so that $\Sigma[v_{\infty}] = 0$. The entropy can be put in an $m$-homogeneous form: for $m \neq 1$,

$$\Sigma[v] = \int_{\mathbb{R}^d} \psi \left(\frac{v}{v_{\infty}}\right) v_{\infty}^m \, dx \quad \text{with} \quad \psi(t) = \frac{t^m - 1 - m(t - 1)}{m - 1}$$

**Entropy – entropy production inequality**

**Theorem 1.** $d \geq 3$, $m \in \left[\frac{d-1}{d}, +\infty\right)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2 \Sigma[v]$$

**Corollary 2.** A solution $v$ with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2 u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies

$$\Sigma[v(t, \cdot)] \leq \Sigma[u_0] e^{-2t} \quad \forall \, t \geq 0$$
An equivalent formulation: Gagliardo-Nirenberg inequalities

\[
\Sigma[v] = \int_{\mathbb{R}^d} \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) \, dx - \Sigma_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \nabla v^m \right| \, dx = \frac{1}{2} I[v]
\]

Rewrite it with \( p = \frac{1}{2m-1}, \ v = w^{2p}, \ v^m = w^{p+1} \) as

\[
\frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \left( \frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} \, dx + K \geq 0
\]

- \( 1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \iff \) Fast diffusion case: \( \frac{d-1}{d} \leq m < 1 ; \ K < 0 \)
- \( 0 < p < 1 \iff \) Porous medium case: \( m > 1, \ K > 0 \)
- for some \( \gamma, \ K = K_0 (\int_{\mathbb{R}^d} v \, dx = \int_{\mathbb{R}^d} w^{2p} \, dx) \gamma \)
- \( w = w_\infty = v_\infty^{1/2p} \) is optimal
- \( m = m_1 := \frac{d-1}{d} \): Sobolev, \( m \to 1 \): logarithmic Sobolev

**Theorem 3.** [Del Pino, J.D.] Assume that \( 1 < p \leq \frac{d}{d-2} \) (fast diffusion case) and \( d \geq 3 \)

\[
\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A \|\nabla w\|_{L^2(\mathbb{R}^d)}^\theta \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}
\]

\[
A = \left( \frac{y(p-1)^2}{2\pi d} \right)^{\theta/2} \left( \frac{2y-d}{2y} \right)^{1/2p} \left( \frac{\Gamma(y)}{\Gamma(y-d/2)} \right)^{\theta/d}, \quad \theta = \frac{d(p-1)}{p(d+2)-(d-2)p}, \quad y = \frac{p+1}{p-1}
\]
Intermediate asymptotics

\[ \Sigma[v] \leq \Sigma[u_0] e^{-2\tau} + \text{Csiszár-Kullback inequalities} \]

Undo the change of variables, with

\[ u_\infty(t, x) = R^{-d}(t) v_\infty(x/R(t)) \]

**Theorem 4.** [Del Pino, J.D.] *Consider a solution of* \( u_t = \Delta u^m \) *with initial data* \( u_0 \in L^1_+(\mathbb{R}^d) \) *such that* \( |x|^2 u_0 \in L^1(\mathbb{R}^d), u_0^m \in L^1(\mathbb{R}^d) \)

- **Fast diffusion case:** \( \frac{d-1}{d} < m < 1 \) if \( d \geq 3 \)

  \[
  \limsup_{t \to +\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \| u^m - u_\infty^m \|_{L^1} < +\infty
  \]

- **Porous medium case:** \( 1 < m < 2 \)

  \[
  \limsup_{t \to +\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \| [u - u_\infty] u_\infty^{m-1} \|_{L^1} < +\infty
  \]
Fast diffusion equations: the finite mass regime

Can we consider $m < m_1$?

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality.

- Displacement convexity holds in the same range of exponents, $m \in (m_1, 1)$, as for the Gagliardo-Nirenberg inequalities.

- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance if $m > \tilde{m}_1 := \frac{d}{d+2}$.

- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1$ and the Barenblatt self-similar solution has finite mass.
...the Bakry-Emery method

Consider the generalized Fisher information

$$I[v] := \int_{\mathbb{R}^d} v |Z|^2 \, dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt} I[v(t, \cdot)] + 2 I[v(t, \cdot)] = -2(m - 1) \int_{\mathbb{R}^d} u^m (\text{div} Z)^2 \, dx - 2 \sum_{i, j=1}^{d} \int_{\mathbb{R}^d} u^m (\partial_i Z^j)^2 \, dx$$

- the Fisher information decays exponentially: $I[v(t, \cdot)] \leq I[u_0] e^{-2t}$
- $\lim_{t \to \infty} I[v(t, \cdot)] = 0$ and $\lim_{t \to \infty} \Sigma[v(t, \cdot)] = 0$
- $\frac{d}{dt} \left( I[v(t, \cdot)] - 2 \Sigma[v(t, \cdot)] \right) \leq 0$ means $I[v] \geq 2 \Sigma[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

$$I[v] \geq 2 \Sigma[v]$$

holds for any $m > m_c$
Fast diffusion: finite mass regime

Inequalities...

Sobolev

logarithmic Sobolev

Gagliardo-Nirenberg

\[
\frac{d-2}{d} < \frac{d}{d+2} < \frac{d-1}{d} < 1
\]

\[v^m \in L^1, \quad |x|^2 v \in L^1\]

Bakry-Emery method (relative entropy)

global existence in \( L^1 \)

... existence of solutions of \( u_t = \Delta u^m \)
More references: Extensions and related results

- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev], [Kim, McCann], [Koch, McCann, Slepčev]
Fast diffusion equations: the infinite mass regime – Linearization of the entropy

If \( m > m_c := \frac{d-2}{d} \leq m < m_1 \), solutions globally exist in \( L^1(\mathbb{R}^d) \) and the Barenblatt self-similar solution has finite mass.

For \( m \leq m_c \), the Barenblatt self-similar solution has infinite mass

**Extension to \( m \leq m_c \)? Work in relative variables!**

\[
\begin{align*}
\Sigma[V_D | V_0] &= \infty \\
v_0 - V_{D*} &\in L^1 \\
V_{D1} - V_{D0} &\notin L^1 \\
\Sigma[V_D | V_0] &< \infty \\
v_0, V_D &\in L^1
\end{align*}
\]

\[
\begin{align*}
\frac{d-4}{d-2} &\rightarrow \frac{d-2}{d} & \frac{d}{d+2} &\rightarrow \frac{d-1}{d} & 1 \\
g &\rightarrow \text{Gagliardo-Nirenberg} \\
g &\rightarrow \text{Bakry-Emery method (relative entropy)} \\
g &\rightarrow \text{global existence in } L^1
\end{align*}
\]
Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez], [J.D., Toscani]

- work in relative variables
- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

⇒ Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions

Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where $m_c := \frac{d-2}{d}$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \to T} u(\tau, y) = 0$

Alternative approach by comparison techniques: [Daskalopoulos, Sesum] (without rates)
Fast diffusion equation and Barenblatt solutions

\[
\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \nabla u^{m-1}) = \frac{1-m}{m} \Delta u^m
\]  

(1)

with \( m < 1 \). We look for positive solutions \( u(\tau, y) \) for \( \tau \geq 0 \) and \( y \in \mathbb{R}^d, d \geq 1, \)
corresponding to nonnegative initial-value data \( u_0 \in L^1_{\text{loc}}(dx) \)
In the limit case \( m = 0 \), \( u^m/m \) has to be replaced by \( \log u \)

**Barenblatt type solutions** are given by

\[
U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d} \left( D + \frac{1-m}{2d|m-m_c|} |y/R(\tau)|^2 \right)^{-\frac{1}{1-m}}
\]

- If \( m > m_c := (d-2)/d \), \( U_{D,T} \) with \( R(\tau) := (T+\tau)^{d/(m-m_c)} \) describes the large time
  asymptotics of the solutions of equation (1) as \( \tau \to \infty \) (mass is conserved)
- If \( m < m_c \) the parameter \( T \) now denotes the **extinction time** and
  \( R(\tau) := (T - \tau)^{-d(m_c-m)} \)
- If \( m = m_c \) take \( R(\tau) = e^\tau, U_{D,T}(\tau, y) = e^{-d \tau} \left( D + e^{-2\tau} |y|^2/2 \right)^{-d/2} \)

Two crucial values of \( m \): \( m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{d} < 1 \)
A time-dependent change of variables

\[ t := \frac{1-m}{2} \log \left( \frac{R(\tau)}{R(0)} \right) \quad \text{and} \quad x := \sqrt{\frac{1}{2d|m-m_c|}} \frac{y}{R(\tau)} \]

If \( m = m_c \), we take \( t = \tau/d \) and \( x = e^{-\tau} y/\sqrt{2} \)

The generalized Barenblatt functions \( U_{D,T}(\tau, y) \) are transformed into stationary generalized Barenblatt profiles \( V_D(x) \)

\[ V_D(x) := (D + |x|^2)^{-\frac{1}{m-1}} \quad x \in \mathbb{R}^d \]

If \( u \) is a solution to (1), the function \( v(t, x) := R(\tau)^d u(\tau, y) \) solves

\[ \partial_t v = -\nabla \cdot \left[ v \nabla \left( v^{m-1} - V_D^{m-1} \right) \right] \quad t > 0, \quad x \in \mathbb{R}^d \]  \hspace{1cm} (2)

with initial condition \( v(t = 0, x) = v_0(x) := R(0)^{-d} u_0(y) \)
Goal

We are concerned with the sharp rate of convergence of a solution $v$ of the rescaled equation to the generalized Barenblatt profile $V_D$ in the whole range $m < 1$. Convergence is measured in terms of the relative entropy

$$E[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[ v^m - V_D^m - m V_D^{m-1}(v - V_D) \right] \, dx$$

for all $m \neq 0$, $m < 1$

Assumptions on the initial datum $v_0$

(H1) $V_{D_0} \leq v_0 \leq V_{D_1}$ for some $D_0 > D_1 > 0$

(H2) if $d \geq 3$ and $m \leq m^*$, $(v_0 - V_D)$ is integrable for a suitable $D \in [D_1, D_0]$

The case $m = m^* = \frac{d-4}{d-2}$ will be discussed later

If $m > m^*$, we define $D$ as the unique value in $[D_1, D_0]$ such that $\int_{\mathbb{R}^d} (v_0 - V_D) \, dx = 0$

Our goal is to find the best possible rate of decay of $E[v]$ if $v$ solves (2)
Sharp rates of convergence

**Theorem 5.** [Bonforte, J.D., Grillo, Vázquez] Under Assumptions (H1)-(H2), if $m < 1$ and $m \neq m_*$, the entropy decays according to

$$
\mathcal{E}[v(t, \cdot)] \leq C e^{-2(1-m)\Lambda t} \quad \forall \ t \geq 0
$$

The sharp decay rate $\Lambda$ is equal to the best constant $\Lambda_{\alpha,d} > 0$ in the Hardy–Poincaré inequality of Theorem 6 with $\alpha := 1/(m - 1) < 0$

The constant $C > 0$ depends only on $m, d, D_0, D_1, D$ and $\mathcal{E}[v_0]$

- Notion of *sharp rate* has to be discussed
- Rates of convergence in more standard norms: $L^q(dx)$ for $q \geq \max\{1, d (1 - m) / [2 (2 - m) + d (1 - m)]\}$, or $C^k$ by interpolation
- By undoing the time-dependent change of variables, we deduce results on the *intermediate asymptotics* of (1), i.e. rates of decay of $u(\tau, y) - U_{D,T}(\tau, y)$ as $\tau \to +\infty$ if $m \in [m_c, 1)$, or as $\tau \to T$ if $m \in (-\infty, m_c)$
Strategy of proof

Assume that $D = 1$ and consider $d\mu_\alpha := h_\alpha \, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$, with $\alpha = 1/(m - 1) < 0$, and $\mathcal{L}_{\alpha,d} := -h_{1-\alpha} \text{ div } [h_\alpha \nabla \cdot]$ on $L^2(d\mu_\alpha)$:

$$
\int_{\mathbb{R}^d} f(\mathcal{L}_{\alpha,d} f) \, d\mu_{\alpha-1} = \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha
$$

A first order expansion of $v(t, x) = h_\alpha(x) \left[ 1 + \varepsilon f(t, x) h_\alpha^{1-m}(x) \right]$ solves

$$
\frac{\partial f}{\partial t} + \mathcal{L}_{\alpha,d} f = 0
$$

Theorem 6. Let $d \geq 3$. For any $\alpha \in (-\infty, 0) \setminus \{\alpha_*\}$, there is a positive constant $\Lambda_{\alpha,d}$ such that

$$
\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall f \in H^1(d\mu_\alpha)
$$

under the additional condition $\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0$ if $\alpha < \alpha_*$

$$
\Lambda_{\alpha,d} = \begin{cases} 
\frac{1}{4} (d - 2 + 2\alpha)^2 & \text{if } \alpha \in \left[-\frac{d+2}{2}, \alpha_*\right) \cup (\alpha_*, 0) \\
-4\alpha - 2d & \text{if } \alpha \in \left[-d, -\frac{d+2}{2}\right) \\
-2\alpha & \text{if } \alpha \in (-\infty, -d)
\end{cases}
$$

[Denzler, McCann], [Blanchet, Bonforte, J.D., Grillo, Vázquez]
Proof: Relative entropy and relative Fisher information and interpolation

For $m \neq 0, 1$, the relative entropy of J. Ralston and W.I. Newmann and the generalized relative Fisher information are given by

$$\mathcal{F}[w] := \frac{m}{1-m} \int_{\mathbb{R}^d} \left[ w - 1 - \frac{1}{m} (w^m - 1) \right] V_D^m \, dx$$

$$\mathcal{I}[w] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \nabla \left[ (w^{m-1} - 1) V_D^{m-1} \right] \right|^2 v \, dx$$

where $w = \frac{v}{V_D} \to 1$. If $v$ is a solution of (2), then $\frac{d}{dt} \mathcal{F}[w(t, \cdot)] = -\mathcal{I}[w(t, \cdot)]$

**Linearization:** $f := (w - 1) V_D^{m-1}$, $h_1(t) := \inf_{\mathbb{R}^d} w(t, \cdot)$, $h_2(t) := \sup_{\mathbb{R}^d} w(t, \cdot)$ and $h := \max\{h_2, 1/h_1\}$. We notice that $h(t) \to 1$ as $t \to +\infty$

$$h^{m-2} \int_{\mathbb{R}^d} |f|^2 V_D^{2-m} \, dx \leq \frac{2}{m} \mathcal{F}[w] \leq h^{2-m} \int_{\mathbb{R}^d} |f|^2 V_D^{2-m} \, dx$$

$$\int_{\mathbb{R}^d} |\nabla f|^2 V_D \, dx \leq [1 + X(h)] \mathcal{I}[w] + Y(h) \int_{\mathbb{R}^d} |f|^2 V_D^{2-m} \, dx$$

where $X$ and $Y$ are functions such that $\lim_{h \to 1} X(h) = \lim_{h \to 1} Y(h) = 0$

$$h_2^{2(2-m)} / h_1 \leq h^{5-2m} =: 1 + X(h)$$

$$[ (h_2 / h_1)^{2(2-m)} - 1 ] \leq d (1 - m) [ h_4^{4(2-m)} - 1 ] =: Y(h)$$
Proof (continued)

A new interpolation inequality: for $h > 0$ small enough

$$\mathcal{F}[w] \leq \frac{h^{2-m}}{2} \left[ 1 + X(h) \right] m \mathcal{I}[w]$$

Another interpolation allows to close the system of estimates: for some $C$, $t$ large enough,

$$0 \leq h - 1 \leq C \mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

Hence we have a nonlinear differential inequality

$$\frac{d}{dt} \mathcal{F}[w(t, \cdot)] \leq -2 \frac{\Lambda_{\alpha,d} - m Y(h)}{1 + X(h)} h^{2-m} \mathcal{F}[w(t, \cdot)]$$

A Gronwall lemma (take $h = 1 + C \mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$) then shows that

$$\limsup_{t \to \infty} e^{2 \Lambda_{\alpha,d} t} \mathcal{F}[w(t, \cdot)] < +\infty$$
Plots ($d = 5$)

Essential spectrum of $L_{\alpha,d}$

\[ \lambda_{01} = -4 \alpha - 2 (d + 2) \]

\[ \lambda_{11} = -6 \alpha - 2 (d + 2) \]

\[ \lambda_{02} = -8 \alpha - 4 (d + 2) \]

\[ \lambda_{20} = -4 \alpha \]

\[ \lambda_{03} = -8 \alpha - 4 (d + 2) \]

\[ \lambda_{10} = -4 \alpha - 2d \]

\[ \lambda_{21} = -6 \alpha - 2d \]

\[ \lambda_{30} = -8 \alpha - 4d \]

\[ \lambda_{22} = -10 \alpha - 6d \]

Spectrum of $L_{\alpha,d}$

\[ \alpha = \sqrt{d - \frac{4}{3}} \]

\[ \alpha = -\sqrt{d - \frac{4}{3}} \]

\[ \alpha = -d \]

\[ \alpha = -\sqrt{d - \frac{4}{3}} \]

\[ \alpha = -d \]

\[ \alpha = -\frac{d+2}{2} \]

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Remarks, improvements

- Optimal constants in interpolation inequalities does not mean optimal asymptotic rates

- The critical case ($m = m_*, d \geq 3$): Slow asymptotics [Bonforte, Grillo, Vázquez] If $|v_0 - V_D|$ is bounded a.e. by a radial $L^1(dx)$ function, then there exists a positive constant $C^*$ such that $E[v(t, \cdot)] \leq C^* t^{-1/2}$ for any $t \geq 0$

- Can we improve the rates of convergence by imposing restrictions on the initial data?
  - [Carrillo, Lederman, Markowich, Toscani (2002)] Poincaré inequalities for linearizations of very fast diffusion equations (radially symmetric solutions)
  - Formal or partial results: [Denzler, McCann (2005)], [McCann, Slepčev (2006)], [Denzler, Koch, McCann (announcement)]

- Faster convergence?
  - Improved Hardy-Poincaré inequality: under the conditions $\int_{\mathbb{R}^d} f \, d\mu_{\alpha-1} = 0$ and $\int_{\mathbb{R}^d} xf \, d\mu_{\alpha-1} = 0$ (center of mas),
  \[
  \Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_{\alpha}
  \]
  - Next? Can we kill other linear modes?
Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if $v$ is a solution of (2) with initial datum $v_0$ such that $\int_{\mathbb{R}^d} x v_0 \, dx = 0$ and if $D$ is chosen so that $\int_{\mathbb{R}^d} (v_0 - V_D) \, dx = 0$, then

$$\mathcal{E}[v(t, \cdot)] \leq \bar{C} e^{-\gamma(m) t} \quad \forall t \geq 0$$

with $\gamma(m) = (1 - m) \tilde{\Lambda}_1/(m-1), d$. 

\[ \begin{align*}
\text{Graph of } v(t, \cdot) \\
\text{with } \gamma(m) = (1 - m) \tilde{\Lambda}_1/(m-1), d
\end{align*} \]
Higher order matching asymptotics

For some $m \in (m_c, 1)$ with $m_c := (d - 2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot (u \nabla u^{m-1}) = 0$$

The strategy is easy to understand using a time-dependent rescaling and the relative entropy formalism. We do not use the scaling of self-similar solutions. Define the function $v$ such that

$$u(\tau, y + x_0) = R^{-d} v(t, x), \quad R = R(\tau), \quad t = \frac{1}{2} \log R, \quad x = \frac{y}{R}$$

Then $v$ has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[ v \left( \sigma^{\frac{d}{2}} (m - m_c) \nabla v^{m-1} - 2x \right) \right] = 0 \quad t > 0, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on $R$)

$$2 \sigma^{-\frac{d}{2}} (m - m_c) = R^{1-d(1-m)} \frac{dR}{d\tau}$$
Refined relative entropy

Consider the family of the Barenblatt profiles

\[ B_\sigma(x) := \sigma^{-\frac{d}{2}} \left( C_M + \frac{1}{\sigma} |x|^2 \right)^{-\frac{1}{m-1}} \quad \forall \, x \in \mathbb{R}^d \quad (3) \]

Note that \( \sigma \) is a function of \( t \): as long as \( \frac{d\sigma}{dt} \neq 0 \), the Barenblatt profile \( B_\sigma \) is not a solution but we may still consider the relative entropy

\[ F_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[ v^m - B_\sigma^m - m B_\sigma^{m-1} (v - B_\sigma) \right] \, dx \]

Let us briefly sketch the strategy of our method before giving all details

The time derivative of this relative entropy is

\[
\frac{d}{dt} F_\sigma(t)[v(t, \cdot)] = \frac{d\sigma}{dt} \left( \frac{d}{d\sigma} F_\sigma[v] \right) \bigg|_{\sigma=\sigma(t)} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left( v^{m-1} - B_\sigma^{m-1} \right) \frac{\partial v}{\partial t} \, dx
\]

choose it = 0

\[ \iff \]

Minimize \( F_\sigma[v] \) w.r.t. \( \sigma \) \iff \[ \int_{\mathbb{R}^d} |x|^2 B_\sigma \, dx = \int_{\mathbb{R}^d} |x|^2 v \, dx \quad (4) \]
Second step: the entropy / entropy production estimate

According to the definition of $B_\sigma$, we know that $2x = \sigma \frac{d}{2}(m-m_c) \nabla B_\sigma^{m-1}$

Using the new change of variables, we know that

$$\frac{d}{dt} F_\sigma(t)[v(t, \cdot)] = -m \sigma(t) \frac{d}{2}(m-m_c) \int_{\mathbb{R}^d} v \left| \nabla \left[v^{m-1} - B_\sigma^{m-1}(t)\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$F_\sigma[v] = \frac{m}{1-m} \int_{\mathbb{R}^d} \left[w - 1 - \frac{1}{m} (w^m - 1)\right] B_\sigma^m dx$$

(Repeating) define the relative Fisher information by

$$I_\sigma[v] := \int_{\mathbb{R}^d} \left| \frac{1}{m-1} \nabla \left[(w^{m-1} - 1) B_\sigma^{m-1}\right]\right|^2 B_\sigma w dx$$

so that

$$\frac{d}{dt} F_\sigma(t)[v(t, \cdot)] = -m (1-m) \sigma(t) \frac{d}{2}(m-m_c) I_\sigma(t)[v(t, \cdot)] \quad \forall \ t > 0$$

When linearizing, one more mode is killed and $\sigma(t)$ scales out
Improved rates of convergence

**Theorem 7.** Let \( m \in (\tilde{m}_1, 1) \), \( d \geq 2 \), \( v_0 \in L^1_+(\mathbb{R}^d) \) such that \( v^m_0, |y|^2 v_0 \in L^1(\mathbb{R}^d) \)

\[
\mathcal{E}[v(t, \cdot)] \leq C e^{-2 \gamma(m) t} \quad \forall \ t \geq 0
\]

where

\[
\gamma(m) = \begin{cases} 
\frac{((d-2)m-(d-4))^2}{4(1-m)} & \text{if } m \in (\tilde{m}_1, \tilde{m}_2] \\
4(d+2)m - 4d & \text{if } m \in [\tilde{m}_2, m_2] \\
4 & \text{if } m \in [m_2, 1)
\end{cases}
\]

[Denzler, Koch, McCann], in progress
Quantum mechanics?

Let $V$ be a smooth bounded nonpositive potential on $\mathbb{R}^d$,

$$H_V = -\frac{\hbar^2}{2m} \Delta + V$$

with eigenvalues $\lambda_1(V) < \lambda_2(V) \leq \lambda_3(V) \leq \ldots \lambda_N(V) < 0$

$$C^{(1)}_{LT}(\gamma) := \inf_{V \in \mathcal{D}(\mathbb{R}^d), V \leq 0} \frac{\lvert \lambda_1(V) \rvert^\gamma}{\int_{\mathbb{R}^d} \lvert V \rvert^{\gamma + \frac{d}{2}} \, dx}$$

Gagliardo-Nirenberg inequality:

$$C_{GN}(\gamma) = \inf_{u \in H^1(\mathbb{R}^d), u \not\equiv 0 \text{ a.e.}} \frac{\| \nabla u \|_{L^2(\mathbb{R}^d)}^{\frac{2\gamma + d}{2\gamma + d - 2}}}{\| u \|_{L^\frac{2\gamma + d}{2\gamma + d - 2}(\mathbb{R}^d)}^{\frac{2\gamma + d}{2\gamma + d - 2}}}$$

**Theorem 8.** Let $d \in \mathbb{N}$, $d \geq 1$. For any $\gamma > 1 - \frac{d}{2}$,

$$C^{(1)}_{LT}(\gamma) = \kappa_1(\gamma) \left[ C_{GN}(\gamma) \right]^{-\kappa_2(\gamma)}$$

where $\kappa_1(\gamma) = \frac{2}{d} \left( \frac{d}{2\gamma + d} \right)^{1 + \frac{d}{2\gamma}}$ and $\kappa_2(\gamma) = 2 + \frac{d}{\gamma}$
Lieb-Thirring inequality and interpolation inequalities

\[ \sum_{i=1}^{N} |\lambda_i(V)|^\gamma \leq C_{LT}(\gamma) \int_{\mathbb{R}^d} |V|^{\gamma + \frac{d}{2}} \, dx \]

can be seen as an interpolation inequality: for any \( m > 1 \) (porous medium type), there exists a constant \( K > 0 \) such that

\[ K \int_{\mathbb{R}^d} n_\rho^q \, dx \leq Tr(-\Delta \rho) + Tr(\rho^m) \]

if \( \rho \) is a trace-class Hilbert-Schmidt operator: \( m := \frac{\gamma}{\gamma - 1} \) and \( q = \frac{2\gamma + d}{2\gamma + d - 2} \) and \( n_\rho \) is the spatial density associated to \( \rho \): if \( \rho = \sum_i \mu_i |\psi_i\rangle \langle \psi_i| \), then \( n_\rho(x) = \sum_i \mu_i |\psi_i(x)|^2 \)

Other inequalities [J.D., Felmer, Loss, Paturel]

\( \bullet \) (fast diffusion type): \( m \in (d/(d+2), 1) \)

\[ K \, Tr(\rho^m) \leq Tr(-\Delta \rho) + \int_{\mathbb{R}^d} n_\rho^q \, dx \]

\( \bullet \) (logarithmic Sobolev type): \( m = 1 \)

\[ \int_{\mathbb{R}^d} n_\rho \log n_\rho \, dx + \frac{d}{2} \log(4\pi) \int_{\mathbb{R}^d} n_\rho \, dx \leq Tr(-\Delta \rho) + Tr(\rho \log \rho) \]
Minimizers of free energy functionals and dynamical stability results

[J.D., P. Felmer, J. Mayorga] Compactness properties for trace-class operators and applications to quantum mechanics


[G.L. Aki, J.D., C. Sparber] Thermal effects in gravitational Hartree systems

but...

which relaxation mechanisms?

what about gradient flows? [Degond, Gallego, Méhats, Ringhofer] [Mayorga]
... Thank you for your attention!