Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem

in collaboration with

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We consider the Brezis-Nirenberg problem

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases} \tag{1}$$

in dimension $N \geq 4$, in the supercritical case: $p = \frac{N+2}{N-2}$, $\varepsilon > 0$.

If $\varepsilon \to 0$ and if, simultaneously, $\lambda \to 0$ at the appropriate rate, then there are radial solutions which behave like a superposition of *bubbles*, namely solutions of the form

$$(N(N-2))^{(N-2)/4} \sum_{j=1}^{k} \left(1 + M_j^{\frac{4}{N-2}}\,|y|^2\right)^{-(N-2)/2} M_j\,(\,1\ +\ o(1)\,)\,,$$

where $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j. These solutions lie close to turning points "to the right" of the associated bifurcation diagram.

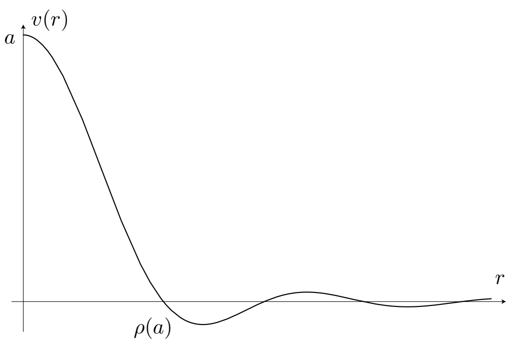
1. Parametrization of the solutions

Let B be the unit ball in $I\!R^N$, $N \geq 4$, and consider for $p = \frac{N+2}{N-2}$ and $\varepsilon \geq 0$ the positive solutions of

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases}$$

Denote by $\rho = \rho(a) > 0$ the first zero of v given by

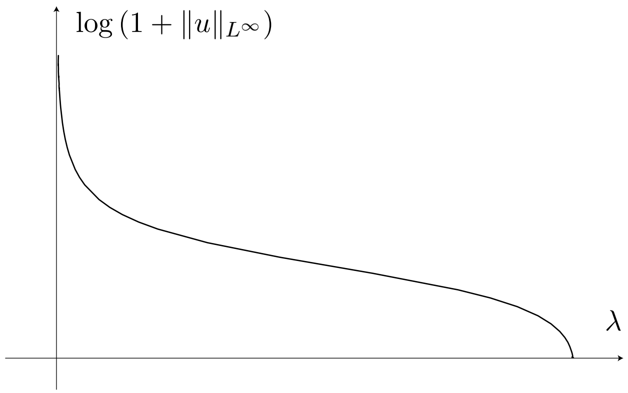
$$\begin{cases} v'' + \frac{N-1}{r}\, v' + v^{p+\varepsilon} + v = 0 & \text{in } [0, +\infty) \\ v(0) = a > 0\,, \quad v'(0) = 0 \end{cases}$$



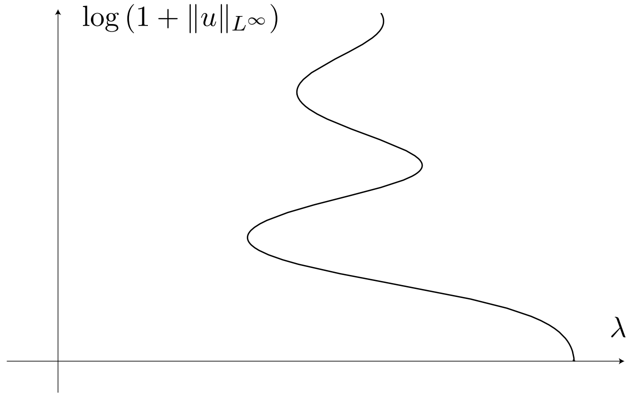
To any solution u of (1) corresponds a function v on $[0, \sqrt{\lambda})$ s.t.

$$v(|x|) = \lambda^{-1/(p+\varepsilon-1)}\, u(x/\sqrt{\lambda}) \Longleftrightarrow u(x) = \rho^{2/(p+\varepsilon-1)}\, v(\rho\,|x|)$$

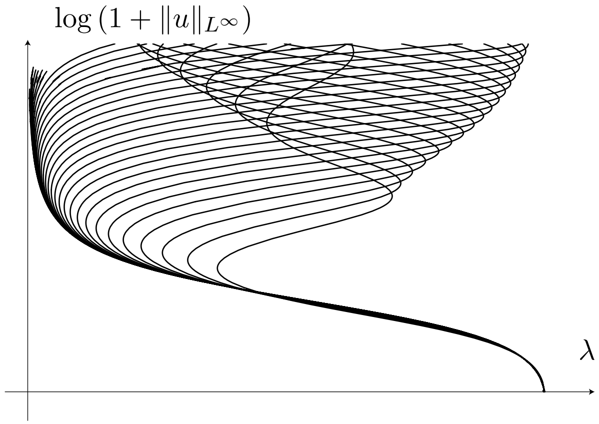
with $\lambda = \rho^2(a)$. The bifurcation diagram $(\lambda, \|u\|_{L^\infty})$ is therefore fully parametrized by $a \mapsto (\rho^2,\, a\,\rho^{2/(p+\varepsilon-1)})$ with $\rho = \rho^2(a)$.



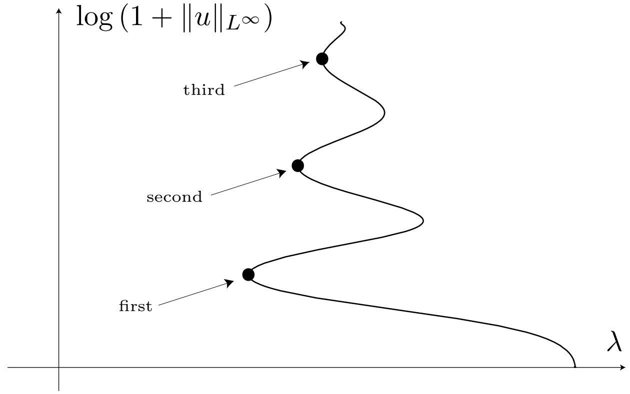
Critical case: $\varepsilon = 0$



Supercritical case: $\varepsilon = 0.2$



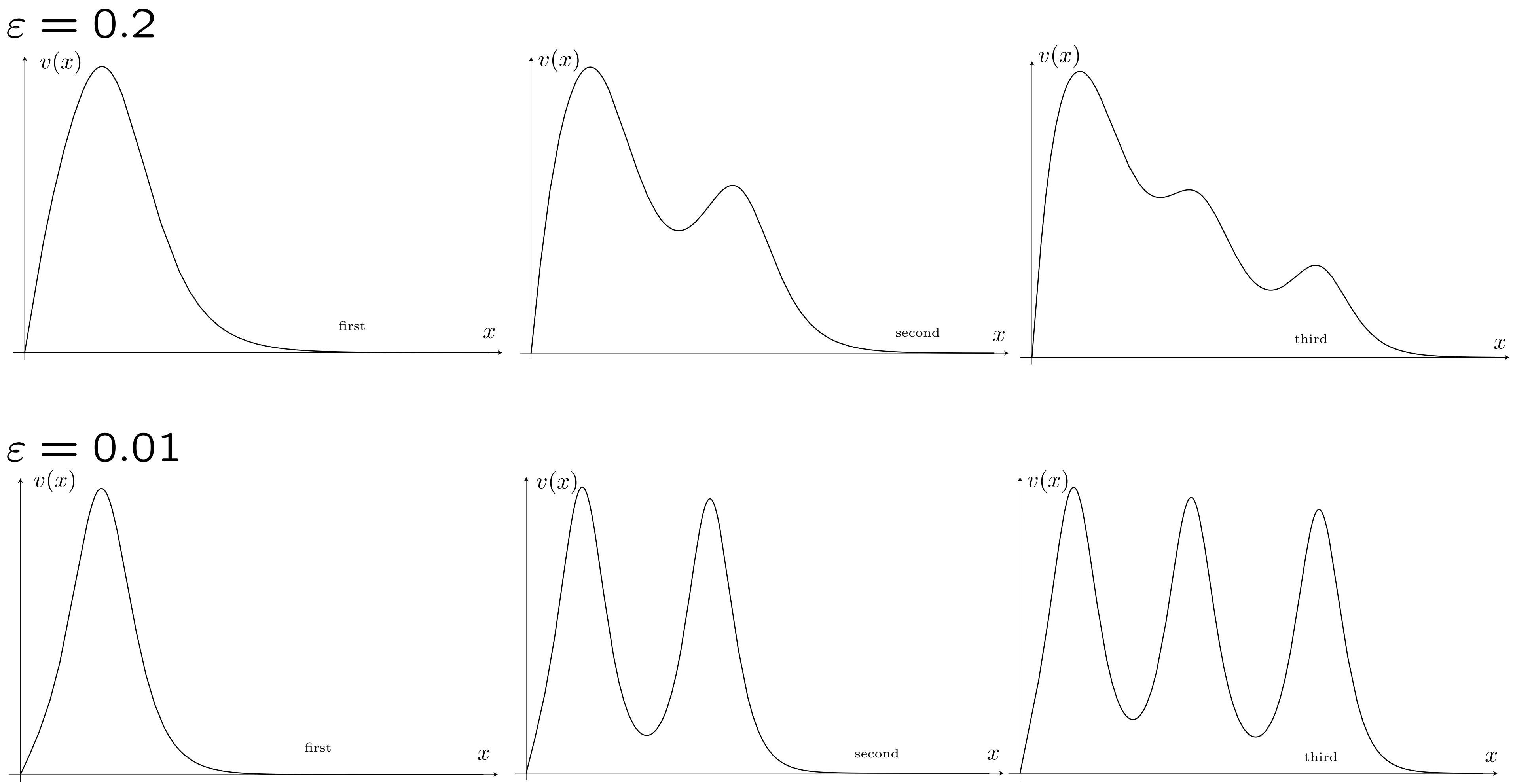
Approximating the critical case:
$\varepsilon = 2^{-q}\varepsilon_0$, $q \to \infty$



$\varepsilon_0 = 0.2$
first three turning points to the right

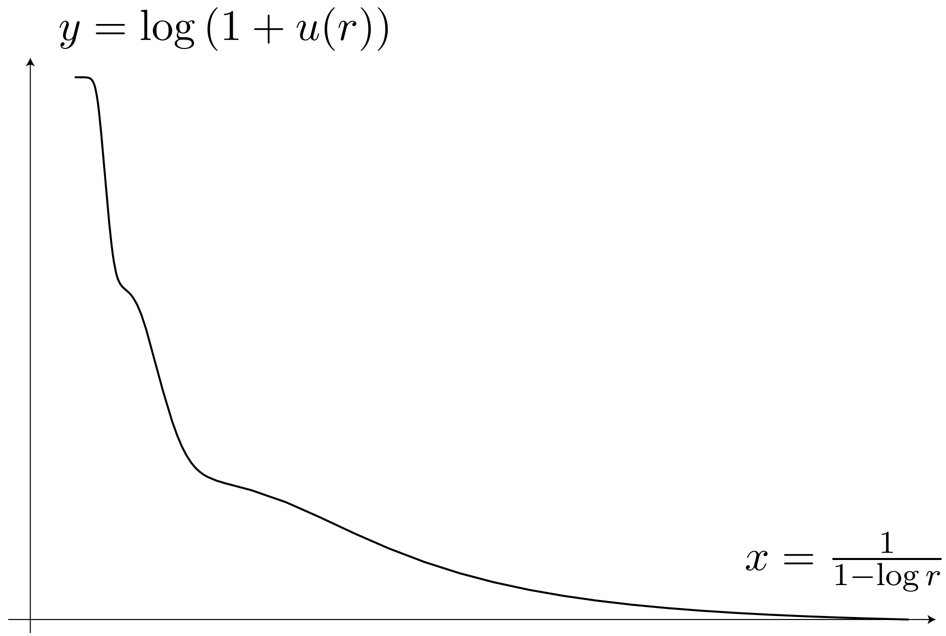
Emden-Fowler transformation:

$$v(x) = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1+\varepsilon}} e^{-x}\, u\left(e^{-\frac{p-1}{2}x}\right)\ , \quad x > 0 \Longleftrightarrow r = e^{-\frac{p-1}{2}x} \in (0,1)$$



$\varepsilon = 0.2$
$v(x)$
first
x
$v(x)$
second
x
$v(x)$
third
x
$\varepsilon = 0.01$
$v(x)$
first
x
$v(x)$
second
x
$v(x)$
third
x

A 3-bubble solution u corresponding to the three bumps solution with $\varepsilon = 0.01$.



2. References, heuristics and main result

$p = \frac{N+2}{N-2}$, $\varepsilon \geq 0$, $N \geq 4$, B is the unit ball in $\mathbb{R}^N$

$$\begin{cases} -\Delta u = u^{p+\varepsilon} + \lambda\, u\,, \quad u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

- <1950: Lane, Emden, Fowler, Chandrasekhar (astrophysics)
- Sobolev, Rellich, Nash, Gagliardo, Nirenberg, Pohozaev
- 1976: Aubin, Talenti
- 1983: Brezis, Nirenberg: case $\varepsilon = 0$ is solvable for $0 < \lambda < \lambda_1 = \lambda_1(-\Delta)$. Uniqueness (Zhang, 1992).
- subcritical case ($0 > \varepsilon \to 0$): Brezis and Peletier, Rey, Han
- supercritical case: Symmetry (Gidas, Ni, Nirenberg, 1979). Budd and Norbury (1987, case $\varepsilon > 0$): formal asymptotics, numerical computations. Merle and Peletier (1991): existence of a unique value $\lambda = \lambda_* > 0$ for which there exists a radial, singular, positive solution. Branch of solutions: Flores (thesis, 2001).

Consider a family of (radial, noincreasing) solutions u_ε of (1) for $\lambda = \lambda_\varepsilon \to 0$. The problem at $\lambda = 0$, $\varepsilon = 0$ has no solution:

$$M_\varepsilon = \gamma^{-1} \max u_\varepsilon = \gamma^{-1} u_\varepsilon(0) \to +\infty$$

for some fixed constant $\gamma > 0$. Let $v_\varepsilon(z) = M_\varepsilon \, u_\varepsilon \left(M_\varepsilon^{(p+\varepsilon-1)/2} \, z \right)$

$$\Delta v_\varepsilon + v_\varepsilon^{p+\varepsilon} + M_\varepsilon^{-(p+\varepsilon-1)} \lambda_\varepsilon v_\varepsilon = 0, \quad |z| < M_\varepsilon^{(p+\varepsilon-1)/2} .$$

Locally over compacts around the origin, $v_\varepsilon \to w$ s.t.

$$\Delta w + w^p = 0$$

with $w(0) = \gamma := (N(N-2))^{\frac{N-2}{4}}$: $w(z) = \gamma \left(\frac{1}{1+|z|^2} \right)^{\frac{N-2}{2}}$.

Guess: $u_\varepsilon(y) = \gamma \left(1 + M_\varepsilon^{\frac{4}{N-2}} \, |y|^2 \right)^{-\frac{N-2}{2}} M_\varepsilon \, (\, 1 + o(1) \,)$ as $\varepsilon \to 0$.

Theorem 1 [k-bubble solution] *Assume $N \geq 5$. Then, given an integer $k \geq 1$, there exists a number $\mu_k > 0$ s.t. if $\mu > \mu_k$ and*

$$\lambda = \mu\, \varepsilon^{\frac{N-4}{N-2}}\,,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$ which depend on k, N and μ and two solutions $u_\varepsilon^\pm$ of Problem (1) of the form

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(1 + \left[\alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j}\right]^{\frac{4}{N-2}} |y|^2\right)^{-(N-2)/2} \alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j}\,(1 + o(1))\,,$$

where $\gamma = (N(N-2))^{\frac{N-2}{4}}$ and $o(1) \to 0$ uniformly on B as $\varepsilon \to 0$.

Bifurcation curve: $\lambda = \varepsilon^{\frac{N-4}{N-2}} f_k\left(c_k^{-1} \varepsilon^{k-\frac{1}{2}}\, m\right)$ for $m \sim \varepsilon^{\frac{1}{2}-k}$.

The numbers $\alpha_j^{\pm}$ can be expressed by the formulae

$$\alpha_j^{\pm} = b_3^{1-j} \frac{(k-j)!}{(k-1)!} s_k^{\pm}(\mu), \quad j = 1, \dots, k \ ,$$

where $b_3 = \frac{(N-2)\sqrt{\pi}\,\Gamma(\frac{N}{2})}{2^{N+2}\,\Gamma(\frac{N+1}{2})}$ and $s_k^{\pm}(\mu)$ are the two solutions of

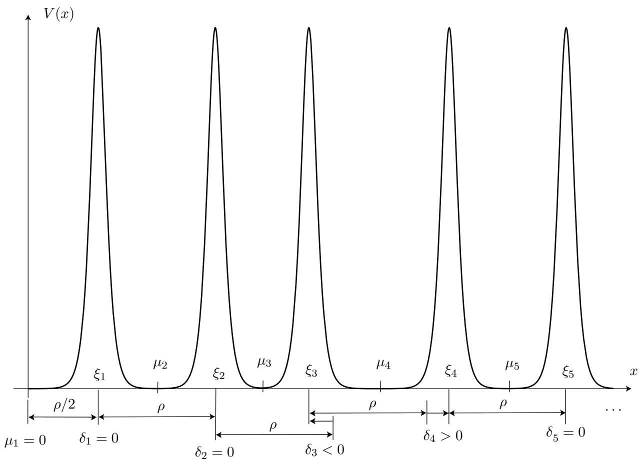
$$\mu = f_k(s) := k b_1 s^{\frac{4}{N-2}} + b_2 s^{-2\frac{N-4}{N-2}}$$

with $b_1 = \left(\frac{N-2}{4}\right)^3 \frac{N-4}{N-1}$ and $b_2 = (N-2)\frac{\Gamma(N-1)}{\Gamma\left(\frac{N-4}{2}\right)\Gamma\left(\frac{N}{2}\right)}$.

Remind that $\mu > \mu_k$ be the minimum value of the function $f_k(s)$:

$$\mu_k = (N-2)\left[\frac{b_1\, k}{N-4}\right]^{\frac{N-4}{N-2}} \left[\frac{b_2}{2}\right]^{\frac{2}{N-2}}$$

Find a k-bump solution after the so-called Emden-Fowler transformation and apply the method for singularly perturbed elliptic equations, introduced by Floer and Weinstein (1986).



Two nondegenerate critical points of Morse indices $k-1$ and k. Open problems: $N = 3$, turning points "to the right", sign changing solutions.

3. The asymptotic expansion

The solution of

$$v'' - v + e^{\varepsilon x}\, v^{p+\varepsilon} + \left(\frac{p-1}{2}\right)^2 \lambda\, e^{-(p-1)x}\, v = 0 \quad \text{on } (0,\infty)$$

with $v(0) = v(\infty) = 0$, $v > 0$ is given as a critical point of

$$E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x} |w|^2\, dx$$

$$I_\varepsilon(w) = \frac{1}{2}\int_0^\infty |w'|^2\, dx + \frac{1}{2}\int_0^\infty |w|^2\, dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon x} |w|^{p+\varepsilon+1}\, dx$$

$U(x) = \left(\frac{4\,N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}\left(1 + e^{-\frac{4}{N-2}\,x}\right)^{-\frac{N-2}{2}}$ is the solution of

$$U'' - U + U^p = 0$$

Ansatz: $v(x) = V(x) + \phi$, $V(x) = \sum_{i=1}^{k} (U(x-\xi_i) - U(\xi_i)\, e^{-x})$.

Further choices:

$$\xi_1 = -\tfrac{1}{2}\log\varepsilon + \log\Lambda_1 \; ,$$
$$\xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_{i+1} \; , \qquad i = 1, \dots, k-1 \; . \tag{2}$$

Lemma 1 *Assume (2). Let* $N \geq 5$ *and* $\lambda = \mu\,\varepsilon^{\frac{N-4}{N-2}}$*. Then*

$$E_\varepsilon(V) \,=\, k\,a_0 + \varepsilon\,\Psi_k(\Lambda) + \frac{k^2}{2}\,a_3\,\varepsilon\,\log\varepsilon + a_5\,\varepsilon + \varepsilon\,\theta_\varepsilon(\Lambda)$$
$$\Psi_k(\Lambda) = a_1\,\Lambda_1^{-2} - k\,a_3\log\Lambda_1 - a_4\,\mu\,\Lambda_1^{-(p-1)} + \sum_{i=2}^{k}\left[(k-i+1)\,a_3\log\Lambda_i - a_2\,\Lambda_i\right] ,$$

and $\lim_{\varepsilon\to 0}\theta_\varepsilon(\Lambda) = 0$ *uniformly and in the* C^1*-sense.*

Constants are explicit:

$$\begin{cases} a_0 = \frac{1}{2}\int_{-\infty}^{\infty}\left(|U'|^2 + U^2\right)\, dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\, dx \\ a_1 = \left(\frac{4N}{N-2}\right)^{(N-2)/2} \\ a_2 = \left(\frac{N}{N-2}\right)^{(N-2)/4} \int_{-\infty}^{\infty} e^{x}\, U^{p}\, dx \\ a_3 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\, dx \\ a_4 = \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \int_{-\infty}^{\infty} e^{-(p-1)\,x}\, U^2\, dx \\ a_5 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1} \log U\, dx + \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1}\, dx \end{cases}$$

$$\Psi_k(\Lambda) = \varphi_k^{\mu}(\Lambda_1) + \sum_{i=2}^{k} \varphi_i(\Lambda_i)$$

$\varphi_k^{\mu}(s) = a_1\, s^{-2} - k\, a_3 \log s \; - \; a_4\, \mu\, s^{-(p-1)}$
$\varphi_i(s) = (k - i + 1)\, a_3 \log s - a_2\, s$
$\varphi_k^{\mu}(s)' = f_k(s) - \mu = 0$ has 2 solutions: $\Psi_k(\Lambda)$ has 2 critical points.

4. The finite dimensional reduction

Let $\mathcal{I}_\varepsilon(\xi) = E_\varepsilon(V+\phi)$ where ϕ is the solution of

$$\mathcal{L}_\varepsilon \phi = h + \sum_{i=1}^{k} c_i\, Z_i \qquad (3)$$

such that $\phi(0) = \phi(\infty) = 0$ and $\int_0^\infty Z_i\, \phi\, dx = 0$,

$$\mathcal{L}_\varepsilon \phi = -\phi'' + \phi - (p+\varepsilon) e^{\varepsilon x} V^{p+\varepsilon-1}\phi - \lambda \left(\tfrac{p-1}{2}\right)^2 e^{-(p-1)x}\phi$$

and $Z_i(x) = U_i'(x) - U_i'(0) e^{-x}$, $i = 1, \ldots, k$. If $h = N_\varepsilon(\phi) + R_\varepsilon$,
$N_\varepsilon(\phi) = e^{\varepsilon x}\left[(V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi\right]$ and
$R_\varepsilon = e^{\varepsilon x}[V^{p+\varepsilon} - V^p] + V^p[e^{\varepsilon x} - 1] + [V^p - \sum_{i=1}^k V_i^p] + \lambda \left(\tfrac{p-1}{2}\right)^2 e^{-(p-1)x}\, V$,

$$\nabla_\xi \mathcal{I}_\varepsilon(\xi) = 0$$

Under technical conditions, one finds a solution to (3) if h is small w.r.t. $\|h\|_* = \sup_{x>0} \left(\sum_{i=1}^k e^{-\sigma|x-\xi_i|}\right)^{-1} |h(x)|$, σ small enough.

Let us consider for a number M large but fixed, the conditions:

$$\left\{ \begin{array}{c} \xi_1 > \frac{1}{2}\log(M\varepsilon)^{-1}\,, \quad \log(M\varepsilon)^{-1} < \min_{1\leq i<k}(\xi_{i+1}-\xi_i)\;, \\ \\ \xi_k < k\log(M\varepsilon)^{-1}\,, \quad \lambda < M\,\varepsilon^{\frac{3-p}{2}}\;. \end{array} \right. \tag{4}$$

For σ chosen small enough:

$$\|N_\varepsilon(\phi)\|_* \leq C\,\|\phi\|_*^{\min\{p,2\}} \quad \text{and} \quad \|R^\varepsilon\|_* \leq C\,\varepsilon^{\frac{3-p}{2}}\;.$$

Lemma 2 *Assume that (4) holds. Then there is a $C > 0$ s.t., for $\varepsilon > 0$ small enough, there exists a unique solution ϕ with*

$$\|\phi\|_* \leq C\varepsilon \quad \textit{and} \quad \|D_\xi \phi\|_* \leq C\varepsilon\;.$$

Lemma 3 *Assume that (4) holds. The following expansion holds*

$$\mathcal{I}_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon)\;,$$

where the term $o(\varepsilon)$ is uniform in the C^1-sense.

5. The case $N = 4$

Theorem 2 *Let $N = 4$. Given a number $k \geq 1$, if*

$$\mu > \mu_k = k \, \frac{\pi}{2^5} \, e^2 \quad \text{and} \quad \lambda \, e^{-2/\lambda} = \mu \, \varepsilon \ ,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$, which depend on k and μ, and two solutions $u_\varepsilon^\pm$:

$$u_\varepsilon^\pm(y) \;=\; \gamma \; \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^2 \, |y|^2} \right) M_j \; (1 + o(1)) \ ,$$

uniformly on B as $\varepsilon \to 0$, with $M_j^\pm = \alpha_j^\pm \varepsilon^{\frac{1}{2} - j} |\log \varepsilon|^{-\frac{1}{2}}$.

The proof is similar to the case $N \geq 5$. For $N = 4$, the order of the height of each bubble is corrected with a logarithmic term.