Entropies in kinetic and nonlinear diffusion equations

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ERWIN SCHRÖDINGER INSTITUTE

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Outline of the talk

- 1. Entropy methods in kinetic equations
- 2. Entropy methods in nonlinear diffusion equations
- 3. New results in entropy methods for nonlinear diffusion equations
- 4. The Keller-Segel system: an illustration
- 5. From kinetic to nonlinear diffusion equations: diffusion limits
- [6. Analogues for systems of quantum mechanics ?]

Entropy method for kinetic equations in a bounded domain

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Charged particles: $f(x, v, t)$, $x \in \omega \subset \mathbb{R}^d$ ($d = 1, 2, 3$), $v \in \mathbb{R}^d$, $t \in \mathbb{R}^+$.
Electric field: $E = E(x, t) = -\nabla \phi_0(x) - \nabla \phi(x, t)$
Phase space $\Omega = \omega \times \mathbb{R}^d$, $\Gamma = \partial\Omega = \partial\omega \times \mathbb{R}^d$
Outward unit normal vector at a point x of $\partial\omega$: $\nu(x)$. For any given $x \in \partial\omega$, we set

$$\Sigma^{\pm}(x) = \{v \in \mathbb{R}^d \ : \ \pm v \cdot \nu(x) > 0\}$$

$$\Gamma^{\pm} = \{(x, v) \in \Gamma \ : \ v \in \Sigma^{\pm}(x)\}$$

On Γ, $d\sigma(x, v) := |\nu(x) \cdot v| \, d_\Gamma(x, v)$ where $d_\Gamma(x, v) = d_{\partial\omega}(x) \, dv$

The model

The Vlasov-Poisson-Boltzmann system:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \phi \; + \nabla_x \phi_0) \cdot \nabla_v f = Q(f) \\ \qquad \text{and} \quad f_{|t=0} = f_0 \,, \qquad f_{|\Gamma^- \times \mathbb{R}^+}(x,v,t) = \gamma(\frac{1}{2}|v|^2 + \phi_0(x)) \\ -\Delta \phi = \rho = \displaystyle\int_{\mathbb{R}^d} f \; dv \,, \qquad (x,t) \in \omega \times \mathbb{R}^+ \\ \qquad \text{and} \quad \phi(x,t) = 0 \,, \qquad (x,t) \in \partial\omega \times \mathbb{R}^+ \end{cases}$$

Assumptions

Property $\mathcal{P}$

The function γ is defined on $(\min_{x\in\omega}\phi_0(x), +\infty)$, bounded, smooth, strictly decreasing with values in $\mathbb{R}_*^+$, and rapidly decreasing at infinity, so that

$$\sup_{x\in\omega}\int_0^{+\infty} s^{d/2}\gamma(s+\phi_0(x))\,ds < +\infty$$

The collision operator Q is assumed to preserve the mass $\int_{\mathbb{R}^d} Q(g)\,dv = 0$, and satisfies the following H-theorem

$$D[g] = -\int_{\mathbb{R}^d} Q(g)\left[\frac{1}{2}|v|^2 - \gamma^{-1}(g)\right]\,dv \geq 0$$

and $D[g] = 0 \iff Q(g) = 0$

Example 1

The Vlasov-Poisson-Fokker-Planck system.

$$Q_{FP,\alpha}(f) = \mathrm{div}_v(vf(1-\alpha f) + \theta\,\nabla_v f)$$

for some $\alpha \geq 0$

$$\int_{\mathbb{R}^d} Q_{FP,\alpha}(f) \log\left(\frac{f}{(1-\alpha f)M_\theta}\right) dv$$

$$= -\int_{\mathbb{R}^d} \theta f(1-\alpha f) \left|\nabla_v \log\left(\frac{f}{(1-\alpha f)M_\theta}\right)\right|^2 dv$$

where $M_\theta = (2\pi\theta)^{-d/2} e^{-|v|^2/(2\theta)}$

Stationary states: $f(v) = \left(\alpha + e^{(|v|^2/2-\mu)/\theta}\right)^{-1}$

Example 2: BGK

BGK approximation of the Boltzmann operator.

$$Q_\alpha(f) = \int_{\mathbb{R}^d} \sigma(v,v') \Big[M_\theta(v) f(v') (1-\alpha f(v)) - M_\theta(v') f(v) (1-\alpha f(v')) \Big]$$

Stationary states: $f(v) = \left(\alpha + e^{(|v|^2/2-\mu)/\theta} \right)^{-1}$

Example 3: Linear elastic collisions

$$Q_E(f) = \int_{\mathbb{R}^d} \chi(v,v')\,(f(v') - f(v))\,\delta(|v'|^2 - |v|^2)\,dv'$$

where χ is a symmetric positive cross-section. Assume that

$$\lambda(v) = \int_{\mathbb{R}^d} \chi(v,v')\,\delta(|v|^2 - |v'|^2)\,dv'$$

is in L^∞ . Then Q_E is bounded on $L^1 \cap L^\infty(\mathbb{R}^d)$. Moreover, for any measurable function ψ and for any increasing function H on $\mathbb{R}$, we have

$$\int_{\mathbb{R}^d} Q_E(f)\cdot\psi(|v|^2)\,dv = 0 \quad \text{and} \quad \mathcal{H}(f) = \int_{\mathbb{R}^d} Q_E(f)\cdot H(f)\,dv \le 0$$

Stationary solutions: $f(v) = \psi(|v|^2)$

Relative entropy

Relative entropy of two functions g, h:

$$\Sigma_\gamma[g|h] = \int_\Omega \big(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h)\big)\ dxdv + \frac{1}{2}\int_\omega |\nabla U[g-h]|^2$$

where β_γ is the real function defined by

$$\beta_\gamma(g) = -\int_0^g \gamma^{-1}(z)\, dz$$

γ is strictly decreasing $\Longrightarrow$ β_γ is strictly convex.

Irreversibility

Theorem 1 *Assume that* $f_0 \in L^1 \cap L^\infty$ *is a nonnegative function such that* $\Sigma_\gamma[f_0|M] < +\infty$*. Then the relative entropy* $\Sigma_\gamma[f(t)|M]$ *where* M *is the (unique) stationary solution, satisfies*

$$\frac{d}{dt}\Sigma_\gamma[f(t)|M] = -\Sigma_\gamma^+[f(t)|M] - \int_\omega D[f](x,t)\,dx$$

where Σ_γ^+ *is the boundary relative entropy flux given by*

$$\Sigma_\gamma^+[g|h] = \int_{\Gamma^+} \big(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta_\gamma'(h)\big)\ d\sigma$$

Solutions to the limit problem

Convergence to a large time limit can be proved assuming the compatibility of the incoming distribution function

$$f_{|\Gamma^- \times \mathbb{R}^+}(x,v,t) = \gamma(\frac{1}{2}|v|^2 + \phi_0(x))$$

with the collision kernel. Otherwise: very difficult question.

Corollary 2 *Let $d = 1$, $Q \equiv 0$. Assume that γ satisfies Property ($\mathcal{P}$) and consider a solution (f,ϕ) achieved by passing to the limit $t \to \infty$.*

If ϕ_0 is analytic in x with C^∞ (in time) coefficients and if ϕ_0 is analytic with $-\frac{d^2\phi_0}{dx^2} \geq 0$ on ω, then (f,ϕ) is the unique stationary solution.

Entropy methods for linear and nonlinear parabolic equations

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I — Entropy methods for linear diffusions
The logarithmic Sobolev inequality
Convex Sobolev inequalities

- *logarithmic Sobolev inequality:* [Gross], [Weissler], [Coulhon],...
- *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...
- *linear diffusions (PDEs):* [Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Otto, Kinderlehrer, Jordan], [Arnold, J.D.]

I-A. Intermediate asymptotics: heat equation

Heat equation: $$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\,dx = 1 \end{cases} \qquad (1)$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$.

Optimal rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1+2\,t}\,, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log\left(\frac{v}{v_\infty}\right)\, dx = \int_{\mathbf{R}^n} \left(v \log v + \frac{1}{2}|x|^2 v\right)\, dx + Const$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\boxed{\int_{\mathbf{R}^n} v \log v\, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \le \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v}\, dx}$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2}\, e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \le \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \le e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \log\left(\frac{u_0}{v_\infty}\right)\, dx$$

- Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v\,dx = \int_{\mathbf{R}^n} \bar{v}\,dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log\left(\frac{v}{\bar{v}}\right)\,dx \geq \frac{1}{4\,M}\|v-\bar{v}\|^2_{L^1(\mathbf{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbf{R}^n)} \leq 4M\boldsymbol{\Sigma}[u_0]\,e^{-2\tau}$$

$\tau(t) = \log\sqrt{1+2t}$

$$\|u(\cdot,t) - u_\infty(\cdot,t)\|^2_{L^1(\mathbf{R}^n)} \leq \frac{4}{1+2t}\boldsymbol{\Sigma}[u_0]$$

$$u_\infty(x,t) = \frac{1}{R^n(t)}\,v_\infty\left(\frac{x}{R(t)},\tau(t)\right)$$

Proof of the Csiszár-Kullback inequality: Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension [Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w=\sqrt{\frac{v}{M\,v_\infty}}$ and optimize on λ for $w_\lambda(x)=\lambda^{n/2}w(\lambda\,x)$

$$\int_{\mathbb{R}^n}|\nabla w_\lambda|^2\,dx-\int_{\mathbb{R}^n}w_\lambda^2\log w_\lambda^2\,dx$$
$$=\lambda^2\int_{\mathbb{R}^n}|\nabla w|^2\,dx-\int_{\mathbb{R}^n}w^2\log w^2\,dx-n\log\lambda\int_{\mathbb{R}^n}w^2\,dx$$

ENTROPY-ENTROPY PRODUCTION METHOD

A method to prove the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)]-2\Sigma[v(\cdot,\tau)]\right)=-C\sum_{i,j=1}^{n}\int_{\mathbb{R}^n}\left|w_{ij}+a\,\frac{w_iw_j}{w}+b\,w\,\delta_{ij}\right|^2dx<0$$

for some $C>0$, a, $b\in\mathbb{R}$. Here $w=\sqrt{v}$.

$I[v(\cdot,\tau)]-2\Sigma[v(\cdot,\tau)]\searrow I[v_\infty]-2\Sigma[v_\infty]=0$

$\Longrightarrow\quad\forall\,u_0\,,\quad I[u_0]-2\Sigma[u_0]\geq I[v(\cdot,\tau)]-2\Sigma[v(\cdot,\tau)]\geq 0$ for $\tau>0$

I-B. Applications...

- Homogeneous and non-homogenous collisional kinetic equations [L. Desvillettes, C. Villani, G. Toscani,...]
- Drift-diffusion-Poisson equations for semi-conductors [A. Arnold, P. Markowich, G. Toscani], [P. Biler, J.D., P. Markowich],...
- The two-dimensional Keller-Segel model [A. Blanchet, B. Perthame, J.D.], [P. Biler, P. Laurençot, G. Karch, T. Nadzieja]
- Streater's models [P. Biler, J.D., M. Esteban, G. Karch]
- Heat equation with a source term [[J.D., G. Karch]
- The flashing ratchet [J.D., D. Kinderlehrer, M. Kowalczyk]
- Models for traffic flow [J.D., Reinhard Illner]
- Navier-Stokes in dimension 2 [T. Gallay, Wayne], [C. Villani], [J.D., A. Munnier]

... and questions under investigation

- Hierarchies of inequalities
- Vlasov-Fokker-Planck [Héraut, Nier, Helffer, Villani]
- Derivation of entropy - entropy-production inequalities in non-standard frameworks:

- singular potentials: [JD, Nazaret, Otto]
- vanishing diffusion coefficients: [Bartier, JD, Illner, Kowalczyk]

- Homogeneization and long time behaviour: [Allaire, Blanchet, Kinderlehrer, Kowalczyk]
- Relaxation and diffusion properties on intermediate time scales, corrections to convex Sobolev inequalities
- Connections with differential geometry [Sturm, Villani]
- etc

II — Porous media / fast diffusion equation and generalizations

[coll. Manuel del Pino (Universidad de Chile)] $\Longrightarrow$ Relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Other approaches:
1) "entropy – entropy-production method"
2) mass transportation techniques
3) hypercontractivity for appropriate semi-groups

• *nonlinear diffusions:* [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vazquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub]

II-A. Porous media / Fast diffusion equation

[Del Pino, JD]

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in } \mathbb{R}^n \\ u_{|t=0} &= u_0 \geq 0 \\ u_0(1+|x|^2) \in L^1\,, &\quad u_0^m \in L^1 \end{aligned} \tag{3}$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0\,dx = 1$, the self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-n/(2-n(1-m))})$ as $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-n/(2-n(1-m))})$$

Rescaling: Take $u(t,x) = R^{-n}(t)\, v\left(\tau(t), x/R(t)\right)$ where

$$\dot{R} = R^{n(1-m)-1}\ , \quad R(0) = 1\ , \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\ , \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi\left(\tfrac{v^m}{v_\infty^m}\right)\, v_\infty^{m-1}\, dx \quad \textit{with } \psi(t) = \tfrac{m t^{1/m} - 1}{1-m} + 1$$

Theorem 1 $m \in [\frac{n-1}{n}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\boxed{I[v] \geq 2\,\Sigma[v]}$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v\right)\, dx - \Sigma_0 \le \frac{1}{2}\int v \left|\frac{\nabla v^m}{v} + x\right|^2\, dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - n\right) \int |w|^{1+p} dx + K \ge 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$

$m = \frac{n-1}{n}$: Sobolev, $m \to 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

$$1 < p \le \frac{n}{n-2} \text{ for } n \ge 3$$

$$\|w\|_{2p} \le A \, \|\nabla w\|_2^{\theta} \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi n}\right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})}\right)^{\frac{\theta}{n}}$$
$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \; y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

$1 < p = \frac{1}{2m-1} \le \frac{n}{n-2} \iff$ Fast diffusion case: $\frac{n-1}{n} \le m < 1$

$0 < p < 1 \iff$ Porous medium case: $m > 1$

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities

$\Rightarrow$ Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{n-1}{n} < m < 1$ if $n \geq 3$

$$\limsup_{t\to+\infty} t^{\frac{1-n(1-m)}{2-n(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+n(m-1)}{2+n(m-1)}} \|\,[u - u_\infty]\; u_\infty^{m-1}\,\|_{L^1} < +\infty$$

The optimal L^p-Euclidean logarithmic Sobolev inequality
(an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 2 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \tfrac{p}{n} \left(\tfrac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

II-B. Consequences for $u_t = \Delta_p u^{1/(p-1)}$

[Del Pino, JD, Gentil]

- Existence
- Uniqueness
- Hypercontractivity, Ultracontractivity
- Large deviations

Existence

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \tag{4}$$

$\Delta_p u^m = \mathrm{div}\left(|\nabla u^m|^{p-2} \nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.

Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 3 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}_t^+, L^1)$ of (4) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}_t^+, W^{1,p}_{\mathrm{loc}})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]
[Bernis, 88], [Ishige, 96]
Crucial remark: [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87]
The functional $u \mapsto \int |\nabla u^\alpha|^p \, dx$ is convex for any $p > 1$, $\alpha \in [\frac{1}{p}, 1]$.

Uniqueness

Consider two solutions u_1 and u_2 of (4).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1+\log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)}\int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \, .$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1}\,\|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \leq \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \leq \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

Hyper- and Ultra-contractivity

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 4 *Let α, $\beta \in [1,+\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in the existence Theorem, if moreover $f \in L^\alpha(\mathbb{R}^n)$, any solution with initial data f satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t>0$$

with $A(n,p,\alpha,\beta) = (\mathcal{C}_1\,(\beta-\alpha))^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}$,

$$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}.$$

Case $p=2$: $\mathcal{L}_2 = \frac{2}{\pi\,n\,e}$, [Gross 75]

As a special case of Theorem 4, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 5 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot, t)\|_\infty \leq \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Proof. Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq}\int u^q\,dx = \int u^q \log u\;dx\;. \tag{5}$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1,+\infty)$,

$$\frac{d}{dt}\int u^q\,dx = -\frac{q(q-1)}{(p-1)^{p-1}}\int u^{q-p}|\nabla u|^p\,dx\;. \tag{6}$$

Assume that q depends on t and let $F(t) = \|u(\cdot,t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (5) and (6) gives

$$\frac{F'}{F} = \frac{q'}{q^2}\left[\int \frac{u^q}{F^q}\log\left(\frac{u^q}{F^q}\right)dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}}\,\frac{1}{F^q}\int u^{q-p}|\nabla u|^p\,dx\right]\;.$$

Since $\int u^{q-p}|\nabla u|^p\,dx = (\frac{p}{q})^p \int |\nabla u^{q/p}|^p\,dx$, Corollary **??** applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } \; A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)\,ds$$

$$\text{and } \; \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}}\;.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 \Longleftrightarrow q(t) = \frac{1}{a\,t+b}\;.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$. $\square$

Conclusion

The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

The Prékopa-Leindler inequality implies (iii).

Entropy-Energy inequalities and improved convergence rates for nonlinear parabolic equations

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Higher order diffusion equations

The one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

The thin film equation

$$u_t = -(u^m \, u_{xxx})_x \,, \quad x \in S^1 \,, \quad t > 0$$

The Derrida-Lebowitz-Speer-Spohn (DLSS) equation

$$u_t = -(u \, (\log u)_{xx})_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

... with initial condition $u(\cdot, 0) = u_0 \geq 0$ in $S^1 \equiv [0, 1)$

Entropies and energies

Averages:

$$\mu_p[v] := \left(\int_{S^1} v^{1/p} \, dx \right)^p \quad \text{and} \quad \bar{v} := \int_{S^1} v \, dx$$

Entropies: $p \in (0, +\infty)$, $q \in \mathbb{R}$, $v \in H^1_+(S^1)$, $v \not\equiv 0 \, a.e.$

$$\Sigma_{p,q}[v] := \frac{1}{p\,q\,(p\,q-1)} \left[\int_{S^1} v^q \, dx - (\mu_p[v])^q \right] \quad \text{if } p\,q \neq 1 \text{ and } q \neq 0$$

$$\Sigma_{1/q,q}[v] := \int_{S^1} v^q \log\left(\frac{v^q}{\int_{S^1} v^q \, dx} \right) dx \quad \text{if } p\,q = 1 \text{ and } q \neq 0$$

$$\Sigma_{p,0}[v] := -\frac{1}{p} \int_{S^1} \log\left(\frac{v}{\mu_p[v]} \right) dx \quad \text{if } q = 0$$

Convexity

$\Sigma_{p,q}[v]$ is non-negative by convexity of

$$u \mapsto \frac{u^{p\,q} - 1 - p\,q\,(u-1)}{p\,q\,(p\,q-1)} =: \sigma_{p,q}(u)$$

By Jensen's inequality,

$$\begin{aligned} \Sigma_{p,q}[v] &= \mu_p[v]^q \int_{S^1} \sigma_{p,q}\left(\frac{v^{1/p}}{(\mu_p[v])^{1/p}}\right) dx \\ &\geq \mu_p[v]^q \sigma_{p,q}\left(\int_{S^1} \frac{v^{1/p}}{(\mu_p[v])^{1/p}}\, dx\right) = \mu_p[v]^q \sigma_{p,q}(1) = 0 \end{aligned}$$

Limit cases

$p\,q = 1$:

$$\lim_{p \to 1/q} \Sigma_{p,q}[v] = \Sigma_{1/q,q}[v] \quad \text{for } q > 0$$

$q = 0$:

$$\lim_{q \to 0} \Sigma_{p,q}[v] = \Sigma_{p,0}[v] \quad \text{for } p > 0$$

$p = q = 0$:

$$\Sigma_{0,0}[v] = -\int_{S^1} \log\left(\frac{v}{\|v\|_\infty}\right) dx$$

Some references (2005 or to appear):
[M. J. Cáceres, J. A. Carrillo, and G. Toscani]
[M. Gualdani, A. Jüngel, and G. Toscani]
[A. Jüngel and D. Matthes]
[R. Laugesen]

Global functional inequalities

Theorem 1 *For all* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*, there exists a positive constant* $\kappa_{p,q}$ *such that, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}}\, J_1[v] := \frac{1}{\kappa_{p,q}} \int_{S^1} |v'|^2 \, dx$$

Corollary 1 *Let* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*. Then, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{4\pi^2\, \kappa_{p,q}}\, J_2[v] := \frac{1}{4\pi^2\, \kappa_{p,q}} \int_{S^1} |v''|^2 \, dx$$

A minimizing sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(S^1)$

$$v_n \rightharpoonup v \quad \text{in } H^1(S^1) \quad \text{and} \quad \Sigma_{p,q}[v_n] \to \Sigma_{p,q}[v] \quad \text{as} \quad n \to \infty$$

If $\Sigma_{p,q}[v] = 0$, $\lim_{n\to\infty} J_1[v_n] = 0$. Let

$$\varepsilon_n := J_1[v_n]\;, \quad w_n := \frac{v_n - 1}{\sqrt{\varepsilon_n}}$$

Taylor expansion

$$\left|\,(1+\sqrt{\varepsilon}\,x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p}\,x\,\right| \le \frac{1}{p}\,r(\varepsilon_0,p)\,\varepsilon \quad \forall\,(x,\varepsilon) \in \left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$$

$$\varepsilon_n := J_1[v_n] \ , \qquad \Sigma_{p,q}[v_n] \leq c(\varepsilon_0, p, q)\, \varepsilon_n$$

Hence, since $q < 2$,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} = \frac{\varepsilon_n \, J_1[w_n]}{\Sigma_{p,q}[v_n]^{2/q}} \geq [c(\varepsilon_0, p, q)]^{-2/q} \, \varepsilon_n^{1-2/q} \to \infty$$

gives a contradiction

Asymptotic functional inequalities

The regime of small entropies:

$$\mathcal{X}_\varepsilon^{p,q} := \left\{ v \in H^1_+(S^1) \; : \; \Sigma_{p,q}[v] \leq \varepsilon \text{ and } \mu_p[v] = 1 \right\}$$

Theorem 2 *For any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\, J_1[v] \quad \forall\, v \in \mathcal{X}_\varepsilon^{p,q}$$

Without the condition $\mu_p[v] = 1$:

$$\Sigma_{p,q}[v] \leq \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\, (\mu_p[v])^{q-2}\, J_1[v]$$

If $J_1[v] \leq 8\,p^2\,\pi^2\,\varepsilon$, define $w := (v-1)/(\kappa_p^\infty\,\sqrt{\varepsilon})$: $J_1[w] \leq 1$.

$$\begin{aligned}
\Sigma_{p,q}[v] &= \frac{1}{pq(pq-1)}\left[\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^q dx - \left(\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^{1/p} dx\right)^{pq}\right] \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\left[\int_{S^1} w^2\,dx - \left(\int_{S^1} w\,dx\right)^2\right] + O(\varepsilon^{3/2}) \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\int_{S^1}(w-\bar{w})^2\,dx + O(\varepsilon^{3/2}) \\
&\leq \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\,\frac{J_1[w]}{(2\pi)^2} + O(\varepsilon^{3/2}) = \frac{J_1[v]}{8\,p^2\,\pi^2} + O(\varepsilon^{3/2})
\end{aligned}$$

using Poincaré's inequality

1^{st} application: Porous media

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1,\ t > 0$$

A one parameter family of entropies :

$$\Sigma_k[u] := \begin{cases} \dfrac{1}{k\,(k+1)} \displaystyle\int_{S^1} \left(u^{k+1} - \bar{u}^{k+1}\right)\, dx & \text{if} \quad k \in \mathbb{R} \setminus \{-1, 0\} \\ \displaystyle\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = 0 \\ -\displaystyle\int_{S^1} \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = -1 \end{cases}$$

With $v := u^p$, $p := \frac{m+k}{2}$, $q := \frac{k+1}{p} = 2\,\frac{k+1}{m+k}$, $\Sigma_k[u] = \Sigma_{p,q}[v]$

Lemma 1 *Let $k \in \mathbb{R}$. If u is a smooth positive solution*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} \left|(u^{(k+m)/2})_x\right|^2 dx = 0$$

with $\lambda := 4\,m/(m+k)^2$ whenever $k+m \neq 0$, and

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} |(\log u)_x|^2 \, dx = 0$$

with $\lambda := m$ for $k+m=0$.

Decay rates

Proposition 1 *Let* $m \in (0, +\infty)$*,* $k \in \mathbb{R} \setminus \{-m\}$*,* $q = 2\,(k+1)/(m+k)$*,* $p = (m+k)/2$ *and* u *be a smooth positive solution*

i) *Short-time Algebraic Decay:* *If* $m > 1$ *and* $k > -1$*, then*

$$\Sigma_k[u(\cdot,t)] \le \left[\Sigma_k[u_0]^{-(2-q)/q} + \frac{2-q}{q}\,\lambda\,\kappa_{p,q}\,t\right]^{-q/(2-q)}$$

ii) *Asymptotically Exponential Decay:* *If* $m > 0$ *and* $m + k > 0$*, there exists* $C > 0$ *and* $t_1 > 0$ *such that for* $t \ge t_1$*,*

$$\Sigma_k[u(\cdot,t)] \le \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{8\,p^2\,\pi^2\,\lambda\,\bar{u}^{p(2-q)}\,(t-t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

2^{nd} Application: fourth order eqs.

$$u_t = -\Big(u^m \;(u_{xxx} + a\,u^{-1}\,u_x\,u_{xx} + b\,u^{-2}\,u_x^3)\Big)_x, \qquad x \in S^1,\; t > 0\,,$$

Example 1. The thin film equation: $a = b = 0$

$$u_t = -(u^m\,u_{xxx})_x,$$

Example 2. The DLSS equation: $m = 0$, $a = -2$, and $b = 1$

$$u_t = -\Big(u\,(\log u)_{xx}\Big)_{xx},$$

$$L_\pm := \tfrac{1}{4}(3\,a + 5) \pm \tfrac{3}{4}\sqrt{(a-1)^2 - 8\,b}$$

$$A := (k + m + 1)^2 - 9\,(k + m - 1)^2 + 12\,a\,(k + m - 2) - 36\,b$$

Theorem 3 *Assume* $(a-1)^2 \geq 8\,b$

i) *Entropy production: If* $L_- \leq k+m \leq L_+$

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] \leq 0 \quad \forall\, t>0$$

ii) *Entropy production: If* $k+m+1 \neq 0$ *and* $L_- < k+m < L_+$,

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} \left|(u^{(k+m+1)/2})_{xx}\right|^2 dx \leq 0 \quad \forall\, t>0$$

If $k+m+1=0$ *and* $a+b+2-\mu \leq 0$ *for some* $0<\mu<1$, *then*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} |(\log u)_{xx}|^2\, dx \leq 0 \quad \forall\, t>0$$

Decay rates

Theorem 4 *Let $k, m \in \mathbb{R}$ be such that $L_- \leq k + m \leq L_+$*

i) Short-time Algebraic Decay: If $k > -1$ and $m > 0$, then

$$\Sigma_k[u(\cdot,t)] \leq \left[\Sigma_k[u_0]^{-(2-q)/q} + 4\pi^2\, \mu\, \kappa_{p,q} \left(\frac{2}{q} - 1\right)\, t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m + k + 1 > 0$, then there exists $C > 0$ and $t_1 > 0$ such that

$$\Sigma_k[u(\cdot,t)] \leq \Sigma_k[u(\cdot,t_1)]\, \exp\left(-\frac{32\, p^2\, \pi^4\, \mu\, \bar{u}^{p(2-q)}\, (t - t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

The Keller-Segel model

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The Keller-Segel model

The Keller-Segel(-Patlak) system for chemotaxis describes the collective motion of cells (bacteria or amoebae) [Othmer-Stevens, Horstman].
The complete Keller-Segel model is a system of two parabolic equations. Simplified two-dimensional version :

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \chi \nabla \cdot (n \nabla c) & x \in \mathbb{R}^2 \, , \; t > 0 \\ -\Delta c = n & x \in \mathbb{R}^2 \, , \; t > 0 \\ n(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases} \tag{1}$$

$n(x,t)$: the cell density
$c(x,t)$: concentration of chemo-attractant
$\chi > 0$: *sensitivity* of the bacteria to the chemo-attractant

I. Main results and *a priori* estimates

Dimension 2 is critical

The total mass of the system

$$M := \int_{\mathbb{R}^2} n_0 \, dx$$

is conserved

There are related models in gravitation which are defined in $\mathbb{R}^3$

The L^1-norm is critical in the sense that there exists a critical mass above which all solution blow-up in finite time and below which they globally exist. The critical space is $L^{d/2}(\mathbb{R}^d)$ for $d \geq 2$, see [Corrias-Perthame-Zaag]. In dimension $d = 2$, the Green kernel associated to the Poisson equation is a logarithm, namely

$$c = -\frac{1}{2\pi} \log|\cdot| * n$$

First main result

Theorem 1. *Assume that* $n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ *and* $n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$. *If* $M < 8\pi/\chi$, *then the Keller-Segel system (1) has a global weak non-negative solution* n *with initial data* n_0 *such that*

$$(1+|x|^2+|\log n|)\,n \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2)) \quad \int_0^t \int_{\mathbb{R}^2} n\,|\nabla \log n - \chi \nabla c|^2\,dx\,dt < \infty$$

and $$\int_{\mathbb{R}^2} |x|^2\,n(x,t)\,dx = \int_{\mathbb{R}^2} |x|^2\,n_0(x)\,dx + 4M\left(1 - \frac{\chi M}{8\pi}\right) t$$

for any $t > 0$. *Moreover* $n \in L^\infty_{\mathrm{loc}}((\varepsilon,\infty), L^p(\mathbb{R}^2))$ *for any* $p \in (1,\infty)$ *and any* $\varepsilon > 0$, *and the following inequality holds for any* $t > 0$ *:*

$$F[n(\cdot,t)] + \int_0^t \int_{\mathbb{R}^2} n\,|\nabla(\log n) - \chi \nabla c|^2\,dx\,ds \le F[n_0]$$

Here $F[n] := \int_{\mathbb{R}^2} n \log n\,dx - \frac{\chi}{2}\int_{\mathbb{R}^2} n\,c\,dx$

Notion of solution

The equation holds in the distributions sense. Indeed, writing

$$\Delta n - \chi \nabla\cdot(n\nabla c) = \nabla\cdot[n(\nabla \log n - \chi\nabla c)]$$

we can see that the flux is well defined in $L^1(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$ since

$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla \log n - \chi\nabla c|\,dx\,dt$$

$$\leq \left(\iint_{[0,T]\times\mathbb{R}^2} n\;dx\,dt\right)^{1/2} \left(\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla \log n - \chi\nabla c|^2\,dx\,dt\right)^{1/2} < \infty$$

Second main result : Large time behavior

Use asymptotically self-similar profiles given in the rescaled variables by the equation

$$u_\infty = M \frac{e^{\chi\, v_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{\chi\, v_\infty - |x|^2/2}\, dx} = -\Delta v_\infty \quad \text{with} \quad v_\infty = -\frac{1}{2\pi} \log|\cdot| * u_\infty \quad (2)$$

In the original variables :

$$\begin{aligned} n_\infty(x,t) &:= \frac{1}{1+2t}\, u_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right) \\ c_\infty(x,t) &:= v_\infty\left(\log(\sqrt{1+2t}), x/\sqrt{1+2t}\right) \end{aligned}$$

Theorem 2. *Under the same assumptions as in Theorem 1, there exists a stationary solution* (u_∞, v_∞) *in the self-similar variables such that*

$$\lim_{t\to\infty} \|n(\cdot,t) - n_\infty(\cdot,t)\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,t) - \nabla c_\infty(\cdot,t)\|_{L^2(\mathbb{R}^2)} = 0$$

Assumptions

We assume that the initial data satisfies the following asssumptions :

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$$

$$n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)$$

The total mass is conserved

$$M := \int_{\mathbb{R}^2} n_0(x)\ dx = \int_{\mathbb{R}^2} n(x,t)\ dx$$

Goal : give a complete existence theory [J.D.-Perthame], [Blanchet-J.D.-Perthame] in the subcritical case, *i.e.* in the case

$$M < 8\pi/\chi$$

Alternatives

There are only two cases :

1. Solutions to (1) blow-up in finite time when $M > 8\pi/\chi$
2. There exists a global in time solution of (1) when $M < 8\pi/\chi$

The case $M = 8\pi/\chi$ is delicate and for radial solutions, some results have been obtained recently [Biler-Karch-Laurençot-Nadzieja]

Our existence theory completes the partial picture established in [Jäger-Luckhaus].

Convention

The solution of the Poisson equation $-\Delta c = n$ is given up to an harmonic function. From the beginning, we have in mind that the concentration of the chemo-attractant is defined by

$$c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\, n(y,t)\, dy$$

$$\nabla c(x,t) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, n(y,t)\, dy$$

Blow-up for super-critical masses

Case $M > 8\pi/\chi$ (Case 1) : use moments estimates

Lemma 3. *Consider a non-negative distributional solution to (1) on an interval* $[0,T]$ *that satisfies the previous assumptions,* $\int_{\mathbb{R}^2} |x|^2\, n_0(x)\, dx < \infty$ *and such that* $(x,t) \mapsto \int_{\mathbb{R}^2} \frac{1+|x|}{|x-y|}\, n(y,t)\, dy \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$ *and* $(x,t) \mapsto (1+|x|)\nabla c(x,t) \in L^\infty\big((0,T)\times\mathbb{R}^2\big)$*. Then it also satisfies*

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx = 4M\left(1 - \frac{\chi\, M}{8\pi}\right)$$

Formal proof.

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\, dx &= \int_{\mathbb{R}^2} |x|^2\, \Delta n(x,t)\, dx \\ &\quad + \frac{\chi}{2\,\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} \frac{x-y}{|x-y|^2}\; n(x,t)\, n(y,t)\, dx\, dy\end{aligned}$$

Justification

Consider a smooth function φ_ε with compact support such that $\lim_{\varepsilon \to 0} \varphi_\varepsilon(|x|) = |x|^2$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} \varphi_\varepsilon \, n \, dx &= \int_{\mathbb{R}^2} \Delta \varphi_\varepsilon \, n \, dx \\ &\quad - \frac{\chi}{4\pi} \int_{\mathbb{R}^2} \underbrace{\frac{(\nabla \varphi_\varepsilon(x) - \nabla \varphi_\varepsilon(y)) \cdot (x-y)}{|x-y|^2}}_{\to 1} n(x,t) \, n(y,t) \, dx \, dy \end{aligned}$$

Since $\frac{d}{dt} \int_{\mathbb{R}^2} \varphi_\varepsilon \, n \, dx \leq C_\varepsilon \int_{\mathbb{R}^2} n_0 \, dx$ where C_ε is some positive constant, as $\varepsilon \to 0$, $\int_{\mathbb{R}^2} \varphi_\varepsilon \, n \, dx \leq c_1 + c_2 \, t$

$$\int_{\mathbb{R}^2} |x|^2 \, n(x,t) \, dx < \infty \quad \forall \, t \in (0,T)$$

$\square$

Weaker notion of solutions

We shall say that n is a solution to (1) if for all test functions $\psi \in \mathcal{D}(\mathbb{R}^2)$

$$\frac{d}{dt}\int_{\mathbb{R}^2} \psi(x)\, n(x,t)\, dx = \int_{\mathbb{R}^2} \Delta\psi(x)\, n(x,t)\, dx$$
$$-\frac{\chi}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} [\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2}\, n(x,t)\, n(y,t)\, dx\, dy$$

Compared to standard distribution solutions, this is an improved concept that can handle measures solutions because the term

$$[\nabla\psi(x) - \nabla\psi(y)] \cdot \frac{x-y}{|x-y|^2}$$

is continuous

However, this notion of solutions does not cover the case of measure valued $n(\cdot, t)$

Finite time blow-up

Corollary 4. *Consider a non-negative distributional solution* $n \in L^\infty(0, T^*; L^1(\mathbb{R}^2))$ *to (1) and assume that* $[0, T^*)$, $T^* \leq \infty$, *is the maximal interval of existence. Let*

$$I_0 := \int_{\mathbb{R}^2} |x|^2 \, n_0(x) \, dx < \infty \quad \textit{and} \quad \int_{\mathbb{R}^2} \frac{1 + |x|}{|x - y|} \, n(y,t) \, dy \in L^\infty\big((0,T) \times \mathbb{R}^2\big)$$

If $\chi \, M > 8\pi$, *then*

$$T^* \leq \frac{2\pi \, I_0}{M(\chi \, M - 8\pi)}$$

If $\chi \, M > 8\pi$ and $I_0 = \infty$: blow-up in finite time ?
Blow-up statements in bounded domains are available
Radial case : there exists a $L^1(\mathbb{R}^2 \times \mathbb{R}^+)$ radial function $\tilde{n}$ such that

$$n(x,t) \to \frac{8\pi}{\chi} \, \delta + \tilde{n}(x,t) \quad \text{as } t \nearrow T^*$$

Comments

1. $\chi M = 8\pi$ [Biler-Karch-Laurençot-Nadzieja] : blow-up only for $T^* = \infty$

2. If the problem is set in dimension $d \geq 3$, the critical norm is $L^p(\mathbb{R}^d)$ with $p = d/2$ [Corrias-Perthame-Zaag]

3. In dimension $d = 2$, the value of the mass M is therefore natural to discriminate between super- and sub-critical regimes. However, the limit of the L^p-norm is rather $\int_{\mathbb{R}^2} n \log n \, dx$ than $\int_{\mathbb{R}^2} n \, dx$, which is preserved by the evolution. This explains why it is natural to introduce the entropy, or better, as we shall see below, the *free energy*

The proof of Jäger and Luckhaus

[Corrias-Perthame-Zaag] Compute $\frac{d}{dt}\int_{\mathbb{R}^2} n \log n \, dx$. Using an integration by parts and the equation for c, we obtain :

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^2} n \log n \, dx &= -4\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2 \, dx + \chi \int_{\mathbb{R}^2} \nabla n \cdot \nabla c \, dx \\ &= -4\int_{\mathbb{R}^2} |\nabla\sqrt{n}|^2 \, dx + \chi \int_{\mathbb{R}^2} n^2 \, dx\end{aligned}$$

The entropy is nonincreasing if $\chi M \leq 4 C_{\mathrm{GNS}}^{-2}$, where $C_{\mathrm{GNS}} = C_{\mathrm{GNS}}^{(4)}$ is the best constant for $p = 4$ in the Gagliardo-Nirenberg-Sobolev inequality :

$$\|u\|^2_{L^p(\mathbb{R}^2)} \leq C_{\mathrm{GNS}}^{(p)} \|\nabla u\|^{2-4/p}_{L^2(\mathbb{R}^2)} \|u\|^{4/p}_{L^2(\mathbb{R}^2)} \quad \forall\, u \in H^1(\mathbb{R}^2) \quad \forall\, p \in [2, \infty)$$

Numerically : $\chi M \leq 4 C_{\mathrm{GNS}}^{-2} \approx 1.862... \times (4\pi) < 8\pi$

A sharper approach : free energy

The *free energy* :

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} n \, c \, dx$$

Lemma 5. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (1) such that $n(1 + |x|^2)$, $n \log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\nabla\sqrt{n} \in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c \in L^\infty_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$. Then*

$$\frac{d}{dt} F[n(\cdot,t)] = -\int_{\mathbb{R}^2} n \left|\nabla\left(\log n\right) - \chi \nabla c\right|^2 \, dx =: \mathcal{I}$$

$\mathcal{I}$ is the *free energy production term* or *generalized relative Fisher information.*

Proof.

$$\frac{d}{dt} F[n(\cdot,t)] = \int_{\mathbb{R}^2} \left[\left(1 + \log n - \chi \, c\right) \nabla \cdot \left(\frac{\nabla n}{n} - \chi \, \nabla c\right)\right] \, dx$$

Hardy-Littlewood-Sobolev inequality

$$F[n(\cdot,t)] = \int_{\mathbb{R}^2} n \log n \, dx + \frac{\chi}{4\pi} \iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x,t)\, n(y,t)\, \log|x-y| \, dx\, dy$$

Lemma 6. [Carlen-Loss, Beckner] *Let f be a non-negative function in $L^1(\mathbb{R}^2)$ such that $f \log f$ and $f \log(1+|x|^2)$ belong to $L^1(\mathbb{R}^2)$. If $\int_{\mathbb{R}^2} f\, dx = M$, then*

$$\int_{\mathbb{R}^2} f \log f \, dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x) f(y) \log|x-y| \, dx\, dy \geq -\, C(M)$$

with $C(M) := M(1 + \log\pi - \log M)$

The above inequality is the key functional inequality

Consequences

$$(1-\theta)\int_{\mathbb{R}^2} n\log n\;dx+\theta\left[\int_{\mathbb{R}^2} n\log n\;dx+\frac{\chi}{4\pi\theta}\iint_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\,n(y)\;\log|x-y|\;dx\,dy\right]$$

Lemma 7. *Consider a non-negative $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$ solution n of (1) such that $n(1+|x|^2)$, $n\log n$ are bounded in $L^\infty_{\rm loc}(\mathbb{R}^+, L^1(\mathbb{R}^2))$, $\int_{\mathbb{R}^2}\frac{1+|x|}{|x-y|}\,n(y,t)\;dy\in L^\infty\big((0,T)\times\mathbb{R}^2\big)$, $\nabla\sqrt{n}\in L^1_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^2))$ and $\nabla c\in L^\infty_{\rm loc}(\mathbb{R}^+\times\mathbb{R}^2)$. If $\chi\,M\leq 8\pi$, then the following estimates hold :*

$$M\;\log M - M\;\log[\pi(1+t)]-K\leq\int_{\mathbb{R}^2} n\log n\;dx\leq\frac{8\pi\,F_0+\chi\,M\,C(M)}{8\pi-\chi\,M}$$

$$0\leq\int_0^t ds\int_{\mathbb{R}^2} n(x,s)\Big|\nabla\left(\log n(x,s)\right)\quad-\quad\chi\nabla c(x,s)\Big|^2 dx$$

$$\leq C_1+C_2\left[M\;\log\left(\frac{\pi(1+t)}{M}\right)+K\right]$$

Lower bound

Because of the bound on the second moment

$$\frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx \le K \quad \forall\, t>0\;,$$

$$\int_{\mathbb{R}^2} n(x,t)\log n(x,t) \;\ge\; \frac{1}{1+t}\int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx - K + \int_{\mathbb{R}^2} n(x,t)\log n(x,t)\, dx$$

$$= \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\left(\frac{n(x,t)}{\mu(x,t)}\right)\mu(x,t)\, dx - M\;\log[\pi(1+t)] - K$$

with $\mu(x,t) := \frac{1}{\pi(1+t)}\, e^{-\frac{|x|^2}{1+t}}$. By Jensen's inequality,

$$\int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\log\left(\frac{n(x,t)}{\mu(x,t)}\right) d\mu(x,t) \ge X\;\log X \text{ where } X = \int_{\mathbb{R}^2} \frac{n(x,t)}{\mu(x,t)}\, d\mu = M$$

$L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R}^2))$ bound of the entropy term

Lemma 8. *For any $u \in L^1_+(\mathbb{R}^2)$, if $\int_{\mathbb{R}^2} |x|^2\, u\, dx$ and $\int_{\mathbb{R}^2} u \,\log u\, dx$ are bounded from above, then $u\,\log u$ is uniformly bounded in $L^\infty(\mathbb{R}^+_{\mathrm{loc}}, L^1(\mathbb{R}^2))$ and*

$$\int_{\mathbb{R}^2} u\,|\log u|\, dx \le \int_{\mathbb{R}^2} u \left(\log u + |x|^2\right) dx + 2\log(2\pi) \int_{\mathbb{R}^2} u\, dx + \frac{2}{e}$$

Proof. Let $\bar u := u\, 1\!\!1_{\{u\le 1\}}$ and $m = \int_{\mathbb{R}^2} \bar u\, dx \le M$. Then

$$\int_{\mathbb{R}^2} \bar u\, (\log \bar u + \frac{1}{2}|x|^2)\, dx = \int_{\mathbb{R}^2} U \log U\, d\mu - m \log(2\pi)$$

$U := \bar u/\mu$, $d\mu(x) = \mu(x)\, dx$, $\mu(x) = (2\pi)^{-1} e^{-|x|^2/2}$. Jensen's inequality :

$$\int_{\mathbb{R}^2} \bar u\, \log \bar u\, dx \ge m \log\left(\frac{m}{2\pi}\right) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\, \bar u\, dx \ge -\frac{1}{e} - M\log(2\pi) - \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\, \bar u\, dx$$

and conclude using

$\int_{\mathbb{R}^2} u\,|\log u|\, dx = \int_{\mathbb{R}^2} u\, \log u\, dx - 2\int_2 \bar u\, \log \bar u\, dx$

II. Proof of the existence result

Weak solutions up to critical mass

Proposition 9. *If $M < 8\pi/\chi$, the Keller-Segel system (1) has a global weak non-negative solution such that, for any $T > 0$,*

$$(1 + |x|^2 + |\log n|)\, n \in L^\infty(0, T; L^1(\mathbb{R}^2))$$

and

$$\iint_{[0,T]\times\mathbb{R}^2} n\, |\nabla \log n - \chi \nabla c|^2\, dx\, dt < \infty$$

For $R > \sqrt{e}$, $R \mapsto R^2/\log R$ is an increasing function, so that

$$0 \le \iint_{|x-y|>R} \log|x-y|\, n(x,t)\, n(y,t)\; dx\, dy \le \frac{2\,\log R}{R^2}\, M \int_{\mathbb{R}^2} |x|^2\, n(x,t)\; dx$$

Since $\iint_{1<|x-y|<R} \log|x-y|\, n(x,t)\, n(y,t)\; dx\, dy \le M^2 \log R$, we only need a uniform bound for $|x-y| < 1$

A regularized model

Let $\mathcal{K}^\varepsilon(z) := \mathcal{K}^1\left(\frac{z}{\varepsilon}\right)$ with

$$\begin{cases} \mathcal{K}^1(z) = -\dfrac{1}{2\pi}\log|z| & \text{if } |z| \geq 4 \\ \mathcal{K}^1(z) = 0 & \text{if } |z| \leq 1 \end{cases}$$

$$0 \leq -\nabla\mathcal{K}^1(z) \leq \frac{1}{2\pi\,|z|} \quad \mathcal{K}^1(z) \leq -\frac{1}{2\pi}\log|z| \quad \text{and} \quad -\Delta\mathcal{K}^1(z) \geq 0$$

Since $\mathcal{K}^\varepsilon(z) = \mathcal{K}^1(z/\varepsilon)$, we also have

$$0 \leq -\nabla\mathcal{K}^\varepsilon(z) \leq \frac{1}{2\pi\,|z|} \quad \forall\, z \in \mathbb{R}^2$$

Proposition 10. *For any fixed positive ε, if $n_0 \in L^2(\mathbb{R}^2)$, then for any $T > 0$ there exists $n^\varepsilon \in L^2(0,T;H^1(\mathbb{R}^2)) \cap C(0,T;L^2(\mathbb{R}^2))$ which solves*

$$\begin{cases} \dfrac{\partial n^\varepsilon}{\partial t} = \Delta n^\varepsilon - \chi \nabla \cdot (n^\varepsilon \nabla c^\varepsilon) \\ \\ c^\varepsilon = \mathcal{K}^\varepsilon * n^\varepsilon \end{cases}$$

1. Regularize the initial data : $n_0 \in L^2(\mathbb{R}^2)$
2. Use the *Aubin-Lions compactness method* with the spaces $H := L^2(\mathbb{R}^2)$, $V := \{v \in H^1(\mathbb{R}^2)\ :\ \sqrt{|x|}\,v \in L^2(\mathbb{R}^2)\}$, $L^2(0,T;V)$, $L^2(0,T;H)$ and $\{v \in L^2(0,T;V)\ :\ \partial v/\partial t \in L^2(0,T;V')\}$
3. Fixed-point method

Uniform a priori estimates

Lemma 11. *Consider a solution n^ε of the regularized equation. If $\chi M < 8\pi$ then, uniformly as $\varepsilon \to 0$, with bounds depending only upon $\int_{\mathbb{R}^2}(1+|x|^2)\, n_0\, dx$ and $\int_{\mathbb{R}^2} n_0 \,\log n_0\, dx$, we have :*

(i) *The function $(t,x) \mapsto |x|^2 n^\varepsilon(x,t)$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*

(ii) *The functions $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t) \log n^\varepsilon(x,t)\, dx$ and $t \mapsto \int_{\mathbb{R}^2} n^\varepsilon(x,t)\, c^\varepsilon(x,t)\, dx$ are bounded.*

(iii) *The function $(t,x) \mapsto n^\varepsilon(x,t) \log(n^\varepsilon(x,t))$ is bounded in $L^\infty(\mathbb{R}^+_{\rm loc}; L^1(\mathbb{R}^2))$.*

(iv) *The function $(t,x) \mapsto \nabla \sqrt{n^\varepsilon}(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(v) *The function $(t,x) \mapsto n^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(vi) *The function $(t,x) \mapsto n^\varepsilon(x,t)\, \Delta c^\varepsilon(x,t)$ is bounded in $L^1(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

(vii) *The function $(t,x) \mapsto \sqrt{n^\varepsilon}(x,t)\, \nabla c^\varepsilon(x,t)$ is bounded in $L^2(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$.*

Proof of (iv)

$$\frac{d}{dt}\int_{\mathbb{R}^2} n^\varepsilon \log n^\varepsilon \, dx \le -4\int_{\mathbb{R}^2} \left|\nabla\sqrt{n^\varepsilon}\right|^2 dx + \chi \int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta c^\varepsilon)\, dx$$

$$\int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta c^\varepsilon)\, dx = \int_{\mathbb{R}^2} n^\varepsilon \cdot (-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon))\, dx = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})$$

with

$$(\mathrm{I}) := \int_{n^\varepsilon < K} n^\varepsilon \cdot (-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon)),\ (\mathrm{II}) := \int_{n^\varepsilon \ge K} n^\varepsilon \cdot (-\Delta(\mathcal{K}^\varepsilon * n^\varepsilon)) - (\mathrm{III}),\ (\mathrm{III}) = \int_{n^\varepsilon \ge K} |n^\varepsilon|^2$$

Let $\frac{1}{\varepsilon^2}\phi_1\left(\frac{\cdot}{\varepsilon}\right) := -\Delta\mathcal{K}^\varepsilon$: $\frac{1}{\varepsilon^2}\phi_1\left(\frac{\cdot}{\varepsilon}\right) = -\Delta\mathcal{K}^\varepsilon \rightharpoonup \delta$ in $\mathcal{D}'$

This heuristically explains why (II) should be small

III. Regularity and free energy

Weak regularity results

Theorem 12. [Goudon2004] *Let $n^\varepsilon : (0,T)\times\mathbb{R}^N \to \mathbb{R}$ be such that for almost all $t\in(0,T)$, $n^\varepsilon(t)$ belongs to a weakly compact set in $L^1(\mathbb{R}^N)$ for almost any $t\in(0,T)$. If $\partial_t n^\varepsilon = \sum_{|\alpha|\le k}\partial_x^\alpha g_\varepsilon^{(\alpha)}$ where, for any compact set $K\subset\mathbb{R}^n$,*

$$\limsup_{\substack{|E|\to 0\\ E\subset\mathbb{R} \text{ is measurable}}} \left(\sup_{\varepsilon>0}\int\int_{E\times K} |g_\varepsilon^{(\alpha)}|\,dt\,dx\right) = 0$$

then $(n^\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T];L^1_{\text{weak}}(\mathbb{R}^N))$.

Corollary 13. *Let n^ε be a solution of the regularized problem with initial data $n_0^\varepsilon = \min\{n_0,\varepsilon^{-1}\}$ such that $n_0\,(1+|x|^2+|\log n_0|)\in L^1(\mathbb{R}^2)$. If n is a solution of (1) with initial data n_0, such that, for a sequence $(\varepsilon_k)_{k\in\mathbb{N}}$ with $\lim_{k\to\infty}\varepsilon_k = 0$, $n^{\varepsilon_k}\rightharpoonup n$ in $L^1((0,T)\times\mathbb{R}^2)$, then n belongs to $C^0(0,T;L^1_{\text{weak}}(\mathbb{R}^2))$.*

L^p uniform estimates

Proposition 14. *Assume that $M < 8\pi/\chi$ hold. If n_0 is bounded in $L^p(\mathbb{R}^2)$ for some $p > 1$, then any solution n of (1) is bounded in $L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^p(\mathbb{R}^2))$.*

$$\begin{aligned}
\frac{1}{2(p-1)} \frac{d}{dt} \int_{\mathbb{R}^2} |n(x,t)|^p \, dx &= -\frac{2}{p} \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2 \, dx + \chi \int_{\mathbb{R}^2} \nabla(n^{p/2}) \cdot n^{p/2} \cdot \nabla c \, dx \\
&= -\frac{2}{p} \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2 \, dx + \chi \int_{\mathbb{R}^2} n^p \, (-\Delta c) \, dx \\
&= -\frac{2}{p} \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2 \, dx + \chi \int_{\mathbb{R}^2} n^{p+1} \, dx
\end{aligned}$$

Gagliardo-Nirenberg-Sobolev inequality with $n = v^{2/p}$:

$$\int_{\mathbb{R}^2} |v|^{2(1+1/p)} \, dx \leq K_p \int_{\mathbb{R}^2} |\nabla v|^2 \, dx \int_{\mathbb{R}^2} |v|^{2/p} \, dx$$

$$\frac{1}{2(p-1)}\,\frac{d}{dt}\int_{\mathbb{R}^2} n^p\,dx \le \int_{\mathbb{R}^2} |\nabla(n^{p/2})|^2\,dx \left(-\frac{2}{p} + K_p\,\chi\,M\right)$$

which proves the decay of $\int_{\mathbb{R}^2} n^p\,dx$ if $M < \frac{2}{p\,K_p\,\chi}$

Otherwise, use the entropy estimate to get a bound : Let $K > 1$

$$\int_{\mathbb{R}^2} n^p\,dx = \int_{n\le K} n^p\,dx + \int_{n>K} n^p\,dx \le K^{p-1}\,M + \int_{n>K} n^p\,dx$$

Let $M(K) := \int_{n>K} n\,dx$:

$$M(K) \le \frac{1}{\log K}\int_{n>K} n\,\log n\,dx \le \frac{1}{\log K}\int_{\mathbb{R}^2} |n\,\log n|\,dx$$

Redo the computation for $\int_{\mathbb{R}^2} (n-K)_+^p\,dx$ [Jäger-Luckhaus]

The free energy inequality in a regular setting

Using the *a priori* estimates of the previous section for $p = 2 + \varepsilon$, we can prove that the free energy inequality holds.

Lemma 15. *Let n_0 be in a bounded set in $L^1_+(\mathbb{R}^2, (1+|x|^2)dx) \cap L^{2+\varepsilon}(\mathbb{R}^2, dx)$, for some $\varepsilon > 0$, eventually small. Then the solution n of (1) found before, with initial data n_0, is in a compact set in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$ and moreover the free energy production estimate holds :*

$$F[n] + \int_0^t \left(\int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) - \chi \nabla c \right|^2 \, dx \right) ds \leq F[n_0]$$

1. n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$
2. ∇n is bounded in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$
3. Compactness in $L^2(\mathbb{R}^+_{\mathrm{loc}} \times \mathbb{R}^2)$

Taking the limit in the Fisher information term

Up to the extraction of subsequences

$$\iint_{[0,T]\times\mathbb{R}^2} |\nabla n|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |\nabla n_k|^2\,dx\,dt$$

$$\iint_{[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt \le \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} n_k\,|\nabla c_k|^2\,dx\,dt$$

$$\iint_{[0,T]\times\mathbb{R}^2} n^2\,dx\,dt = \liminf_{k\to\infty} \iint_{[0,T]\times\mathbb{R}^2} |n_k|^2\,dx\,dt$$

Fisher information term :

$$\iint_{[[0,T]\times\mathbb{R}^2} n\left|\nabla\left(\log n\right) - \chi\nabla c\right|^2\,dx\,dt$$

$$= 4\iint_{[[0,T]\times\mathbb{R}^2} |\nabla\sqrt{n}|^2\,dx\,dt + \chi^2\iint_{[[0,T]\times\mathbb{R}^2} n\,|\nabla c|^2\,dx\,dt - 2\chi\iint_{[[0,T]\times\mathbb{R}^2} n^2\,dx\,dt \qquad \square$$

Hypercontractivity

Theorem 16. *Consider a solution n of (1) such that $\chi\, M < 8\pi$. Then for any $p \in (1,\infty)$, there exists a continuous function h_p on $(0,\infty)$ such that for almost any $t>0$, $\|n(\cdot,t)\|_{L^p(\mathbb{R}^2)} \leq h_p(t)$.*

Notice that unless n_0 is bounded in $L^p(\mathbb{R}^2)$, $\lim_{t\to 0_+} h_p(t) = +\infty$. Such a result is called an *hypercontractivity* result, since to an initial data which is originally in $L^1(\mathbb{R}^2)$ but not in $L^p(\mathbb{R}^2)$, we associate a solution which at almost any time $t>0$ is in $L^p(\mathbb{R}^2)$ with p arbitrarily large.

Proof. Fix $t>0$ and $p \in (1,\infty)$ and consider $q(s) := 1 + (p-1)\,\frac{s}{t}$. Define : $M(K) := \sup_{s\in(0,t)} \int_{n>K} n(\cdot,s)\,dx$

$$\int_{n>K} n(\cdot,s)\,dx \leq \frac{1}{\log K}\int_{\mathbb{R}^2} |n(\cdot,s)\,\log n(\cdot,s)|\,dx$$

and

$$F(s) := \left[\int_{\mathbb{R}^2} (n-K)_+^{q(s)}(x,s)\,dx\right]^{1/q(s)}$$

$$F' \, F^{q-1} = \frac{q'}{q^2} \int_{\mathbb{R}^2} (n-K)_+^q \log\left(\frac{(n-K)_+^q}{F^q}\right) + \int_{\mathbb{R}^2} n_t \, (n-K)_+^{q-1}$$

$$\int_{\mathbb{R}^2} (n-K)_+^{q-1} \, n_t \, dx = -4 \, \frac{q-1}{q^2} \int_{\mathbb{R}^2} |\nabla v|^2 \, dx + \chi \, \frac{q-1}{q} \int_{\mathbb{R}^2} v^{2(1+\frac{1}{q})} \, dx$$

with $v := (n-K)_+^{q/2}$

Logarithmic Sobolev inequality

$$\int_{\mathbb{R}^2} v^2 \log\left(\frac{v^2}{\int_{\mathbb{R}^2} v^2 \, dx}\right) dx \leq 2 \, \sigma \int_{\mathbb{R}^2} |\nabla v|^2 \, dx - (2 + \log(2 \, \pi \, \sigma)) \int_{\mathbb{R}^2} v^2 \, dx$$

Gagliardo-Nirenberg-Sobolev inequality

$$\int_{\mathbb{R}^2} |v|^{2(1+1/q)} \, dx \leq \mathcal{K}(q) \, \|\nabla v\|^2_{L^2(\mathbb{R}^2)} \int_{\mathbb{R}^2} |v|^{2/q} \, dx \quad \forall \, q \in [2, \infty)$$

□

The free energy inequality for weak solutions

Corollary 17. *Let $(n^k)_{k\in\mathbb{N}}$ be a sequence of solutions of (1) with regularized initial data n_0^k . For any $t_0 > 0$, $T \in \mathbb{R}^+$ such that $0 < t_0 < T$, $(n^k)_{k\in\mathbb{N}}$ is relatively compact in $L^2((t_0, T) \times \mathbb{R}^2)$, and if n is the limit of $(n^k)_{k\in\mathbb{N}}$, then n is a solution of (1) such that the free energy inequality holds.*

Proof.

$$F[n^k(\cdot,t)] + \int_{t_0}^{t} \left(\int_{\mathbb{R}^2} n^k \left| \nabla \left(\log n^k \right) - \chi \nabla c^k \right|^2 dx \right) ds \leq F[n^k(\cdot,t_0)]$$

Passing to the limit as $k \to \infty$, we get

$$F[n(\cdot,t)] + \int_{t_0}^{t} \left(\int_{\mathbb{R}^2} n \left| \nabla \left(\log n \right) - \chi \nabla c \right|^2 dx \right) ds \leq F[n(\cdot,t_0)]$$

Let $t_0 \to 0_+$ and conclude □

IV. Large time behaviour

Self-similar variables

$$n(x,t) = \frac{1}{R^2(t)}\, u\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad c(x,t) = v\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla\cdot(u(x + \chi\nabla v)) & x \in \mathbb{R}^2\,,\ t > 0 \\ v = -\dfrac{1}{2\pi}\log|\cdot| * u & x \in \mathbb{R}^2\,,\ t > 0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Free energy : $F^R[u] := \int_{\mathbb{R}^2} u \log u\, dx - \frac{\chi}{2}\int_{\mathbb{R}^2} u\, v\, dx + \frac{1}{2}\int_{\mathbb{R}^2} |x|^2\, u\, dx$

$$\frac{d}{dt}F^R[u(\cdot,t)] \leq -\int_{\mathbb{R}^2} u\, \left|\nabla \log u - \chi\nabla v + x\right|^2\, dx$$

Self-similar solutions : Free energy

Lemma 18. *The functional F^R is bounded from below on the set*

$$\left\{u \in L^1_+(\mathbb{R}^2)\ :\ |x|^2\, u \in L^1(\mathbb{R}^2)\ \int_{\mathbb{R}^2} u\, \log u\, dx < \infty\right\}$$

if and only if $\chi\, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$.

Proof. If $\chi\, \|u\|_{L^1(\mathbb{R}^2)} \leq 8\pi$, the bound is a consequence of the Hardy-Littlewood-Sobolev inequality

Scaling property. For a given u, let $u_\lambda(x) = \lambda^{-2} u(\lambda^{-1} x)$: $\|u_\lambda\|_{L^1(\mathbb{R}^2)} =: M$ does not depend on $\lambda > 0$ and

$$F^R[u_\lambda] = F^R[u] - 2M\left(1 - \frac{\chi\, M}{8\pi}\right)\log\lambda + \frac{\lambda - 1}{2}\int_{\mathbb{R}^2} |x|^2\, u\, dx$$

□

Strong convergence

Lemma 19. *Let $\chi M < 8\pi$. As $t \to \infty$, $(s,x) \mapsto u(x,t+s)$ converges in $L^\infty(0,T;L^1(\mathbb{R}^2))$ for any positive T to a stationary solution self-similar equation and*

$$\lim_{t\to\infty} \int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, u_\infty\, dx = 2M \left(1 - \frac{\chi M}{8\pi}\right)$$

Proof. We use the free energy production term :

$$F^R[u_0] - \liminf_{t\to\infty} F^R[u(\cdot,t)] = \lim_{t\to\infty} \int_0^t \left(\int_{\mathbb{R}^2} u\, |\nabla \log u - \chi \nabla v + x|^2\, dx \right) ds$$

and compute $\int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx$:

$$\int_{\mathbb{R}^2} |x|^2\, u(x,t)\, dx = \int_{\mathbb{R}^2} |x|^2\, n_0\, dx\, e^{-2t} + 2M \left(1 - \frac{\chi M}{8\pi}\right) \left(1 - e^{-2t}\right) \quad \square$$

Stationary solutions

Notice that under the constraint $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$, u_∞ is a critical point of the free energy.

Lemma 20. *Let $u \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)$ with $M := \int_{\mathbb{R}^2} u\,dx$, such that $\int_{\mathbb{R}^2} u \log u\,dx < \infty$, and define $v(x) := -\frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y|\,u(y)\,dy$. Then there exists a positive constant C such that, for any $x \in \mathbb{R}^2$ with $|x| > 1$,*

$$\left| v(x) + \frac{M}{2\pi} \log|x| \right| \leq C$$

Lemma 21. [Naito-Suzuki] *Assume that V is a non-negative non-trivial radial function on $\mathbb{R}^2$ such that $\lim_{|x|\to\infty} |x|^\alpha V(x) < \infty$ for some $\alpha \geq 0$. If u is a solution of*

$$\Delta u + V(x)\,e^u = 0 \quad x \in \mathbb{R}^2$$

such that $u_+ \in L^\infty(\mathbb{R}^2)$, then u is radially symmetric decreasing w.r.t. the origin

Because of the asymptotic logarithmic behavior of v_∞, the result of Gidas, Ni and Nirenberg does not directly apply. The boundedness from above is essential, otherwise non-radial solutions can be found, even with no singularity. Consider for instance the perturbation $\delta(x) = \frac{1}{2}\,\theta\,(x_1^2 - x_2^2)$ for any $x = (x_1, x_2)$, for some fixed $\theta \in (0,1)$, and define the potential $\phi(x) = \frac{1}{2}\,|x|^2 - \delta(x)$. By a fixed-point method we can find a solution of

$$w(x) = -\frac{1}{2\pi}\,\log|\cdot| * M\,\frac{e^{\chi w - \phi(x)}}{\int_{\mathbb{R}^2} e^{\chi w(y) - \phi(y)}\,dy}$$

since, as $|x| \to \infty$, $\phi(x) \sim \frac{1}{2}\left[(1-\theta)x_1^2 + (1+\theta)x_2^2\right] \to +\infty$. This solution is such that $w(x) \sim -\frac{M}{2\pi}\,\log|x|$. Hence $v(x) := w(x) + \delta(x)/\chi$ is a non-radial solution of the self-similar equation, which behaves like $\delta(x)/\chi$ as $|x| \to \infty$ with $|x_1| \neq |x_2|$.

Lemma 22. *If $\chi M > 8\pi$, the rescaled equation has no stationary solution (u_∞, v_∞) such that $\|u_\infty\|_{L^1(\mathbb{R}^2)} = M$ and $\int_{\mathbb{R}^2} |x|^2 u_\infty \, dx < \infty$. If $\chi M < 8\pi$, the self-similar equation has at least one radial stationary solution. This solution is C^∞ and u_∞ is dominated as $|x| \to \infty$ by $e^{-(1-\varepsilon)|x|^2/2}$ for any $\varepsilon \in (0,1)$.*

Non-existence for $\chi M > 8\pi$:

$$0 = \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u_\infty \, dx = 4M \left(1 - \frac{\chi M}{8\pi}\right) - 2 \int_{\mathbb{R}^2} |x|^2 u_\infty \, dx$$

□

Intermediate asymptotics

Lemma 23.

$$\lim_{t\to\infty} F^R[u(\cdot,\cdot+t)] = F^R[u_\infty]$$

Proof. We know that $u(\cdot,\cdot+t)$ converges to u_∞ in $L^2((0,1)\times\mathbb{R}^2)$ and that $\int_{\mathbb{R}^2} u(\cdot,\cdot+t)\, v(\cdot,\cdot+t)\; dx$ converges to $\int_{\mathbb{R}^2} u_\infty\, v_\infty\; dx$. Concerning the entropy, it is sufficient to prove that $u(\cdot,\cdot+t)\,\log u(\cdot,\cdot+t)$ weakly converges in $L^1((0,1)\times\mathbb{R}^2)$ to $u_\infty\,\log u_\infty$. Concentration is prohibited by the convergence in $L^2((0,1)\times\mathbb{R}^2)$. Vanishing or dichotomy cannot occur either : Take indeed $R>0$, large, and compute $\int_{|x|>R} u\,|\log u| = (\mathrm{I}) + (\mathrm{II})$, with $m := \int_{|x|>R,\; u<1} u\; dx$ and

$$\begin{aligned}(\mathrm{I}) &= \int_{|x|>R,\; u\geq 1} u\,\log u\; dx \leq \frac{1}{2}\int_{|x|>R,\; u\geq 1} |u|^2\; dx \\ (\mathrm{II}) &= -\int_{|x|>R,\; u<1} u\,\log u\; dx \leq \frac{1}{2}\int_{|x|>R,\; u<1} |x|^2\, u\; dx - m\log\left(\frac{m}{2\pi}\right)\end{aligned}$$

Conclusion

The result we have shown above is actually slightly better : all terms converge to the corresponding values for the limiting stationary solution

$$F^R[u] - F^R[u_\infty] = \int_{\mathbb{R}^2} u \, \log\left(\frac{u}{u_\infty}\right) \, dx - \frac{\chi}{2} \int_{\mathbb{R}^2} |\nabla v - \nabla v_\infty|^2 \, dx$$

Csiszár-Kullback inequality : for any nonnegative functions f, $g \in L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} f \, dx = \int_{\mathbb{R}^2} g \, dx = M$,

$$\|f - g\|^2_{L^1(\mathbb{R}^2)} \leq \frac{1}{4\,M} \int_{\mathbb{R}^2} f \, \log\left(\frac{f}{g}\right) \, dx$$

Corollary 24.

$$\lim_{t\to\infty} \|u(\cdot, \cdot + t) - u_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \textit{and} \quad \lim_{t\to\infty} \|\nabla v(\cdot, \cdot + t) - \nabla v_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

Nonlinear diffusions as limits of BGK-type kinetic equations

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Outline of the talk

1. Kinetic BGK Model: Formulation
2. Motivations and references
3. Main results and assumptions
4. Existence and uniqueness
5. Drift diffusion limit
6. Convergence to equilibrium
7. Examples
    - Porous medium flow
    - Fast diffusion
    - Fermi-Dirac statistics
    - Bose-Einstein statistics

BGK models

- BGK model of gas dynamics

$$\partial_t f + v \cdot \nabla_x f = \frac{\rho(x,t)}{(2\pi T)^{n/2}} \exp\left(\frac{-|v-u(x,t)|^2}{2T(x,t)}\right) - f \ ,$$

where $\rho(x,t)$ (position density), $u(x,t)$ (local mean velocity) and $T(x,t)$ (temperature) are chosen such that they equal the corresponding quantities associated to f.

[Perthame,Pulvirenti]: Weighted L^∞ bounds and uniqueness for the Boltzmann BGK model, 1993

- Linear BGK model in semiconductor physics

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \frac{\rho(x,t)}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}|v|^2\right) - f \ ,$$

where $\rho(x,t)$ equals the position density of f.

[Poupaud]: Mathematical theory of kinetic equations for transport modelling in semiconductors, 1994

BGK-type kinetic equation

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon - \varepsilon \nabla_x V(x) \cdot \nabla_v f^\varepsilon = G_{f^\varepsilon} - f^\varepsilon \,,$$
$$f^\varepsilon(x,v,t=0) = f_I(x,v) \,, \quad x,v \in \mathbb{R}^3 \,,$$

with the Gibbs equilibrium $G_f := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right)$.

The Fermi energy $\mu_{\rho_f}(x,t)$ is implicitly defined by

$$\int_{\mathbb{R}^3} \gamma\left(\frac{|v|^2}{2} + V(x) - \mu_{\rho_f}(x,t)\right) dv = \int_{\mathbb{R}^3} f(x,v,t)dv =: \rho_f(x,t).$$

$f^\varepsilon(x,v,t)$ ... phase space particle density

$V(x)$ ... potential

ε ... mean free path.

Motivations, I

- Local Gibbs states in stellar dynamics (polytropic distribution functions) and semiconductor theory (Fermi-Dirac distributions).
  Collisions : short time scale
- Monotone energy profiles are natural for the study of stability: monotonicity $\Leftrightarrow$ convex Lyapunov functional, Global Gibbs states
- Goal: derive the nonlinear diffusion limit consistently with the Gibbs state: a relaxation-time kernel

Motivations, II

- Gibbs states $\Longleftrightarrow$ generalized entropies
- nonlinear diffusion equations are difficult to justify directly
- global Gibbs states have the same macroscopic density at the kinetic / diffusion levels
- they have the 'same' Lyapunov functionals

References

- Formal expansions (generalized Smoluchowski equation):
  [Ben Abdallah, J.D.], [Chavanis-Laurençot, Lemou], [Chavanis et al.], [Degond, Ringhofer]
- Astrophysics:
  [Binney,Tremaine], [Guo,Rein], [Chavanis et al.]
- Fermi-Dirac statistics in semiconductors models:
  [Goudon-Poupaud]

Main result

Theorem 1. *For any $\varepsilon > 0$, the equation has a unique weak solution $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$. As $\varepsilon \to 0$, f^ε weakly converges to a local Gibbs state f^0 given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho(x,t))\right)$$

where ρ is a solution of the nonlinear diffusion equation

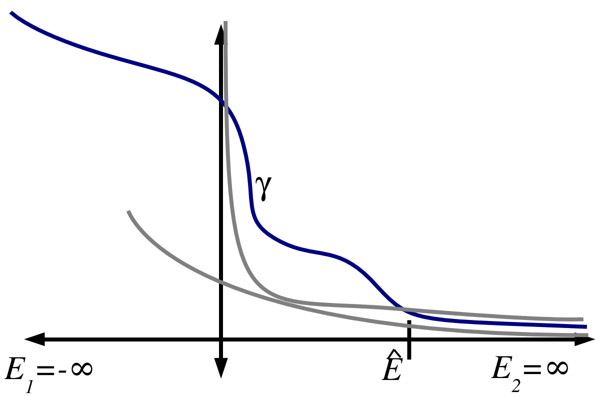
$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\, dv$

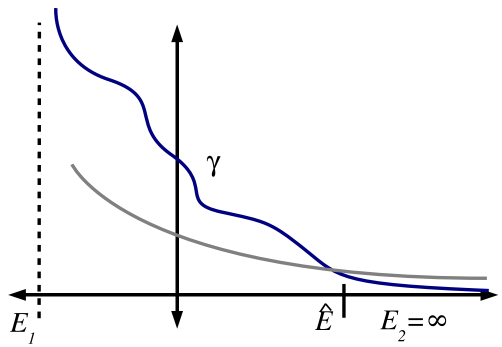
$$\nu(\rho) = \int_0^\rho s\, \bar{\mu}'(s)\, ds$$

Assumptions on the energy profile

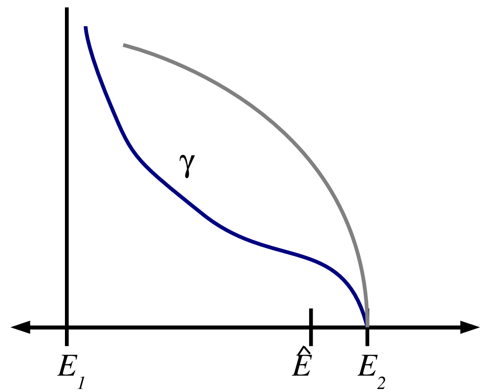
- $\gamma(E) \in \mathcal{C}^1((E_1, E_2), \mathbb{R}^+)$ where $-\infty \leq E_1 < E_2 \leq \infty$.
- γ monotonically decreasing and $\lim_{E \to E_2} \gamma(E) = 0$.



(a) Asymptotically exponential lower bound.



(b) Asymptotically exponential upper bound..



(c) $E_2 < \infty$.

Initial condition

- $f(x, v, t = 0) = f_I(x, v)$
- The total mass $M := \iint_{\mathbb{R}^6} f_I(x, v)\, dv\, dx$ is preserved by the evolution.
- $\exists \mu^*$ s.t. $0 \leq f_I(x, v) \leq f^*(x, v) := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^*\right)$
- Maximal macroscopic density

$$\bar{\rho} := \lim_{\theta \to -E_1^+} \int_{\mathbb{R}^3} \gamma\left(\frac{1}{2}|v|^2 - \theta\right) dv\ .$$

  Observe $\bar{\rho} = \infty$ if $E_1 = -\infty$.
- If $\bar{\rho} < \infty$ we require $\rho^*(x) := \int_{\mathbb{R}^3} f^*\, dv \leq \bar{\rho}\ \forall\, x \in \mathbb{R}^3$.

Fermi energy

The Fermi-energy $\mu_{\rho_f}(x,t)$ ensures local mass conservation,

$$\int_{\mathbb{R}^3} G_f \, dv = \int_{\mathbb{R}^3} \gamma\Bigg(\frac{|v|^2}{2} + \underbrace{V(x) - \mu_{\rho_f}(x,t)}_{=:-\bar{\mu}(\rho_f(x,t)) \text{ ('quasi Fermi level')}}\Bigg) dv = \rho_f(x,t)$$

- Compute $\bar{\mu}$ in terms of γ
$$(\bar{\mu}^{-1})(\theta) = 4\pi\sqrt{2}\int_0^\infty \gamma(p-\theta)\sqrt{p}\, dp$$
$\Rightarrow \bar{\mu}(\rho) : (0,\bar{\rho}) \to (-E_2, -E_1)$, increasing.
- Differentiation leads to an Abelian equation $\Rightarrow \gamma$ in terms of $\bar{\mu}$:
$$\gamma(E) = \frac{1}{\sqrt{2}\, 2\pi^2} \frac{d^2}{dE^2} \int_{-\infty}^{-E} \frac{(\bar{\mu}^{-1})(\theta)}{\sqrt{-E-\theta}}\, d\theta$$

Assumptions on the potential

- Boundedness from below

$$V(x) \geq V_{\text{min}} = 0,$$

- Regularity

$$V \in C^{1,1}(\mathbb{R}^3)\ .$$

- Potential is confining in the sense that

$$\iint_{\mathbb{R}^6} \left(1 + \frac{|v|^2}{2} + V(x)\right) \underbrace{\gamma\left(\frac{|v|^2}{2} + V(x) - \mu^*\right)}_{=f^*} dv\, dx < \infty.$$

Existence and uniqueness

Proposition 1. *Let $1 \leq p < \infty$, then the problem has a unique solution in $\mathcal{V} := \{f \in \mathcal{C}(0,\infty;(L^1 \cap L^p)(\mathbb{R}^6)) : 0 \leq f \leq f^*, \forall t > 0$ a.e.$\}$.*

The proof uses a fixpoint argument on the map $f \mapsto g$, where g satisfies

$$\begin{aligned}
\varepsilon^2 \partial_t g + \varepsilon\, v \cdot \nabla_x g - \varepsilon\, \nabla_x V \cdot \nabla_v g &= \gamma\left(\frac{|v|^2}{2} - \bar{\mu}(\rho_f)\right) - g\,, \\
g(t=0,x,v) &= f_I(x,v)\,, \\
\text{where} \quad \rho_f(x,t) &:= \int_{\mathbb{R}^3} f(x,v,t)\,dv\,.
\end{aligned}$$

$f \leq f^* \Rightarrow f \in L^\infty_{x,v,t}$, $\rho \in L^\infty_{x,t}$ and if f^* has compact support in $\mathbb{R}^3_v$, this will also be true for f (porous medium case).

Formal asymptotics

$$\varepsilon^2 \partial_t f + \varepsilon v \cdot \nabla_x f - \varepsilon \nabla_x V(x) \cdot \nabla_v f = Q[f]$$

Expand $f = \sum_{i=0}^{\infty} f^i \varepsilon^i$, $\rho^i = \int_{\mathbb{R}^3} f^i \, dv$, $G_f = \sum_{i=1}^{\infty} G^i \varepsilon^i$. Then $G^0 = \gamma(|v|^2/2 - \bar{\mu}(\rho^0)) = \gamma(|v|^2/2 + V - \mu^0)$.

$$\mathcal{O}(1) : G^0 = f^0.$$

$$\mathcal{O}(\varepsilon) : v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1$$

$$\Rightarrow \quad f^1 = v \cdot \nabla_x \mu^0 \, \gamma' \left(\tfrac{1}{2} v^2 + V(x) - \mu^0(x,t)\right) + G^1$$

$$\Rightarrow \quad \int_{\mathbb{R}^3} v f^1 \, dv = -\rho^0 \nabla_x \mu^0$$

$$\mathcal{O}(\varepsilon^2) : \partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x V \cdot \nabla_v f^1 = G^2 - f^2$$

$$\Rightarrow \quad \partial_t \rho^0 = \nabla \cdot (\rho^0 \nabla \mu^0) = \Delta \nu(\rho^0) + \nabla \cdot (\rho^0 \nabla_x V)$$

where $\rho^0(x,t) = \int_{\mathbb{R}^3} f^0(x,v,t) dv, \quad \rho^0(x,0) = \int_{\mathbb{R}^3} f_I(x,v) dv$.
The nonlinearity ν is given by $\nu(\rho) := \int_0^{\rho} \tilde{\rho} \, \bar{\mu}'(\tilde{\rho}) \, d\tilde{\rho}$

Free energy

- Define the free energy (convex functional)

$$\mathcal{F}(f) := \iint_{\mathbb{R}^6} \left[\left(\frac{|v|^2}{2} + V(x) \right) f - \int_0^f \gamma^{-1}(\tilde{f})\, d\tilde{f} \right] dv\, dx.$$

- Production of free energy

$$\varepsilon^2 \frac{d}{dt}\mathcal{F}(f^\varepsilon) = \iint_{\mathbb{R}^6} \left(\gamma(E_{f^\varepsilon}) - f^\varepsilon\right)\left(E_{f^\varepsilon} - (\gamma^{-1})(f^\varepsilon)\right) dv\, dx \le 0,$$

with $E_{f^\varepsilon} := \dfrac{|v|^2}{2} + V(x) - \mu_{\rho_{f^\varepsilon}}(x,t)$, $G_{f^\varepsilon} = \gamma(E_{f^\varepsilon})$

- Free energy is finite, $\forall\, t \in \mathbb{R}_+$:

$$-\infty < \mathcal{F}(f^\infty) \le \mathcal{F}(G_{f^\varepsilon}(.,.,t)) \le \mathcal{F}(f^\varepsilon(.,.,t)) \le \mathcal{F}(f_I) < \infty$$

as $\mathcal{F}(f^\infty) = \iint_{\mathbb{R}^6} \gamma \left(\frac{|v|^2}{2} + V - \mu^\infty \right)\left(\mu^\infty - \frac{|v|^2}{3} \right) < \infty$ by assumptions on the potential.

Perturbations of moments

- Perturbations of 1st and 2nd moments

$$j^{\varepsilon} := \int_{\mathbb{R}^3} v \, \frac{f^{\varepsilon} - G_{f^{\varepsilon}}}{\varepsilon} \, dv \quad \text{and} \quad \kappa^{\varepsilon} := \int_{\mathbb{R}^3} v \otimes v \, \frac{f^{\varepsilon} - G_{f^{\varepsilon}}}{\varepsilon} \, dv.$$

- $\Rightarrow \forall\, U$ open and bounded $\exists$ uniform bounds,

$$\|j^{\varepsilon}\|_{L^2_{x,t}(U)} \leq M^1_U \quad \text{and} \quad \|\kappa^{\varepsilon}\|_{L^2_{x,t}(U)} \leq M^2_U$$

- Proof uses production of free energy

$$\mathcal{O}(\varepsilon^2) = \iiint_{\{G_{f^{\varepsilon}}>0\}} (G_{f^{\varepsilon}} - f^{\varepsilon})^2 (-\gamma^{-1})'(f^*) dx\, dv\, dt +$$

$$+ \iiint_{\{G_{f^{\varepsilon}}=0\}} \Big(\underbrace{E_{f^{\varepsilon}} - E_2}_{\geq 0} + \underbrace{E_2 - \gamma^{-1}(f^{\varepsilon})}_{\geq 0} \Big) f^{\varepsilon}\, dx\, dv\, dt.$$

2^{nd} moments of local Gibbs states

- Let

$$\nu(\rho) := \int_{\mathbb{R}^3} v_i^2 \, \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}(\rho)\right) \, dv \, .$$
$$\Rightarrow \nu'(\rho) = \rho \, \bar{\mu}'(\rho) \, .$$

- On $[0, \rho^{\max} := \bar{\mu}^{-1}(\mu^*)]$ for some $C > 0$:

either $\quad \nu'(\rho) > C \quad$ or $\quad 1/\nu'(\rho) > C$.

- If $E_2 < \infty$ ("porous medium case"): $\lim_{\rho \to 0} \nu'(\rho) = 0$.

Strong convergence of ρ, I

Proposition 2. $\rho^\varepsilon \to \rho^0$ *in* L^p_{loc} *strongly for all* $p \in (1, \infty)$.

The proof uses compensated compactness theory applying the Div-Curl-Lemma to

$$U^\varepsilon := (\rho^\varepsilon, j^\varepsilon), \quad V^\varepsilon := (\nu(\rho^\varepsilon), 0, 0, 0).$$

Rewrite the equations for the mass and momentum densities (using $(\text{curl}\, w)_{ij} := w^i_{x_j} - w^j_{x_i}$)

$$\begin{cases} \text{div}_{t,x} U^\varepsilon = \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0, \\ (\text{curl}_{t,x} V^\varepsilon)_{1,2\ldots4} = \nabla_x \nu(\rho^\varepsilon) = \underbrace{-j^\varepsilon - \rho^\varepsilon \nabla_x V - \varepsilon \nabla_x \cdot \kappa^\varepsilon - \varepsilon^2 \partial_t j^\varepsilon}_{\text{precompact in } H^{-1,\text{loc}}_{x,t}}, \end{cases}$$

as j^ε, κ^ε and $\rho^\varepsilon \in L^{2,\text{loc}}_{x,t}$.

Strong convergence of ρ, II

The Div-Curl-Lemma yields

$$\overline{\rho\,\nu} = \overline{\rho}\;\overline{\nu}.$$

where

$$\begin{cases} \nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\nu} = \displaystyle\int_0^{\rho^{\text{max}}} \nu(\rho)\, d\eta_{x,t}(\rho), \\ \rho^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\rho} = \displaystyle\int_0^{\rho^{\text{max}}} \rho\, d\eta_{x,t}(\rho), \\ \rho^{\varepsilon_i}\, \nu(\rho^{\varepsilon_i}) \overset{*}{\rightharpoonup} \overline{\rho\nu} = \displaystyle\int_0^{\rho^{\text{max}}} \rho\nu(\rho)\, d\eta_{x,t}(\rho). \end{cases}$$

$\eta_{x,t}$ ... Young measure associated with $\rho^{\varepsilon_i} \overset{*}{\rightharpoonup} \overline{\rho}$

Strong convergence of ρ, III

The mean value theorem yields

$$\nu(\rho) = \nu(\overline{\rho}) + \nu'(\tilde{\rho})(\rho - \overline{\rho})$$

for some $\tilde{\rho} \in (0, \rho^{\text{max}})$. Conclude

$$0 = \overline{\rho\,\nu} - \overline{\rho}\,\overline{\nu} =$$

$$= \int_0^{\rho^{\text{max}}} \nu(\rho)(\rho - \overline{\rho})d\eta_{x,t}(\rho) = \underbrace{\int_0^{\rho^{\text{max}}} \nu(\overline{\rho})(\rho - \overline{\rho})d\eta_{x,t}(\rho)}_{=0} +$$

$$+ \int_0^{\rho^{\text{max}}} \nu'(\tilde{\rho})(\rho - \overline{\rho})^2 d\eta_{x,t}(\rho) \geq C \int_0^{\rho^{\text{max}}} (\rho - \overline{\rho})^2 d\eta_{x,t}(\rho),$$

assuming $\nu'(\tilde{\rho}) \geq C \Rightarrow \eta_{x,t} = \delta_{\overline{\rho}(x,t)} \Rightarrow \nu(\overline{\rho}) = \overline{\nu}$.

Weak formulation of the pde I

Lemma 1. *Let* $f^{\varepsilon_i} \rightharpoonup f^0$*, then* $f^0 = G_{f^0}$ *a.e. .*

Lemma 2. *Let* $j^{\varepsilon_i} \to j^0$ *in* $\mathcal{D}'_{x,t}$*, then* $j^0 = -\nabla_x \nu(\rho^0) - \rho^0 \nabla_x V$*.*

Proof. Multiply the kinetic equation by $\frac{1}{\varepsilon}$,

$$\begin{aligned} \varepsilon \partial_t f^\varepsilon + v \cdot \nabla_x f^\varepsilon - \nabla_x V \cdot \nabla_v f^\varepsilon &= -\frac{f^\varepsilon - G_{f^\varepsilon}}{\varepsilon} \\ \downarrow \text{ in } \quad & \mathcal{D}'(\mathbb{R}^7) \\ v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 &= \\ = v \cdot \nabla_x G_{f^0} - \nabla_x V \cdot \nabla_v G_{f^0} &=: -r^0 \end{aligned}$$

Using uniform boundedness of κ^ε we prove

$$j^{\varepsilon_i} \xrightarrow{\mathcal{D}'_{x,t}} \int_{\mathbb{R}^3} v\, r^0 \, dv = -\Big(\rho^0 \nabla_x V + \nabla_x \nu(\rho^0)\Big) \, .$$

Weak formulation of the pde II

Proposition 3. *$\rho^0 := \int_{\mathbb{R}^3} f^0\, dv$ satisfies a weak formulation of the formal macroscopic limit.*

Integrate the kinetic equation w.r.t. to v,

$$\partial_t \rho^\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^3} v \frac{f^\varepsilon - G_{f^\varepsilon}}{\varepsilon} dv = \partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon = 0\,.$$

In the limit as $\varepsilon \to 0$ we obtain

$$\begin{aligned} \partial_t \rho^0 &= \Delta \nu\left(\rho^0\right) + \nabla_x \cdot \left(\rho^0\, \nabla_x V\right)\,, \\ \rho^0(x, t=0) &= \int_{\mathbb{R}^3} f_I(x,v)\, dv\,. \end{aligned}$$

with $\quad \nu(\rho) = \int_0^\rho \tilde{\rho}\, \bar{\mu}'(\tilde{\rho})\, d\tilde{\rho}\,.$

Convergence to equilibrium, I

If $E_2 < \infty$ we additionally require that V is uniformly convex.

We consider the evolution in time of solutions of the problem with $\varepsilon = 1$

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V(x) \cdot \nabla_v f = G_f - f \ .$$

Proposition 4. *For every sequence $t_n \longrightarrow \infty$, there exists a subsequence (again denoted by t_n) such that*

$$f^n(t,x,v) := f(t_n+t,x,v) \rightharpoonup f^\infty = G^\infty := \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^\infty\right)$$

where μ^∞ is the unique constant Fermi energy which satisfies $\int_{\mathbb{R}^3} \bar{\mu}^{-1}(\mu^\infty - V(x))\, dx = M = \int_{\mathbb{R}^6} f_I\, dv\, dx$.

Velocity averaging

- Let $\phi \in \mathcal{D}_{x,v,t}$, then $(\phi f) \in L^2_{x,v,t}$ and

$$\partial_t(\phi f^n) + v \cdot \nabla_x(\phi f^n) =$$
$$= \phi G^n + f^n \left(\partial_t \phi + v \cdot \nabla_x \phi - \phi - \nabla_x V \cdot \nabla_v \phi\right) +$$
$$+ \nabla_v \cdot (\phi f^n \nabla_x V) =: g^n \ \in \ L^2_{x,t}(H_v^{-1}) \ .$$

- Golse, Perthame, Sentis '85:

$$\rho_R^n := \int_{|v| \leq R} f^n \, dv \xrightarrow{L^2_{x,t}(U)} \rho_R^\infty \ .$$

- As $(f^n)_n$ is weakly precompact in $L^1(U \times \mathbb{R}^3)$:

$$\exists \rho^\infty = \lim_{R \to \infty} \rho_R^\infty = \lim_{n \to \infty} \rho^n \quad \text{in} \quad L^2_{x,t}(U) \ .$$

Convergence to equilibrium, II

By boundedness of the free energy from below and integrating the production of free energy we obtain

$$0 \le -\int_0^\infty \iint_{\mathbb{R}^6} (\gamma(E_f) - f)(E_f - \gamma^{-1}(f)) \, dv \, dx \, dt < \infty \, .$$

Hence

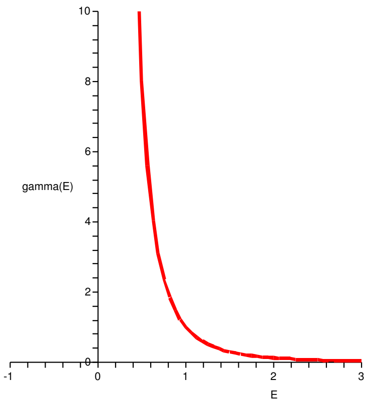
$$0 = \lim_{n\to\infty} \int_0^\infty \iint_{\mathbb{R}^6} (\gamma(E_{f^n}) - f^n)(E_{f^n} - \gamma^{-1}(f^n)) \, dv \, dx \, dt \, .$$

Finally implying $f^\infty = G^\infty$. Boundedness in L^1 and L^∞ on $\mathbb{R}^6 \times [0,T)$ and choosing particular test-functions in the weak formulation of the problem yields

$$f^n \rightharpoonup G^\infty := \gamma\left(\frac{|v|^2}{2} - \bar{\mu}(\rho^\infty(x,t))\right) = \gamma\left(\frac{|v|^2}{2} + V(x) - \mu^\infty\right) \, .$$

Ex. 1, fast diffusion case

- Maxwellian is a negative power of the energy, $\gamma(E) := \frac{D}{E^k}$, $D > 0$ and $k > 5/2$.



- $$\Rightarrow \partial_t \rho = \nabla \cdot \left(\Theta(k) \nabla (\rho^{\frac{k-5/2}{k-3/2}}) + \rho \nabla V \right).$$

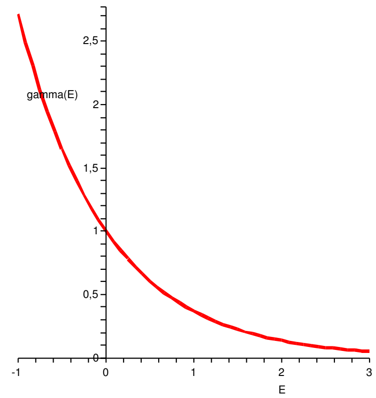
  Observe $0 < \frac{k-5/2}{k-3/2} < 1$ and $\nu'(\rho) = \Theta \frac{2k-5}{2k-3} \rho^{\frac{-1}{k-\frac{3}{2}}} \xrightarrow{\rho \to 0} \infty$.

- Sufficient confinement of the potential

$$V(x) \geq C|x|^q, \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > \frac{3}{k - \frac{5}{2}}.$$

Ex. 2, borderline case

- Maxwell distribution $\gamma(E) = \exp(-E)$ leads to the linear kinetic BGK model (simplified version).



- $\Rightarrow$ Linear drift-diffusion equation

$$\partial_t \rho = \nabla \cdot \big(\nabla \rho + \rho \nabla V\big).$$

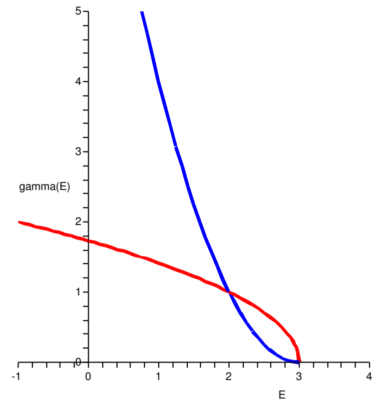
- $\nu(\rho) = \rho$ and the diffusivity $\nu'(\rho) \equiv 1$.
- Growth of the potential

$$V(x) \geq q \log(|x|), \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > 3.$$

Ex. 3, porous medium case

- Cut-off power as Gibbs state:

$$\gamma(E) = (E_2 - E)_+^k \,, \quad k > 0$$



- $\Rightarrow$ Porous medium equation

$$\partial_t \rho = \nabla \cdot \left(\Theta(k) \nabla (\rho^{\frac{k+5/2}{k+3/2}}) + \rho \nabla V \right)$$

$1 < \frac{k+5/2}{k+3/2} < \frac{5}{3}$ and $\nu'(\rho) = \Theta \frac{2k+5}{2k+3} \rho^{\frac{1}{k+\frac{3}{2}}} \xrightarrow{\rho \to 0} 0.$

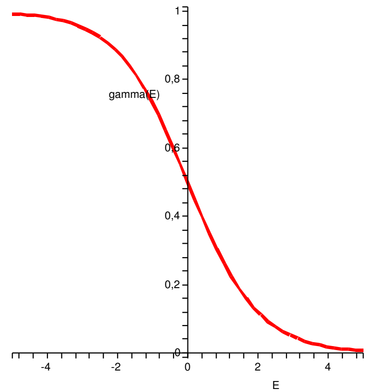
- Potential (μ^* is the upper bound for the Fermi enery)

$\left(E_2 + \mu^* - V(x)\right)_+ = \mathcal{O}\left(\frac{1}{|x|^q}\right)$ a.e., $q > \frac{3}{k+3/2}$ as $|x| \to \infty$

Ex. 4, Fermi-Dirac statistics

For the Fermi-Dirac distribution

$$\gamma(E) = \frac{1}{\exp(E) + \alpha}$$



we obtain $\quad \partial_t \rho = \nabla \cdot \big(D(\rho) \nabla \rho + \rho \nabla V \big).$

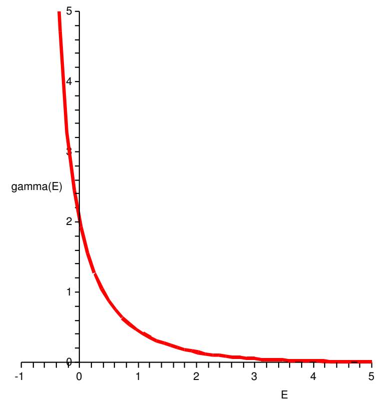
$$\begin{aligned} D(\rho) &= \nu'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}} \frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1}) (\frac{-\alpha\rho}{(2\pi)^{3/2}}) \big)} \\ &= 1 + \frac{\sqrt{2}}{4} \frac{\alpha\rho}{(2\pi)^{3/2}} + \mathcal{O}(\rho^2) \,, \quad \text{as} \quad \rho \to 0 \,. \end{aligned}$$

with the polylogarithmic function $\mathrm{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n}$.

Ex. 5, Bose-Einstein statistics

For the Bose-Einstein distribution

$$\gamma(E) = \frac{1}{\exp(E) - \alpha}$$



the diffusivity is given by

$$\begin{aligned} D(\rho) &= \nu'(\rho) = \frac{+\alpha}{(2\pi)^{3/2}} \frac{\rho}{\mathrm{Li}_{1/2}\left((\mathrm{Li}_{3/2}^{-1})(\frac{+\alpha\rho}{(2\pi)^{3/2}})\right)} \\ &= 1 - \frac{\sqrt{2}}{4} \frac{\alpha\rho}{(2\pi)^{3/2}} + \mathcal{O}(\rho^2) \,, \quad \text{as} \quad \rho \to 0 \,. \end{aligned}$$

The maximal density $\bar{\rho}$ is given by $\bar{\rho} = \frac{(2\pi)^{3/2}\zeta(\frac{3}{2})}{\alpha}$. (Riemann Zeta function $\zeta(s) := \mathrm{Li}_s(1) = \sum_{k=1}^{\infty} \frac{1}{k^s}$).

Observe: $\lim_{\rho\to\bar{\rho}} \nu'(\rho) = 0$ and $\lim_{\rho\to 0} \nu'(\rho) = 1$

Extension

- An extended model with local energy conservation:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f =$$
$$= \gamma \left(\alpha_f(x,t) \left(\frac{1}{2}|v|^2 + V(x) \right) + \mu_f(x,t) \right) - f \ ,$$

  where the parameter functions $\mu_f(x,t)$ and $\alpha_f(x,t)$ are adjusted to the position density and to the energy density of f.

- The diffusion limit of this equation is an energy transport model, see [Degond, Génieys, Jüngel, 1997].