Asymptotic behaviour for the Vlasov-Poisson System in the stellar dynamics case

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Consider (VP)

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \phi + \nabla_x \phi_0) \cdot \nabla_v f = 0 \\ \\ \Delta_x \phi = 4\pi\gamma\rho \,, \quad \lim_{|x|\to\infty} \phi(t,x) = 0 \end{cases} \tag{VP}$$

in presence of an external potential ϕ_0. If we look for stationary solutions taking the form

$$f(x,v) = g\left(\frac{1}{2}|v|^2 + \phi(x) + \phi_0(x) - \kappa\right)$$

Vlasov's equation is satisfied and the problem is reduced to solve the nonlinear Poisson equation

$$\Delta\phi = 4\pi\gamma\, G(\phi + \phi_0 - \kappa)$$

with

$$G(u) := \int_{\mathbb{R}^3} g\left(\frac{1}{2}|v|^2 + u\right)\, dv = \frac{4\pi}{3} \int_0^{+\infty} g(s+u)\sqrt{2\,s}\, ds$$

An example: let $g(u) := (2\pi)^{-3/2} e^{-u}$ and $M > 0$ fixed. The distribution function f is a gaussian and

$$\Delta\phi = 4\pi\gamma\, e^{\kappa}\, e^{-\phi-\phi_0} \quad \text{for} \quad 4\pi\, e^{\kappa} \int_{\mathbb{R}^3} e^{-\phi-\phi_0}\, dx = M$$

For $\gamma = \pm 1$, $\varphi = \pm\phi$, $F(u) = e^{\mp u}$, $e^{-\phi_0} = \chi_\Omega$, we have to solve

$$\Delta\varphi = M\, \frac{F(\varphi)}{\int_\Omega F(\varphi)\, dx}$$

Plasma case, $\gamma = -1$: if $\mathcal{G}' = G$, then ϕ is a critical point of

$$\phi \mapsto \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\phi|^2\, dx + \int_{\mathbb{R}^3} \mathcal{G}(\phi + \phi_0 - \kappa[\phi + \phi_0])\, dx - M\, \kappa[\phi + \phi_0]$$

which is convex if g is nonincreasing

The parameter κ is a functional of ϕ, implicitly defined by

$$4\pi\gamma \int_{\mathbb{R}^3} G(\phi + \phi_0 - \kappa[\phi + \phi_0])\, dx = M$$

where $M > 0$ is the mass of the solution: $M = \int_{\mathbb{R}^6} f(x,v)\, dxdv$
The solution is unique if one assumes that $g \searrow$ ($\mathcal{G}$ convex)

Dual approach: look for a critical point of the functional

$$f \mapsto \int_{\mathbb{R}^6} \left(\frac{1}{2}|v|^2 + \phi(x) + \phi_0(x) - \kappa\right) f(x,v)\, dxdv + \int_{\mathbb{R}^6} H(f(x,v))\, dxdv$$

where $H'(\cdot) = g^{(-1)}(-\cdot)$ and κ now appears as the Lagrange multiplier associated to the mass constraint.

$\gamma = -1$: [Batt, Morrison, Rein], [Braasch, Rein, Vukadinović], [Cáceres, Carrillo, J.D.]
$\gamma = +1$: [Wolansky], [Rein], [Guo], [J.D., Sánchez, Soler], [Sánchez, Soler]

The three-dimensional *Vlasov-Poisson system* can be written in terms of a distribution function $f : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^+ \cup \{0\}$ and the corresponding mass density $\rho(x,t) := \int_{\mathbb{R}^3} f(t,x,v)\, dv$ as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = 0 \\ f(t=0,x,v) = f_0(x,v) \\ \Delta_x \phi = 4\pi\gamma\rho\,, \quad \lim_{|x|\to\infty} \phi(t,x) = 0 \end{cases} \qquad (VP)$$

where $\gamma = +1$ corresponds to the gravitational case and $\gamma = -1$ to the plasma physical case.

Polytropic gas spheres ($\gamma = +1$):

$$f(x,v) = (E_0 - |v|^2/2 - \phi(x))_+^{\mu}\, |x \times v|^{2k}$$

Existence results

Classical solutions: K. Pfaffelmoser, Schaeffer, R. Glassey] Take f_0 with compact support and control the growth of the size of the support:

$$Q(t) = 1 + \sup\Big\{|v| : \exists (t,x) \in (0,t) \times \mathbb{R}^3 s.t.\ f(t,x,v) \neq 0\Big\}$$

Then the Cauchy problem for (VP) has a unique C^1 solution and $Q(t) \ \leq \ C_p(1+t)^p$ with $p > \frac{33}{17}$

$\Longrightarrow$Strong solutions: $f_0 \in L^1 \cap L^\infty$, finite energy

Weak solutions: [Horst, Hunze] $f_0 \in L^1 \cap L^p$, $p > (12+3\sqrt{5})/11$, finite energy

Renormalized solutions: [DiPerna, Lions] $f_0 \in L^1$, $f_0 \log f_0 \in L^1$, finite energy

Other results using moments [Perthame, Lions], [Castella]

1. Optimal bounds for the kinetic and potential energies

Total energy associated to f:

$$E(f) := E_{KIN}(f) - \gamma\, E_{POT}(f)$$

Kinetic energy:

$$E_{KIN}(f) := \frac{1}{2}\int_{\mathbb{R}^6} |v|^2\, f(t,x,v)\, dx\, dv$$

Potential energy :

$$E_{POT}(f) := \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx$$

The potential ϕ associated to f is given by the Poisson equation:

$$\phi = -\frac{\gamma}{|\cdot|} * \int_{\mathbb{R}^3} f(\cdot, v)\, dv$$

For a smooth solution the total energy remains constant. Total mass is also preserved

$$\|f(t,\cdot,\cdot)\|_{\mathsf{L}^1(\mathbb{R}^6)} = \int_{\mathbb{R}^6} f(t,x,v)\,dx\,dv$$

The transport also preserves uniform bounds :

$$\|f(t,\cdot,\cdot)\|_{\mathsf{L}^\infty(\mathbb{R}^6)} \le \|f(0,\cdot,\cdot)\|_{\mathsf{L}^\infty(\mathbb{R}^6)} \,.$$

Functional setting: for any $M > 0$, let

$$\Gamma_M = \{f \in \mathsf{L}^1 \cap \mathsf{L}^\infty(\mathbb{R}^6) \;:\; f(x,v) \ge 0\,,\ \|f\|_{\mathsf{L}^1(\mathbb{R}^6)} = M\,,\ \|f\|_{\mathsf{L}^\infty(\mathbb{R}^6)} \le 1\}$$

and consider $E_M := \inf\{E(f) \;:\; f \in \Gamma_M\}$

Let $\gamma = 1$ (gravitational case). For any $E \ge E_M$

$$K_\pm(E,M) := -2E_M \left(1 - \frac{E}{2E_M} \pm \sqrt{1 - \frac{E}{E_M}}\right)$$

$$P_\pm(E,M) := -2E_M \left(1 \pm \sqrt{1 - \frac{E}{E_M}}\right)$$

Theorem 1 *E_M is negative, bounded from below and for any $f \in \Gamma_M$, with $E = E(f)$, the following properties hold:*

(i) $E_{KIN}(f) \in \Big[\, K_-(E,M),\ K_+(E,M) \,\Big]$

(ii) $E_{POT}(f) \in \Big[\, \max\{0, P_-(E,M)\},\ P_+(E,M) \,\Big]$

(iii) $E_{POT}(f) \in \Big[\, 0,\ \sqrt{-4E_M E_{KIN}(f)} \,\Big]$

Moreover, there exist functions which minimize the energy on Γ_M. They are stationary solutions to the VP system for which $E = E_M$ and

$$K_\pm(E,M) = E_{KIN}(f) = \frac{1}{2} E_{POT}(f) = P_\pm(E,M)$$

A potential energy estimate

Lemma 1 *There exists a positive constant* C *such that* $\forall\, f \in \mathsf{L}^1_+ \cap \mathsf{L}^\infty(\mathbb{R}^6)$ *with* $|v|^2 f \in L^1(\mathbb{R}^6)$

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx \le C\, \|f\|^{7/6}_{\mathsf{L}^1(\mathbb{R}^6)}\, \|f\|^{1/3}_{\mathsf{L}^\infty(\mathbb{R}^6)} \left(\int_{\mathbb{R}^6} |v|^2 f(x,v)\, dx\, dv\right)^{1/2}$$

Proof. From the definition of ϕ we have

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx = \int_{\mathbb{R}^3} (-\Delta\phi)\,\phi\, dx = 4\,\pi \int_{\mathbb{R}^6} \frac{\rho(y)\rho(x)}{|x-y|}\, dx\, dy$$

According to the Hardy-Littlewood-Sobolev inequalities,

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\, dx \le 4\pi\,\Sigma\, \|\rho\|^2_{\mathsf{L}^{\frac{6}{5}}(\mathbb{R}^3)}$$

Because of Hölder's inequality, $\|\rho\|_{\mathsf{L}^{6/5}(\mathbf{R}^3)} \le \|\rho\|^{7/12}_{\mathsf{L}^1(\mathbf{R}^3)} \|\rho\|^{5/12}_{\mathsf{L}^{5/3}(\mathbf{R}^3)}$

Interpolation of the $L^{5/3}$-norm of ρ

$$\int_{\mathbb{R}^3} |\rho|^{5/3}\, dx \le C\, \|f\|_{L^\infty(\mathbb{R}^6)}^{2/3} \int_{\mathbb{R}^6} |v|^2\, f(x,v)\, dx\, dv$$

For any $R > 0$,

$$\rho(x,t) := \int_{\mathbb{R}^3} f(t,x,v)\, dv = \int_{|v|\le R} f(t,x,v)\, dv + \int_{v \ge R} f(t,x,v)\, dv$$

$$\rho(x,t) := \int_{\mathbb{R}^3} f(t,x,v)\, dv \le \frac{4\pi}{3}\, R^3\, \|f\|_{L^\infty(\mathbb{R}^6)} + \frac{1}{R^2} \int_{\mathbb{R}^3} |v|^2\, f(t,x,v)\, dv$$

Optimize on $R = R(t,x)$ for t, x fixed: $R^5 = \frac{1}{2\pi} \frac{\int_{\mathbb{R}^3} |v|^2\, f(t,x,v)\, dv}{\|f\|_{L^\infty(\mathbb{R}^6)}}$.

$$\rho(x,t) \le C\, \|f\|_{L^\infty(\mathbb{R}^6)}^{2/5} \left(\int_{\mathbb{R}^3} |v|^2\, f(t,x,v)\, dv \right)^{3/5} \qquad \square$$

An equivalent minimization problem

Let $J_M = \inf\{J(f) \ : \ f \in \Gamma_M\}$,

$$J(f) = \frac{\frac{1}{2}\int_{\mathbb{R}^6} |v|^2 f \, dx \, dv}{\left(\frac{1}{8\pi}\int_{\mathbb{R}^6} |\nabla\phi|^2 \, dx\right)^2} \equiv \frac{E_{KIN}(f)}{(E_{POT}(f))^2}$$

Lemma 2 *The minimization problems $E(f) = E_M$ and $J(f) = J_M$ over the set Γ_M are equivalent*
(i) *Their respective minima satisfy*

$$4\, J_M\, E_M = -1$$

(ii) *If $f_M \in \Gamma_M$ is a minimizer of the functional E, then it is also a minimizer of the functional J.*

If $J(g_M) = J_M$ for some $g_M \in \Gamma_M$, then $E(g_M^\sigma) = E_M$ where $g_M^\sigma(x,v) := g_M(\sigma x, v/\sigma)$ and $\sigma = \frac{E_{POT}(g_M)}{2E_{KIN}(g_M)}$

Proof. The set Γ_M is stable under $f \mapsto f^\sigma(x,v) = f(\sigma x, v/\sigma)$, $\sigma > 0$. Optimize on σ:

$$E(f^\sigma) = \sigma^2 E_{KIN}(f) - \sigma\, E_{POT}(f)$$

$$\sigma_{min} := \frac{E_{POT}(f)}{2\, E_{KIN}(f)}\,, \quad E(f) \geq E(f^{\sigma_{min}}) = -\frac{1}{4}\frac{(E_{POT}(f))^2}{E_{KIN}(f)} = -\frac{1}{4J(f)}$$

Proof of Assertions of (i)-(iii) of Theorem 1.

$$E := E(f) = E_{KIN} - E_{POT} \quad \text{and} \quad \frac{E_{KIN}(f)}{(E_{POT}(f))^2} = J(f) \geq -\frac{1}{4\, E_M}$$

$$-\frac{E_{POT}(f))^2}{4\, E_M} - E_{POT}(f) \leq E\,, \quad (E_{KIN} - E)^2 \leq -4\, E_M\, E_{KIN}$$

Existence of minimizers ?

Corollary 2 *Let f_M be a minimizing function for the functional E on Γ_M. Then*

$$E_{POT}(f_M) = 2\,E_{KIN}(f_M) = -2\,E_M$$

Note that $E_M = -E_{KIN}(f_M) = -\frac{1}{2}\,E_{POT}(f_M) < 0$
This property is shared with any stationary solution.

Nonincreasing symmetric rearrangements

The symmetric rearrangement A^* of the Borel set $A \subset \mathbb{R}^n$ is the open ball in $\mathbb{R}^n$ centered at the origin whose volume is that of A:

$$\chi_A^* := \chi_{A^*} = \begin{cases} 1 & \text{if } \frac{1}{n}|S^{n-1}|\,|x|^n \le \|\chi_A\|_{\mathsf{L}^1(\mathbb{R}^n)} = |A| \\ 0 & \text{otherwise} \end{cases}$$

Let $h : \mathbb{R}^n \to \mathbb{C}$ be a Borel measurable function:

$$h^*(x) := \int_0^\infty \chi^*_{\{|h|>t\}}(x)\, dt$$

Partial symmetric nonincreasing rearrangement with respect to the x variable only:

$$g^{*x}(x,v) := \int_0^\infty \chi^*_{\{x\in\mathbb{R}^n\,:\,g(x,v)>t\}}\, dt$$

Elementary properties: use Fubini's theorem

$$\begin{aligned}
\int_{\mathbb{R}^{2n}} g^{*x}(x,v)\,dx\,dv &= \int_{\mathbb{R}^{2n}} g(x,v)\,dx\,dv \\
\|g^{*x}\|_{L^\infty(\mathbb{R}^{2n})} &= \|g\|_{L^\infty(\mathbb{R}^{2n})} \\
\int_{\mathbb{R}^{2n}} |v|^2 g^{*x}(x,v)\,dx\,dv &= \int_{\mathbb{R}^{2n}} |v|^2 g(x,v)\,dx\,dv \\
\int_{\mathbb{R}^n} g^{*x}(x,v)\,dv &= \int_{\mathbb{R}^n} g(x',v)\,dv \quad \text{if } |x| = |x'| \\
\int_{\mathbb{R}^n} g^{*x}(|x|,v)\,dv &\geq \int_{\mathbb{R}^n} g^{*x}(|y|,v)\,dv \quad \text{if } |x| \leq |y| \\
\int_{\mathbb{R}^n} \psi(|x|)\,g^{*x}(|x|,v)\,dv &< \int_{\mathbb{R}^n} \psi(|x|)\,g(x,v)\,dv \quad \text{for } \psi \nearrow
\end{aligned}$$

Theorem 3 [Riesz' theorem]

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(x)\,g(x-y)\,h(y)\,dx\,dy =: I(f,g,h) \leq I(f^*,g^*,h^*)$$

If g is radially symmetric and strictly decreasing, equality holds only if $\exists\, y \in \mathbb{R}^n$ $f(x) = f^*(x-y)$ and $h(x) = h^*(x-y)$

Spherical symmetry and regularity of the potential

Lemma 3 *Let $M > 0$. There exists a minimizing sequence $(f_n)_{n\in\mathbb{N}} \in \Gamma_M^{\mathbb{N}}$ of the functional E such that $\rho_n(x) = \int_{\mathbb{R}^3} f_n(x,v)\,dv$ is a radial nonincreasing function.*

$$\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx \;=\; 4\,\pi \int_{(\mathbf{R}^3)^4} \frac{f(x,v)\,f(x',v')}{|x-x'|}\,dx\,dx'\,dv\,dv'$$

$$\leq\; 4\,\pi \int_{(\mathbf{R}^3)^4} \frac{f^{*x}(x,v)\,f^{*x}(x',v')}{|x-x'|}\,dx\,dx'\,dv\,dv'$$

Lemma 4 *Let $\rho \in \mathsf{L}^1(\mathbb{R}^3) \cap \mathsf{L}^{5/3}(\mathbb{R}^3)$ be a spherically symmetric function. Then $\phi \in W^{2,5/3}_{\mathrm{loc}}(\mathbb{R}^3)$ and $\exists\,\eta > 0$, $\forall\, R > 0$*

$$\int_{|x|<R} |\nabla\phi|^{2+\eta}\,dx \leq C(M,R)\left(\int_{|x|<R} \rho^{5/3}\,dx + 1\right)$$

Poisson equation

$$\Delta\phi = 4\,\pi\,\rho\,, \qquad \lim_{|x|\to+\infty} \phi(x) = 0$$

ϕ is radial, nondecreasing and strictly increasing in the interior of the support of ρ if $\rho \in L^1_+$ is radial. Let $\int_{\mathbb{R}^3} \rho(x)\,dx =: M > 0$

(i) For any $r > 0$,

$$\phi'(r) \le \frac{M}{r^2} \quad \text{and} \quad -\frac{M}{r} \le \phi(r) \le 0$$

(ii) For any $R > 0$,

$$\int_{|x|\ge R} |\nabla\phi(x)|^2\,dx \le 4\,\pi\,\frac{M^2}{R}$$

Lemma 5 [Scaling] *[Guo, Rein]*

$$E_M = M^{7/3}\, E_1$$

Lemma 6 [Splitting] *[Guo, Rein] Let $f \in \Gamma_M$ be such that the mass density $\rho(x) = \int_{\mathbb{R}^3} f(x,v)\, dv$ is spherically symmetric. Given $R > 0$, if $M - \lambda = \int_{|x|<R} \int_{\mathbb{R}^3} f(x,v)\, dv\, dx$ for some $\lambda \in [0, M]$, then*

$$E(f) - E_M \geq -\left(\frac{7}{3} \frac{E_M}{M^2} + \frac{1}{4\,\pi\, R} \right) (M - \lambda)\, \lambda$$

Proof. Let $\phi = \phi_1 + \phi_2$, with

$$\Delta \phi_1(x) = \int_{\mathbb{R}^3} \chi_{B_R}(x)\, f(x,v)\, dv\ , \quad \Delta \phi_2(x) = \int_{\mathbb{R}^3} (1 - \chi_{B_R}(x))\, f(x,v)\, dv$$

$$
\begin{aligned}
E(f) &= E_{KIN}(\chi_{B_R} f) + E_{KIN}((1-\chi_{B_R})f) \\
&\quad -\frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_1|^2\,dx - \frac{1}{8\pi}\int_{\mathbb{R}^3} |\nabla\phi_2|^2\,dx - \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\phi_1\cdot\nabla\phi_2\,dx \\
&= E(\chi_{B_R} f) + E((1-\chi_{B_R})f) - \frac{1}{4\pi}\int_{\mathbb{R}^3} \nabla\phi_1\cdot\nabla\phi_2\,dx \\
&\geq E_{M-\lambda} + E_\lambda + \frac{1}{4\pi}\int_{\mathbb{R}^3} \phi_2\,\Delta\phi_1\,dx \\
&\geq \left[\left(1-\frac{\lambda}{M}\right)^{\frac{7}{3}} + \left(\frac{\lambda}{M}\right)^{\frac{7}{3}} - 1\right] E_M + \frac{1}{4\pi}\int_{\mathbb{R}^3} \phi_2\,\Delta\phi_1\,dx
\end{aligned}
$$

1) $\forall\, x \in [0,1],\ (1-x)^{7/3} + x^{7/3} - 1 \leq -\frac{7}{3}\,x\,(1-x)$

2) $\left|\frac{1}{4\pi}\int_{\mathbb{R}^3} \phi_2\,\Delta\phi_1\,dx\right| \leq \|\phi_2\|_{L^\infty(\mathbb{R}^3)}\,(M-\lambda)$

where $\|\phi_2\|_{L^\infty(\mathbb{R}^3)} = |\phi_2(0)| = |\phi_2(R)| \leq \frac{\lambda}{4\,\pi\,R}$ $\square$

Lemma 7 [No vanishing] *Let* $R_0 > \frac{3\,M^2}{28\,\pi\,|E_M|}$, $(f_n)_{n\in\mathbb{N}} \in \Gamma_M^{\mathbb{N}}$ *a minimizing sequence as in Lemma 3. Then*

$$\limsup_{n\to\infty} \int_{|x|\geq R_0} \int_{\mathbb{R}^3} f_n \, dv \, dx = 0$$

Proof. By contradiction: $\lim_{n\to\infty} \int_{|x|\geq R_0} \int_{\mathbb{R}^3} f_n \, dv \, dx = \lambda > 0$.

$$\exists\, R_n > R_0 \quad \frac{\lambda}{2} = \int_{|x|\geq R_n} \int_{\mathbb{R}^3} f_n \, dv \, dx$$

Apply now Lemma 6 to each f_n with $R = R_n$:

$$\begin{aligned} E(f_n) - E_M \;&\geq\; -\left(\frac{7\,E_M}{3\,M^2} + \frac{1}{4\pi R_n}\right)\left(M - \frac{\lambda}{2}\right)\frac{\lambda}{2} \\ &\geq\; -\left(\frac{7\,E_M}{3\,M^2} + \frac{1}{4\pi R_0}\right)\left(M - \frac{\lambda}{2}\right)\frac{\lambda}{2} > 0 \end{aligned}$$

□

Convergence of a minimizing sequence

Proposition 8 *Let* $(f_n)_{n\in\mathbb{N}} \in \Gamma_M^{\mathbb{N}}$ *be a minimizing sequence for the functional* E*, with radial nonincreasing mass densities. Up to a subsequence, the sequence converges to a minimizer* $f_M \in \Gamma_M$ *s.t.* $E_M = E(f_M)$, $\mathrm{supp}\,(f_M) \subset B_{R_0} \times \mathbb{R}^3$ *where* $R_0 = \frac{3\,M^2}{28\,\pi\,|E_M|}$

Proof. Lemma 7 $\Rightarrow \lim_{n\to\infty} \int_{B_{R_0}} \int_{\mathbb{R}^3} f_n\,dv\,dx = M$. Dunford-Pettis:

(i) [boundedness] $(f_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{L}^1(\mathbb{R}^6)$

(ii) [no concentration] for any measurable set A

$$\int_A f_n\,dx\,dv \;\le\; \|f_n\|_{\mathrm{L}^\infty(\mathbb{R}^6)}\,|A| \;\le\; |A|$$

(iii) [no vanishing] for any K_1, K_2, either $K_1 \ge R_0$ or

$$\int_{|x|>K_1}\int_{|v|>K_2} f_n\,dx\,dv \le \int_{\mathbb{R}^3}\int_{|v|>K_2} f_n\,dx\,dv \le \frac{1}{K_2^2}\,E_{KIN}(f_n) \qquad \square$$

2. Solutions of the VP system with minimal energy

Theorem 4 *Let f_M be a minimizing function for the functional E on Γ_M, with radial mass density. Then*

$$f_M(x,v) = \begin{cases} 1 & \textit{if} \quad \frac{1}{2}|v|^2 + \phi_{f_M}(x) < \frac{7}{3}\frac{E(f_M)}{M} \\ \\ 0 & \textit{otherwise} \end{cases}$$

where ϕ_{f_M} is the unique radial solution on $\mathbb{R}^3$ of

$$\Delta \phi_{f_M} = \frac{1}{3}(4\pi)^2 \left[2 \left(\frac{7}{3}\frac{E_M}{M} - \phi_{f_M} \right)_+ \right]^{3/2}$$

It is the unique minimizer with radial mass density and it is also a steady-state solution to the VP system. Moreover, if f is another minimizing function, then with $\bar{x} := \frac{1}{M}\int_{\mathbb{R}^6} x\, f(x,v)\, dx\, dv$

$$f(x,v) = f_M(x - \bar{x}, v) \quad \forall\, (x,v) \in \mathbb{R}^6$$

Explicit form of the minimizers

Lemma 9

$$f_M(x,v) = \begin{cases} 1 & \textit{for } (x,v) \textit{ such that } |v| \le \left(\frac{3}{4\pi}\,\rho_M(x)\right)^{1/3} \textit{a.e.} \\ 0 & \textit{otherwise} \end{cases}$$

Proof. 1) $\|f_M\|_{\mathrm{L}^\infty(\mathbb{R}^6)} = 1$ otherwise let:

$$\bar{f}(x,v) = \kappa f(\kappa^{2/3} x, \kappa^{-1/3} v)$$

$$\|\bar{f}\|_{\mathrm{L}^1(\mathbb{R}^6)} = \|f\|_{\mathrm{L}^1(\mathbb{R}^6)}\,, \quad \|\bar{f}\|_{\mathrm{L}^\infty(\mathbb{R}^6)} = \kappa\,\|f\|_{\mathrm{L}^\infty(\mathbb{R}^6)}\,, \quad E(\bar{f}) = \kappa^{2/3}\,E(f)$$

for $\kappa = \|f_M\|^{-1}_{\mathrm{L}^\infty(\mathbb{R}^6)} > 1$, a contradiction

2) Let $\epsilon \in (0,1)$ and $g(x,v) \in \mathsf{L}^1(\mathbb{R}^6) \cap \mathsf{L}^\infty(\mathbb{R}^6)$ be a test function such that $g \geq 0$ a.e. in $\mathbb{R}^6 \setminus \text{supp}\,(f_M)$, with compact support contained inside

$$(\text{supp}\,(f_M) \setminus S_\epsilon)^c \equiv \left(\mathbb{R}^6 \setminus \text{supp}\,(f_M)\right) \cup S_\epsilon$$

where $S_\epsilon = \{(x,v) \in \mathbb{R}^6 \ : \ \epsilon \leq f_M(x,v) \leq 1-\epsilon\}$. Let

$$g(t) := M \frac{tg + f_M}{\|tg + f_M\|_{\mathsf{L}^1(\mathbb{R}^6)}}, \qquad 0 < t < T := M\,\epsilon \left(M\|g\|_{\mathsf{L}^\infty} + \|g\|_{\mathsf{L}^1}\right)^{-1}$$

$$\begin{array}{llll} 0 \leq -T\|g\|_{\mathsf{L}^\infty(\mathbb{R}^6)} + \epsilon & \leq tg + f_M & \text{in } S_\epsilon \\ 0 \leq \quad f_M & = tg + f_M & \text{in supp}\,(f_M) \setminus S_\epsilon \\ 0 \leq \quad tg & = tg + f_M & \text{in } \mathbb{R}^6 \setminus \text{supp}\,(f_M) \end{array}$$

$$\left.\begin{array}{r} \|g(t)\|_{\mathsf{L}^\infty(S_\epsilon)} \leq M \frac{T\|g\|_{L^\infty(\mathbb{R}^6)} + 1 - \epsilon}{M - T\|g\|_{L^1(\mathbb{R}^6)}} = 1 \\ \|g(t)\|_{\mathsf{L}^1(\mathbb{R}^6)} = M \end{array}\right\} \Rightarrow g(t) \in \Gamma_M$$

Taylor expansion at $t = 0_+$

$$g(t) - f_M = t\, g'(0) + \frac{t^2}{2}\, g''(\theta) = t \left(g - \frac{1}{M} \left[\int_{\mathbb{R}^6} g\; dx\, dv \right] f_M \right) + \frac{t^2}{2}\, g''(\theta)$$

$$E(g(t)) - E_M = t \int_{\mathbb{R}^6} \left(\frac{1}{2}\, |v|^2 + \phi_{f_M} - \frac{3E_M}{M} \right) g\; dx\, dv + O(t^2)$$

Since f_M minimizes $E(\cdot) - E_M$ on Γ_M, we have that $E(g(t)) - E_M \geq 0$ for any $t \in [0, T]$ and consequently

$$\int_{\mathbb{R}^6} \left(\frac{1}{2}\, |v|^2 + \phi_{f_M} - \frac{3E_M}{M} \right) g\; dx\, dv \geq 0$$

for every g and ϵ.

(i) $g \geq 0$ on $\mathbb{R}^6 \setminus \mathrm{supp}\,(f_M)$:

$$\frac{1}{2}|v|^2 + \phi_{f_M}(x) \geq \frac{3E_M}{M} \quad \forall\,(x,v) \in \mathbb{R}^6 \setminus \mathrm{supp}\,(f_M)$$

or equivalently

$$\left\{(x,v) \in \mathbb{R}^6 \ : \ \frac{1}{2}|v|^2 + \phi_{f_M}(x) \leq \frac{3E_M}{M}\right\} \subset \mathrm{supp}\,(f_M)$$

(ii) On the other hand, g has no determined sign on S_ϵ

$$\frac{1}{2}|v|^2 + \phi_{f_M}(x) = \frac{3E_M}{M} \quad \forall\,(x,v) \in S_\epsilon \cap \mathrm{supp}\,(f_M)$$

This set and S_ϵ and have 0 Lebesgue measure:

$$f_M \equiv 1 \quad \text{on} \quad \mathrm{supp}\,(f_M)$$

3) f_M minimizes $\{E(f) \;:\; f \in \gamma_M\}$ where

$$\gamma_M = \left\{ f \in \Gamma_M \;:\; f \equiv 1 \text{ a.e. on supp}(f)\,,\; \int_{\mathbb{R}^3} f(x,v)\,dv = \rho_M(x) \;\; \forall\, x \in \mathbb{R}^3 \right\}$$

The problem is therefore reduced to the minimization of

$$\{E_{KIN}(f) \;:\; f \in \gamma_M\}$$

$$\int_{\mathbf{R}^3} |v|^2 f^{*v}\,dv \le \int_{\mathbf{R}^3} |v|^2 f\,dv$$

with a strict inequality unless $f \equiv f^{*v}$ a.e. Thus

$$f_M \equiv \chi_{|v| \le \left(\frac{3}{4\pi}\rho_M(x)\right)^{1/3}} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}^3 \text{ a.e.}$$

$\square$

Lemma 10 *Let f_M be a minimizing function for E on Γ_M with radial mass density. Then $\rho_M = \int_{\mathbb{R}^3} f_M(\cdot, v)\, dv$ and $\phi_{f_M} = |\cdot|^{-1} * \rho_M$ are related by*

$$\rho_M(x) = \begin{cases} \frac{4\pi}{3}\left[2\left(\frac{7}{3}\frac{E_M}{M} - \phi_{f_M}(x)\right)\right]^{3/2} & \text{if} \quad \phi_{f_M}(x) \leq \frac{7}{3}\frac{E_M}{M} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Let $\rho(x) \in \mathrm{L}^1(\mathbb{R}^3) \cap \mathrm{L}^{5/3}(\mathbb{R}^3)$ be a nonnegative function such that $\|\rho\|_{\mathrm{L}^1(\mathbb{R}^3)} = M$ and define $f_\rho(x,v) := \chi_{\left\{|v| \leq \left(\frac{3}{4\pi}\rho(x)\right)^{1/3}\right\}}$

$$E_{POT}(f_\rho) = \frac{1}{2}\int_{\mathbb{R}^6} \frac{\rho(y)\,\rho(x)}{|x-y|}\, dx\, dy$$

$$E_{KIN}(f_\rho) = \frac{3^{5/3}}{10\,(4\pi)^{2/3}}\int_{\mathbb{R}^3} \left[\rho(x)\right]^{5/3} dx$$

and write the Euler-Lagrange equations.

Lemma 11 *ϕ_{f_M} is unique and continuously differentiable. If f is another minimum of E on Γ_M, then there exists $y \in \mathbb{R}^3$ such that*

$$\int_{\mathbb{R}^3} f(x,v)\,dv = \rho_M(x-y) \quad a.e. \quad x \in \mathbb{R}^3\,.$$

Proof. Rewrite the Poisson equation for ϕ_{f_M}:

$$\Delta\phi_{f_M} = 4\pi\rho_M(x) = \begin{cases} \frac{1}{3}(4\pi)^2\left[2\left(\frac{7}{3}\frac{E_M}{M} - \phi_{f_M}\right)\right]^{3/2} & \text{if } \phi_{f_M}(x) \le \frac{7}{3}\frac{E_M}{M} \\ 0 & \text{otherwise} \end{cases}$$

with $w(r) := \frac{7}{3}\frac{E_M}{M} - \phi_{f_M}(r/\sqrt{c})$, $c = \frac{1}{3}32\sqrt{2}\,\pi^2$:

$$(r^2 w'(r))' + r^2 w_+^{3/2}(r) = 0$$

At $R = \frac{3M^2}{7|E_M|}\sqrt{c}$, $w(R) = 0$ and $w'(R) = -\frac{1}{\sqrt{c}}\phi'_{f_M}\left(\frac{R}{\sqrt{c}}\right) = -\frac{M\sqrt{c}}{R^2}$

Nonlinear stability for the evolution problem

We follow the strategy of Guo in [Guo99]. Consider for any g, $h \in \Gamma_M$ the distance d defined by

$$d(g,h) = E(g) - E(h) + \frac{1}{4\pi} \|\nabla \phi_g - \nabla \phi_h\|^2_{L^2(\mathbb{R}^3)}$$

Theorem 5 *For every $\epsilon > 0$, there exists a $\delta > 0$ such that, if f is a solution of the VP system with an initial condition $f_0 \in \Gamma_M$, then*

$$d(f_0, f_M) \leq \delta \quad \Longrightarrow \quad d(f^*(t), f_M) \leq \epsilon \quad \forall\, t \geq 0$$

Proof. The result is easily achieved by contradiction since $E(f^*(t)) - E(f_M) \leq E(f_0) - E(f_M) \searrow 0$ implies $\|\nabla \phi_{f^*(t)} - \nabla \phi_{f_M}\|_{L^2(\mathbb{R}^3)} \searrow 0$ $\square$

3. Large time behaviour

The *Galilean invariance* of a classical solution f to the VP system with initial data $f_0(x,v)$ means that for any $u \in \mathbb{R}^3$, the solution with initial data $f_0^u(x,v) = f_0(x, v-u)$ is given by

$$f^u(t,x,v) = f(t, x - t\,u, v-u) \quad \forall\,(t,x,v) \in (0,+\infty) \times \mathbb{R}^3 \times \mathbb{R}^3$$

Total momentum:

$$\langle v\rangle(f^u) := \int_{\mathbb{R}^6} v f^u(t,x,v)\,dx\,dv = \langle v\rangle(f) + u\,\|f(t)\|_{L^1(\mathbb{R}^6)}$$

Total energy:

$$E(f^u) = E(f) + u \cdot \langle v\rangle(f) + \frac{1}{2}\,|u|^2\,\|f\|_{L^1(\mathbb{R}^6)}\,.$$

Among the family $(f^u)_{u\in\mathbb{R}^3}$, $\min_{u\in\mathbb{R}^3} E_{KIN}(f^u)$ is reached by

$$E_{KIN}(f^{\bar u}) = E_{KIN}(f) - \frac{\langle v\rangle^2(f)}{2\,\|f\|_{L^1(\mathbb{R}^6)}} \quad \text{for } \bar u = -\frac{\langle v\rangle(f)}{\|f\|_{L^1(\mathbb{R}^6)}}$$

Galilean invariance and asymptotic behaviour

Lemma 12 $E(f) < \frac{1}{2} \frac{\langle v \rangle^2(f)}{\|f\|_{L^1(\mathbb{R}^6)}} \quad \Longrightarrow \quad E(f^{\bar{u}}) < 0$

Proposition 13 *Under the above condition, there exists three constants C_1, C_2, $C_3 > 0$ such that*

$$C_1 \leq E_{POT}(f(t,\cdot,\cdot)) \leq C_2$$
$$\|\rho_f(t,\cdot)\|_{L^{5/3}(\mathbb{R}^3)} \geq C_3$$

Variance and dispersion estimates The solutions to the VP system in the gravitational case have a qualitative behaviour which strongly differs from the behaviour in the plasma physics case since, for instance, stationary solutions exist. Assume that

$$E(f) > \frac{1}{2}\,\frac{\langle v\rangle^2(f)}{\|f\|_{\mathrm{L}^1(\mathbb{R}^6)}}$$

$$\begin{aligned} <(\Delta x)^2> \quad &:= \textstyle\int_{\mathbb{R}^6} |x|^2 f(t,x,v)\,dx\,dv - \left(\int_{\mathbb{R}^6} x f(t,x,v)\,dx\,dv\right)^2 \\ <(\Delta v)^2> \quad &:= \textstyle\int_{\mathbb{R}^6} |v|^2 f(t,x,v)\,dx\,dv - \left(\int_{\mathbb{R}^6} v f(t,x,v)\,dx\,dv\right)^2 \end{aligned}$$

Lemma 14

$$\frac{1}{2}\,\frac{d^2}{dt^2} <(\Delta x)^2> \quad = \quad E(f) + \frac{1}{2} <(\Delta v)^2> - \frac{1}{2}\,\langle v\rangle^2(f)$$

Proof. A straightforward calculation using the VP system gives

$$\begin{aligned}
\frac{1}{2}\frac{d^2}{dt^2}\int_{\mathbb{R}^6}|x|^2 f(t,x,v)\,dx\,dv &= \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^6}|x|^2\left(-v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f\right)\,dx\,dv \\
&= \int_{\mathbb{R}^6}(v\cdot x)\left(-v\cdot\nabla_x f+\nabla_x\phi\cdot\nabla_v f\right)\,dx\,dv \\
&= \int_{\mathbb{R}^6}|v|^2 f\,dx\,dv-\frac{1}{4\gamma\pi}\int_{\mathbb{R}^3}(x\cdot\nabla_x\phi)\,\Delta\phi\,dx \\
&= \int_{\mathbb{R}^6}|v|^2 f\,dx\,dv-\frac{1}{8\gamma\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx \\
&= E(f)+\frac{1}{2}\int_{\mathbb{R}^6}|v|^2 f\,dx\,dv
\end{aligned}$$

This equivalent to the *pseudo-conformal law*

$$\frac{d}{dt}\left(\int_{\mathbb{R}^6}|x-tv|^2 f(t,x,v)\,dx\,dv-\frac{t^2}{4\pi\gamma}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx\right)=-\frac{t}{4\pi\gamma}\int_{\mathbb{R}^3}|\nabla\phi|^2\,dx$$

established by R. Illner, G. Rein [IlRe96], and B. Perthame [Pert96].

Proposition 15 *If* $E(f) > \frac{1}{2}\frac{\langle v\rangle^2(f)}{\|f\|_{L^1(\mathbb{R}^6)}}$*, then there exists constants* C*,* C_1*,* $C_2 > 0$ *such that for some* $t_0 > 0$*,*

$$C_1\, t^2 \leq \int_{\mathbb{R}^6} |x|^2 f(t,x,v)\, dx\, dv \leq C_2\, t^2 \quad \forall\, t \geq t_0 > 0$$

and, for any $p \in [1,\infty)$*,*

$$\|\rho(t,x)\|_{L^p(\mathbb{R}^3)} \geq \frac{C}{t^{3(p-1)/p}} \quad \forall\, t > t_0 > 0$$

Proof. $$\frac{1}{2}\frac{d^2}{dt^2}\left(\int_{\mathbb{R}^6} |x|^2\, f(t,x,v)\, dx\, dv\right) = 2E(f) + E_{POT}(f)$$

$$\begin{aligned}\int_{\mathbb{R}^6} f(t,x,v)\, dx\, dv &\leq \int_{\mathbb{R}^3}\int_{|x|\leq R} f(t,x,v)\, dx\, dv + \int_{\mathbb{R}^3}\int_{|x|>R} f(t,x,v)\, dx\, dv \\ &\leq C\, \|\rho(t,x)\|_{L^p(\mathbb{R}^3)}^{\frac{2p}{5p-3}} \left(\int_{\mathbb{R}^6} |x|^2 f(t,x,v)\, dx\, dv\right)^{\frac{3p-3}{5p-3}} \quad \square\end{aligned}$$

CONCLUSION

• Rotating systems [Guo, Rein], [Schaeffer]: mathematically, rotations do not induce a loss of compactness

• Other norms (or convex functionals) than L^∞: stability w.r.t. other stationary solutions

• Stability in L^1 [Sánchez, Soler] can be achieved by concentration-compactness methods (no rotation)

• Concentration-compactness offers 3 scenarii (after concentration has been discarded)
[J.D., Poupaud, Sánchez, Soler, in progress]
1) compactness
2) vanishing
3) dichotomy
Notion of "eternal" solution

• Several notions of dispersion :
1) local convergence to 0 in L^1
2) local convergence to 0 in L^q, $q > 1$, or in $L^p_t(L^q_x(L^1_v))$
3) statistical dispersion: $\langle x^2 \rangle - \langle x \rangle^2 \sim t^2$ as $t \to +\infty$. This occurs whenever $E(f^{\bar{u}}) > 0$