New results on entropy methods for (non) linear diffusions

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I — Entropy methods for linear diffusions

The logarithmic Sobolev inequality

Convex Sobolev inequalities
I-A. Entropy method for getting the intermediate asymptotics of the heat equation

Consider the heat equation:

\[
\begin{align*}
& u_t = \Delta u \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+ \\
& u(\cdot, t = 0) = u_0 \geq 0 \quad \int_{\mathbb{R}^n} u_0 \, dx = 1
\end{align*}
\]  

(1)

As \( t \to +\infty \), \( u(x, t) \sim U(x, t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}} \). What is the (optimal) rate of convergence of \( \|u(\cdot, t) - U(\cdot, t)\|_{L^1(\mathbb{R}^n)} \)?
The time dependent rescaling

\[ u(x, t) = \frac{1}{R^n(t)} v \left( \xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0) \right) \]

allows to transform this question into that of the convergence to the stationary solution \( v_\infty(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2} \).

• Ansatz: \( \frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0: \)

\[ R(t) = \sqrt{1 + 2t} , \quad \tau(t) = \log R(t) \]

As a consequence: \( v(\tau = 0) = u_0. \)

• Fokker-Planck equation:

\[
\begin{aligned}
\left\{ 
\begin{array}{l}
\nu_\tau = \Delta v + \nabla(\xi v) \\
v(\cdot, \tau = 0) = u_0 \geq 0 \\
\int_{\mathbb{R}^n} u_0 \, dx = 1
\end{array}
\right.
\end{aligned}
\]
Entropy (relative to the stationary solution $v_\infty$):

$$
\Sigma[v] := \int_{\mathbb{R}^n} v \log \left( \frac{v}{v_\infty} \right) \, dx
$$

If $v$ is a solution of (2), then ($I$ is the Fisher information)

$$
\frac{d}{d\tau} \Sigma[v(\cdot, \tau)] = -\int_{\mathbb{R}^n} v \left| \nabla \log \left( \frac{v}{v_\infty} \right) \right|^2 \, dx =: -I[v(\cdot, \tau)]
$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$
\int_{\mathbb{R}^n} v \log v \, dx + n \left( 1 + \frac{1}{2} \log(2\pi) \right) \leq \frac{1}{2} \int_{\mathbb{R}^n} \frac{|\nabla v|^2}{v} \, dx
$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

$$
\implies \Sigma[v(\cdot, \tau)] \leq \frac{1}{2} I[v(\cdot, \tau)]
$$

$$
\Sigma[v(\cdot, \tau)] \leq e^{-2\tau} \Sigma[u_0] = e^{-2\tau} \int_{\mathbb{R}^n} u_0 \log \left( \frac{u_0}{v_\infty} \right) \, dx
$$
**Csiszár-Kullback inequality:** Consider \( v \geq 0, \bar{v} \geq 0 \) such that \( \int_{\mathbb{R}^n} v \, dx = \int_{\mathbb{R}^n} \bar{v} \, dx =: M > 0 \)

\[
\int_{\mathbb{R}^n} v \log \left( \frac{v}{\bar{v}} \right) \, dx \geq \frac{1}{4M} \| v - \bar{v} \|_{L^1(\mathbb{R}^n)}^2
\]

\[\iff \| v - v_\infty \|_{L^1(\mathbb{R}^n)}^2 \leq 4M \sum [u_0] e^{-2\tau} \]

\[\tau(t) = \log \sqrt{1 + 2t} \]

\[\| u(\cdot, t) - u_\infty(\cdot, t) \|_{L^1(\mathbb{R}^n)}^2 \leq \frac{4}{1 + 2t} \sum [u_0] \]

\[u_\infty(x, t) = \frac{1}{R^n(t)} v_\infty \left( \frac{x}{R(t)}, \tau(t) \right) \]

The proof of the Csiszár-Kullback inequality is given by a Taylor development at second order.
Euclidean logarithmic Sobolev inequality: other formulations
1) independent from the dimension [Gross, 75]

\[
\int_{\mathbb{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^n} w |\nabla \log w|^2 \, d\mu(x)
\]

with \( w = \frac{v}{M v_\infty} \), \( \|v\|_{L^1} = M \), \( d\mu(x) = v_\infty(x) \, dx \).

2) invariant under scaling [Weissler, 78]

\[
\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left( \frac{2}{\pi n e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)
\]

for any \( w \in H^1(\mathbb{R}^n) \) such that \( \int w^2 \, dx = 1 \)
Proof: take $w = \sqrt{\frac{v}{M v_\infty}}$ and optimize for $w_\lambda(x) = \lambda^{n/2} w(\lambda x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2 \, dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2 \, dx = \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2 \, dx - \int_{\mathbb{R}^n} w^2 \log w^2 \, dx - n \log \lambda \int_{\mathbb{R}^n} w^2 \, dx \quad \Box$$

Entropy-entropy production method: a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau} (I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)]) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a \frac{w_i w_j}{w} + b w \delta_{ij} \right|^2 \, dx < 0$$

for some $C > 0$, $a$, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$$\implies \forall u_0, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)] \geq 0 \text{ for } \tau > 0$$
I-B. Entropy-entropy production method: improvements of convex Sobolev inequalities

goal: large time behavior of parabolic equations:

\[
\begin{aligned}
  v_t &= \text{div}_x [D(x)(\nabla_x v + v \nabla_x A(x))] = \text{div}[e^{-A}\nabla(v e^A)] \\
  v(x, t = 0) &= v_0(x) \in L^1_+(\mathbb{R}^n) \\
  v(x, t > 0) &= v_\infty(x) \in L^1_+(\mathbb{R}^n)
\end{aligned}
\]  

(3)

A(x) ... given ‘potential’

\[v_\infty(x) = e^{-A(x)} \in L^1 \ldots \text{(unique) steady state}\]

mass conservation: \(\int_{\mathbb{R}^d} v(t) \, dx = \int_{\mathbb{R}^d} v_\infty \, dx = 1\)

questions: exponential rate? connection to logarithmic Sobolev inequalities?  
[Bakry-Emery '84, Gross '75, Toscani '96, AMTU...]

[Anton Arnold, J.D.]
**Entropy-entropy production method**

[Bakry, Emery, 84]
[Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of \( v(x) \) w.r.t. \( v_\infty(x) \):

\[
\Sigma[v|v_\infty] := \int_{\mathbb{R}^d} \psi \left( \frac{v}{v_\infty} \right) v_\infty \, dx \geq 0
\]

with

\[
\psi(w) \geq 0 \text{ for } w \geq 0, \text{ convex}
\]

\[
\psi(1) = \psi'(1) = 0
\]

Admissibility condition:

\[
(\psi''')^2 \leq \frac{1}{2} \psi'' \psi^{IV}
\]

Examples:

\[
\psi_1 = w \ln w - w + 1, \quad \Sigma_1(v|v_\infty) = \int v \ln \left( \frac{v}{v_\infty} \right) \, dx \ldots \text{ physical entropy}
\]

\[
\psi_p = w^p - p(w - 1) - 1, \quad 1 < p \leq 2, \quad \Sigma_2(v|v_\infty) = \int_{\mathbb{R}^d} (v - v_\infty)^2 v_\infty^{-1} \, dx
\]
**Exponential decay of entropy production**

\[
I(v(t)|v_\infty) := \frac{d}{dt} \Sigma v(t)|v_\infty] = - \int \psi'' \left( \frac{v}{v_\infty} \right) \left| \nabla \left( \frac{v}{v_\infty} \right) \right|^2 v_\infty \, dx \leq 0
\]

Assume: \( D \equiv 1, \quad \frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id, \quad \lambda_1 > 0 \quad (A(x) \ldots \text{unif. convex}) \)

entropy production rate:

\[
I' = 2 \int \psi'' \left( \frac{v}{v_\infty} \right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty \, dx + 2 \int \text{Tr} (XY) v_\infty \, dx \geq 0
\]

\[
\geq -2\lambda_1 I
\]
with

\[ X = \begin{pmatrix} \psi'' \left( \frac{v}{v_\infty} \right) & \psi''' \left( \frac{v}{v_\infty} \right) \\ \psi''' \left( \frac{v}{v_\infty} \right) & \frac{1}{2} \psi^IV \left( \frac{v}{v_\infty} \right) \end{pmatrix} \geq 0 \]

\[ Y = \begin{pmatrix} \sum_{ij} \frac{\partial u_i}{\partial x_j}^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0 \]

\[ \Rightarrow |I(t)| \leq e^{-2\lambda_1 t} |I(t = 0)| \quad t > 0 \]

\[ \forall v_0 \text{ with } |I(v_0|v_\infty)| < \infty \]
**Exponential decay of relative entropy**

known: \[ I' \geq -2\lambda_1 I \quad \Rightarrow \quad \int_t^\infty \ldots dt = \Sigma' \]

\[ \Rightarrow \quad \Sigma' = I \leq -2\lambda_1 \Sigma \]  \hspace{1cm} (4)

**Theorem 1**  [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

\[ \frac{\partial^2 A}{\partial x^2} \geq \lambda_1 \text{Id} \quad \text{("Bakry–Emery condition")}, \quad \Sigma[v_0|v_\infty] < \infty \]

\[ \Rightarrow \quad \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty] e^{-2\lambda_1 t}, \quad t > 0 \]

\[ \|v(t) - v_\infty\|_{L^1}^2 \leq C \Sigma[v(t)|v_\infty] \ldots \text{Csizár-Kullback} \]
**Convex Sobolev Inequalities**

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

**Example 1:** Logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln \left(\frac{v}{v_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int v \left|\nabla \ln \left(\frac{v}{v_\infty}\right)\right|^2 dx$$

$\forall v, v_\infty \in L^1_+(\mathbb{R}^n), \int v dx = \int v_\infty dx = 1$

Logarithmic Sobolev inequality – “entropy version”
Set $f^2 = \frac{v}{v_\infty} \Rightarrow$

\[
\int f^2 \ln f^2 \, dv_\infty \leq \frac{2}{\lambda_1} \int |\nabla f|^2 \, dv_\infty
\]

$\forall f \in L^2(\mathbb{R}^n, dv_\infty), \int f^2 \, dv_\infty = 1$

logarithmic Sobolev inequality–$dv_\infty$ measure version [Gross ’75]

**Example 2**: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w - 1) - 1, \quad 1 < p \leq 2$

\[
(B_p) \quad \frac{p}{p-1} \left[ \int f^2 \, dv_\infty - \left( \int |f|^\frac{2}{p} \, dv_\infty \right)^p \right] \leq \frac{2}{\lambda_1} \int |\nabla f|^2 \, dv_\infty
\]

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^\frac{2}{p} \, dv_\infty}$

$\forall f \in L^\frac{2}{p}(\mathbb{R}^n, v_\infty \, dx)$

Poincaré-type inequality [Beckner ’89], $(B_p) \Rightarrow (B_2), \quad 1 < p \leq 2$
Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

\[ I' = 2 \int \psi'' \left( \frac{v}{v_\infty} \right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot uv_\infty dx + 2 \int \text{Tr} (XY)v_\infty dx \geq 0 \geq -2\lambda_1 I \]

[Arnold, J.D.]: Observation for \( \psi_p(w) = w^p - p(w - 1) - 1 \), \( 1 < p < 2 \):

\[
X = \begin{pmatrix}
\psi'' \left( \frac{v}{v_\infty} \right) & \psi''' \left( \frac{v}{v_\infty} \right) \\
\psi''' \left( \frac{v}{v_\infty} \right) & \frac{1}{2} \psi IV \left( \frac{v}{v_\infty} \right)
\end{pmatrix} > 0
\]
• Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 \text{Id} \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1 + \Sigma}$, $\kappa = \frac{2-p}{p} < 1$

$\Rightarrow \left[ k(\Sigma[v|v_\infty]) \right] \leq \frac{1}{2\lambda_1} |\Sigma'| = \frac{1}{2\lambda_1} \int \psi'' \left( \frac{v}{v_\infty} \right) |\nabla \frac{v}{v_\infty}|^2 dv_\infty$

“refined convex Sobolev inequality” with $x \leq k(x) = \frac{1 + x - (1 + x)^\kappa}{1 - \kappa}$

• Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

\[
\frac{1}{2} \left( \frac{p}{p - 1} \right)^2 \left[ \int f^2 dv_\infty - \left( \int |f|^{\frac{2}{p}} dv_\infty \right)^{2(p - 1)} \left( \int f^2 dv_\infty \right)^{\frac{2-p}{p}} \right]
\leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty \quad \forall f \in L^p(\mathbb{R}^n, dv_\infty)
\]

“refined Beckner inequality” [Arnold, J.D. ’00]

$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$
I-C. An example of application: the flashing ratchet. Long time behavior and dynamical systems interpretation

[M. Chipot, D. Heath, D. Kinderlehrer, M. Kowalczyk, N. Walkington,...]

[J.D., David Kinderlehrer, Michał Kowalczyk]

Flashing ratchet: a simple model for a molecular motor (Brownian motors, molecular ratchets, or Brownian ratchets)

Diffusion tends to spread and dissipate density / transport concentrates density at specific sites determined by the energy landscape: unidirectional transport of mass.

Fokker-Planck type problem

\[ u_t = (u_x + \psi_x u)_x \quad (x, t) \in \Omega \times (0, \infty) \]
\[ u_x + \psi_x u = 0 \quad (x, t) \in \partial \Omega \times (0, \infty) \]
\[ u(x, 0) = u_0(x) \quad x \in \Omega \]

\[ u_0 > 0, \int_\Omega u_0 = 1, \psi = \psi(x, t) \]
**Periodic state and asymptotic behaviour**

**Theorem 1** Let $\psi \in L^\infty([0, T) \times \Omega)$ be a $T$-periodic potential and assume that there exists a finite partition of $[0, T)$ into intervals $[T_i, T_{i+1}), \ i = 0, \ldots, n$ with $T_0 = 0, T_n = T$ such that $\psi[T_i, T_{i+1}) \in L^\infty([T_i, T_{i+1}), W^{1,\infty}(\Omega))$. Then there exists a unique nonnegative $T$-periodic solution $U$ to (5) such that $\int_\Omega U(x, t) \, dx = 1$ for any $t \in [0, T)$.

Entropy and entropy production:

$\sigma_q(u) = \begin{cases} 
\frac{u^{q-1}}{q-1} & \text{if } q > 1, \\
\ln u & \text{if } q = 1.
\end{cases}$

\[ \Sigma_q[u|v] = \int_\Omega \left[ \sigma_q \left( \frac{u}{v} \right) - \sigma'_q(1) \left( \frac{u}{v} - 1 \right) \right] v \, dx \]

\[ I_q[u|v] = \int_\Omega \sigma''_q \left( \frac{u}{v} \right) \left| \nabla \left( \frac{u}{v} \right) \right|^2 v \, dx, \]
Theorem 2 Let $u_1, u_2$ be any two solutions to (5).

$$\Sigma_q[u_1(t)|u_2(t)] \leq e^{-C_q t} \Sigma_q[u_1(0)|u_2(0)]$$

Proposition 3 $\Omega$ is a bounded domain in $\mathbb{R}^d$ with $C^1$ boundary. Let $u$ and $v$ be two nonnegative functions in $L^1 \cap L^q(\Omega)$ if $q \in (1, 2]$ and in $L^1(\Omega)$ with $u \log u$ and $u \log v$ in $L^1(\Omega)$ ($q = 1$).

$$\Sigma_q[u|v] \geq 2^{-2/q} \left[ \max \left(\|u\|_{L^q(\Omega)}^{2-q}, \|v\|_{L^q(\Omega)}^{2-q}\right) \right]^{-1} \|u - v\|_{L^q(\Omega)}^2$$

Corollary 4 Let $q \in [1, 2]$. Any solution of (5) with initial data $u_0 \in L^1 \cap L^q(0, 1)$ $u_0 \log u_0 \in L^1(0, 1)$ if $q = 1$, converges to $\|u_0\|_{L^1 U(x,t)}$, (periodic solution):

$$\|u(x,t) - u_0\|_{L^1 U(x,t)}\|L^q(0,1; dx) \leq K e^{-C_{q,\psi} t} \quad \forall t \geq 0$$
Let \( u_\psi := \|u_0\|_{L^1} \frac{e^{-\psi}}{\int_\Omega e^{-\psi} \, dx} \).

\[
\frac{d}{dt} \Sigma_1[u|u_\psi] = \int_\Omega \left[ 1 + \log \left( \frac{u}{u_\psi} \right) \right] u_t \, dx - \int_\Omega \frac{u}{u_\psi} u_{\psi,t} \, dx
\]

\[
= -I_1[u|u_\psi] - \int_\Omega \frac{u}{u_\psi} u_{\psi,t} \, dx
\]

**Lemma 5** Let \( u \geq 0 \) be a solution to (5) such that \( \int_\Omega u \, dx = 1 \). With the above notations, the following estimate holds:

\[
\frac{d}{dt} \Sigma_1[u|u_\psi] \leq -C_\psi \Sigma_1[u|u_\psi] + K_\psi.
\]

Fixed-point for the map \( T(u(\cdot,0)) = u(\cdot,T) \) in

\[ y = \{ u \in H^1(\Omega) \mid u \geq 0, \|u\|_{L^1(\Omega)} = 1, \Sigma_1[u|u_0(\cdot,0)] \leq K_\psi/C_\psi \}. \]

Flashing potentials: same on each time interval.
Let $\mathcal{X}$ be the set of bounded nonnegative functions $u$ in $L^1 \cap L^q(\Omega)$ (resp. in $L^1(\Omega)$ with $u \log u$ in $L^1(\Omega)$) if $q \in (1, 2]$ (resp. if $q = 1$) such that $\int_\Omega u \, dx = 1$.

**Theorem 6** Assume that $v \in \mathcal{X}$ with $0 < m := \inf_\Omega v \leq v \leq \sup_\Omega v =: M < \infty$. For any $q \in [1, 2]$

$$J = \frac{q}{q - 1} \inf_{u \in \mathcal{X}, \ u \neq v \text{ a.e.}} \frac{I_q[u|v]}{\sum_q[u|v]} \quad \text{if } q > 1, \quad J = \inf_{u \in \mathcal{X}, \ u \neq v \text{ a.e.}} \frac{I_1[u|v]}{\sum_1[u|v]} \quad \text{if } q = 1$$

(6)

*can be estimated by*

$$J \geq 4 \lambda_1(\Omega) \frac{m}{M}$$

(7)

where $\lambda_1(\Omega)$ is Poincaré’s constant of $\Omega$ (with weight 1).

Relation between entropy and entropy production: exponential decay of the relative entropy.
**Dissipation principle**

[Jordan, Kinderlehrer, Otto], [Chipot, Kinderlehrer, Kowalczyk]

**Wasserstein distance** between Borel probability measures $\mu, \mu^*$:

$$d(\mu, \mu^*)^2 = \inf_{p \in \mathcal{P}(\mu, \mu^*)} \int_{\Omega \times \Omega} |x - \xi|^2 p(dx d\xi),$$

$\phi : \Omega \to \Omega$, $\phi(0) = 0$, $\phi(1) = 1$, strictly increasing continuous

$$\int_{\Omega} \zeta f d\xi = \int_{\Omega} \zeta(\phi(x)) f^*(x) dx,$$

for any $\zeta \in C^0(\Omega)$.

$f = F'$ is the push forward of $f^* = (F^*)'$, $\phi$ is the transfer function. In particular if $\zeta = \chi_{[0,x]}$, then

$$\int_{0}^{\phi(x)} f(\xi) \, d\xi = F(\phi(x)) = \int_{0}^{x} f^*(x') \, dx' = F^*(x) \implies \phi = F^{-1} \circ F^*.$$

**Wasserstein distance:**

$$d(f, f^*)^2 = \int_{\Omega} |x - \phi(x)|^2 f^*(x) \, dx.$$
[Benamou and Brenier]: convex duality. Differentiating with respect to $t$ and $x$ yields

$$f_\xi(\phi(x,t),t) \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} + f_t(\phi(x,t),t) \frac{\partial \phi}{\partial x} + f(\phi(x,t),t) \frac{\partial^2 \phi}{\partial x \partial t} = 0.$$  

We implicitly define a velocity $\nu$ by $\nu(\phi,t) = \frac{\partial \phi}{\partial t}$. Using $\frac{\partial^2 \phi}{\partial x \partial t} = \nu_\xi(\phi,t) \frac{\partial \phi}{\partial x}$ we find a continuity equation for $f(x,t)$:

$$f_t + (\nu f)_x = 0, \quad \text{in } \Omega \times (0, \tau).$$  

$$d(f^{**}, f^*)^2 = \tau \min_\nu \int_0^\tau \int_\Omega \nu(x,t)^2 f(x,t) \, dx \, dt,$$

where the minimum is taken over all velocities $\nu$ such that

$$f_t + (\nu f)_x = 0, \quad \text{in } \Omega \times (0, \tau),$$

$$f(x,0) = f^*, \quad f(x,\tau) = f^{**}(x) \quad x \in \Omega.$$  

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Free energy functional:

\[ F(u) = \int_{\Omega} (\psi u + \sigma u \log u) \, dx. \]

[Kinderlehrer, Otto, Jordan]: Determine \( u^{(k)} \) such that

\[
\frac{1}{2} d(u^{(k-1)}, u^{(k)})^2 + \tau F(u^{(k)}) = \min_u \left[ \frac{1}{2} d(u^{(k-1)}, u)^2 + \tau F(u) \right].
\]

Then \( u_\tau(x, t) := u^{(k)}(x) \) if \( t \in [k \tau, (k + 1) \tau), \ x \in \Omega. \)

(1) There exists a unique solution to the above scheme.
(2) As \( \tau \to 0 \), \( u_\tau \) converges strongly in \( L^1((0, t) \times \Omega) \) to the unique solution to (5).

Observe that in the limit \( \tau \to 0 \), \( \nu(x, t) = - (\sigma \log u + \psi)_x. \)

After [Kinderlehrer and Walkington], a new numerical scheme, based on the spatial discretization of

\[ U_t = u(\log u + \psi)_x. \]
I-D. Heat equation with a source term

[J.D., G. Karch]
I-E. A model for traffic flow

[J.D., Reinhard Illner] \( f = f(t, v) \) is an homogeneous distribution function, with velocities ranging in \((0, 1)\):

\[
f_t = (-B(t, v) f + D(t, v) f')', \quad (t, v) \in \mathbb{R}^+ \times (0, 1)
\]

where \( f_t = \partial f/\partial t \), \( f' = \partial f/\partial v \). Let \( C(t, v) := -\int_0^v \frac{B(t, w)}{D(t, w)} \, dw \)

\[
g(t, v) = \rho \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,w)} \, dw}
\]

is a local equilibrium

Zero flux: \(-B(t, v) g + D(t, v) g' = 0\) but \( g_t \equiv 0 \) is not granted.

Relative entropy:

\[
e[t, f] := \int_0^1 \left( f \log f - g \log g + C(t, v)(f - g) \right) \, dv
- \iint_{(0,1) \times (0,t)} C_t(s, v)(f - g)(s, v) \, dv \, ds
\]

Then \( \frac{d}{dt} e[t, f(t, .)] = -\int_0^1 D(t, v) f \left| \frac{f'}{f} - \frac{g'}{g} \right|^2 \, dv \) \dots but we don't have a lower bound for \( e[t, f(t, .)] \).
Density: $\rho = \int_0^1 f(t, v) \, dv$ does not depend on $t$

Mean velocity: $u(t) = \frac{1}{\rho} \int_0^1 v f(t, v) \, dv$

Braking term:
\[
B(t, v) = \begin{cases} 
- C_B |v - u(t)|^2 \rho \left( 1 - \left| \frac{v-u(t)}{1-u(t)} \right|^\delta \right) & \text{if } v > u(t) \\
C_A |v - u(t)|^2 (1 - \rho) & \text{if } v \leq u(t)
\end{cases}
\]

Diffusion term: $D(t, v) = \sigma m_1(\rho) m_2(u(t)) |v - u(t)|^\gamma$

**Proposition 7** [Illner-Klar-Materne02] Any stationary solution is uniquely determined by $\rho$ and its average velocity $u$. The set $(\rho, u[\rho])$ is in general multivalued. For any $\rho \in (0, 1]$.

**Example.** The Maxwellian case.
Convex entropies

Relative entropy of $f$ w.r.t. $g$ by $E[f \mid g] = \int_0^1 \Phi \left( \frac{f}{g} \right) g \, dv$

"Standard" example: $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ for some $\alpha > 1$, $\Phi(x) = x \log x$ if "$\alpha = 1$"

\[
\begin{cases}
  f_t = \left[ D(t, v) f \left( \frac{f'}{f} - \frac{g'}{g} \right) \right]' = \left[ D(t, v) g \left( \frac{f}{g} \right)' \right]' & \forall (t, v) \in \mathbb{R}^+ \times (0, 1) \\
  \left( \frac{f}{g} \right)'(t, v) = 0 & \forall t \in \mathbb{R}^+, \, v = 0, 1
\end{cases}
\]

$g(t, v) := \kappa(t) e^{-C(t,v)}$ for some $\kappa(t) \neq 0$.

\[
\frac{d}{dt} E[f(t, \cdot) \mid g(t, \cdot)] = \int_0^1 \Phi' \left( \frac{f}{g} \right) f_t \, dv + \int_0^1 \left[ \Phi \left( \frac{f}{g} \right) - \frac{f}{g} \Phi' \left( \frac{f}{g} \right) \right] g_t \, dv
\]

\[= 0 \quad \text{if} \quad \kappa = \kappa \frac{\int_0^1 \psi \left( \frac{f}{g} \right) g C_t(t,v) \, dv}{\int_0^1 \psi \left( \frac{f}{g} \right) g \, dv}, \quad \kappa(0) = 1\]

with $\psi(x) := \Phi(x) - x\Phi'(x) < 0$
Convergence to a stationary solution

\[ \lim_{t \to +\infty} \sup \kappa(t) < +\infty. \]

Theorem 8  Let \( \Phi = \Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1) \), \( f \) be a smooth global in time solution and assume that \( E[f|g] \) is well defined and \( C^1 \) in \( t \). If \( \exists \varepsilon \in (0, \frac{1}{2}) \) s.t. \( \varepsilon < u(t) = \frac{1}{\rho} \int_0^1 v f(t, v) \, dv < 1 - \varepsilon \) \( \forall t > 0 \), then, as \( t \to +\infty \), \( f(t, \cdot) \) converges a.e. to a stationary solution \( f_\infty \).
I-F. Navier-Stokes in dimension 2

[J.D., T. Gallay, Wayne, C. Villani, A. Munnier]
II – Entropy methods for (non)linear diffusions

The logarithmic Sobolev inequality in $W^{1,p}$

[coll. Manuel del Pino (Universidad de Chile), Ivan Gentil (Ceremade)]
Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 9 \(1 < p < n, \ 1 < a \leq \frac{p(n-1)}{n-p}, \ b = p \frac{a-1}{p-1}\)

\[
\begin{align*}
\|w\|_b & \leq S \|\nabla w\|_p^\theta \|w\|_a^{1-\theta} \quad \text{if } a > p \\
\|w\|_a & \leq S \|\nabla w\|_p^\theta \|w\|_b^{1-\theta} \quad \text{if } a < p \\
\end{align*}
\]

Equality if \(w(x) = A (1 + B |x|^{p-1})^{\frac{p}{a-p}}\)

\[
\begin{align*}
a > p: \ & \theta = \frac{(q-p)n}{(q-1)(n-p-(n-p)q)} \\
a < p: \ & \theta = \frac{(p-q)n}{q(n(p-q)+p(q-1))} \\
\end{align*}
\]

Proof based on [Serrin, Tang]
Nonhomogeneous version – Gagliardo-Nirenberg ineq.

\[ b = \frac{p(p-1)}{p^2-p-1}, \quad a = bq, \quad v = w^b. \]  
For \( p \neq 2 \), let

\[ \mathcal{F}[v] = \int v^{-\frac{1}{p-1}}|\nabla v|^p \, dx - \frac{1}{q} \left( \frac{n}{1 - \kappa_p} + \frac{p}{p - 2} \right) \int v^q \, dx \]

\[ \kappa_p = \frac{1}{p} (p - 1)^{\frac{p-1}{p}} \]

**Corollary 3** \( n \geq 2, (2n + 1)/(n + 1) \leq p < n. \) \( \forall v \) s.t. \( \|v\|_{L^1} = \|v_\infty\|_{L^1} \)

\[ \mathcal{F}[v] \geq \mathcal{F}[v_\infty] \]
The optimal $L^p$-Euclidean logarithmic Sobolev inequality (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

**Theorem 10** If $\|u\|_{L^p} = 1$, then

\[
\int |u|^p \log |u| \, dx \leq \frac{n}{2} \log \left( L_p \int |\nabla u|^p \, dx \right)
\]

\[
L_p = \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[ \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( n \frac{p-1}{p} + 1 \right)} \right]^{\frac{p}{n}}
\]

**Equality:** $u(x) = \left( \frac{n}{2} \left( \frac{\sigma}{p} \right)^{\frac{n}{p^*}} \frac{\Gamma \left( \frac{n}{p^*} + 1 \right)}{\Gamma \left( \frac{n}{2} + 1 \right)} \right)^{-1/p} e^{-\frac{1}{\sigma} |x - \bar{x}|^{p^*}}$

$p = 2$: Gross’ logarithmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]
For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that
\[
\inf_{\mu > 0} \left[ \frac{n}{p} \log \left( \frac{n}{p\mu} \right) + \mu \frac{\|\nabla w\|_p^p}{\|w\|_p^p} \right] = n \log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{n}{p}.
\]

**Corollary 11** *For any* \( w \in W^{1,p}(\mathbb{R}^n) \), \( w \neq 0 \), *for any* \( \mu > 0 \),
\[
p \int |w|^p \log \left( \frac{|w|}{\|w\|_p} \right) \, dx + \frac{n}{p} \log \left( \frac{p \mu e}{n \mathcal{L}_p} \right) \int |w|^p \, dx \leq \mu \int |\nabla w|^p \, dx.
\]
Consequences

II-A. Existence and uniqueness: [M. Del Pino, J.D., I. Gentil]
Cauchy problem for \( u_t = \Delta_p(u^{1/(p-1)}) \) \((x,t) \in \mathbb{R}^n \times \mathbb{R}^+\)

II-B. Applications to nonlinear diffusions: [M. Del Pino, J.D.] intermediate asymptotics for \( u_t = \Delta_p u^m \)

II-C. Hypercontractivity, Ultracontractivity, Large deviations: [M. Del Pino, J.D., I. Gentil] Connections with \( u_t + |\nabla v|^p = 0 \)
II-A. Existence and uniqueness

[Manuel Del Pino, J.D., Ivan Gentil] Consider the Cauchy problem

\[
\begin{aligned}
&u_t = \Delta_p(u^{1/(p-1)}) & (x,t) &\in \mathbb{R}^n \times \mathbb{R}^+ \\
u(\cdot, t = 0) = f &\geq 0
\end{aligned}
\] (9)

\(\Delta_p u^m = \text{div} \left( |\nabla u^m|^{p-2} \nabla u^m \right)\) is 1-homogeneous \(\iff m = 1/(p-1)\).

Notations: \(\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \, dx)^{1/q}, \quad q \neq 0\). \(p^* = p/(p-1), \quad p > 1\).

**Theorem 12** Let \(p > 1\), \(f \in L^1(\mathbb{R}^n)\) s.t. \(|x|^{p^*} f, f \log f \in L^1(\mathbb{R}^n)\). Then there exists a unique weak nonnegative solution \(u \in C(\mathbb{R}^+_t, L^1)\) of (9) with initial data \(f\), such that \(u^{1/p} \in L^1_{\text{loc}}(\mathbb{R}^+_t, W^{1,p}_{\text{loc}})\).

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]
[Bernis, 88], [Ishige, 96]

The *a priori* estimate on the entropy term \(\int u \log u \, dx\) plays a crucial role in the proof.
(9) is 1-homogenous: we assume that $\int f\,dx = 1$. $u$ is a solution of (9) if and only if $v$ is a solution of

$$\begin{cases}
  v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi (\xi v) & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\
  v(\cdot, \tau = 0) = f
\end{cases}$$

(10)

provided $u$ and $v$ are related by the transformation

$$u(x, t) = \frac{1}{R(t)^n} v(\xi, \tau), \quad \xi = \frac{x}{R(t)}, \quad \tau(t) = \log R(t), \quad R(t) = (1 + pt)^{1/p}$$

[DelPino, J.D., 01]. Let

$$v_\infty(\xi) = \pi^{-n/2} (\frac{p}{\sigma})^{n/p^*} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n}{p^*} + 1)} \exp(-\frac{p}{\sigma} |x|^{p^*}), \quad \sigma = (p^*)^2$$

$\forall \mu > 0$, $\mu v_\infty$ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi (\xi v) = 0$$
In the original variables, $t$ and $x$: consider $u_{\infty} = \frac{1}{R(t)^n} v_{\infty}(\frac{x}{R(t)}, \log R(t))$.

$$\int u \log \left( \frac{u}{u_{\infty}} \right) dx = \int u \log u \, dx + (p-1)(R(t))^{-p^*} \int |x|^{p^*} u \, dx + \sigma(t) \int u \, dx$$

Note that:

$$\frac{d}{dt} \int u \log u \, dx = -\frac{1}{p-1} \int |p^* \nabla u^{1/p}|^p \, dx .$$

**Lemma 13** [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz, Saa, 87]

*On the space* $\{u \in L^1(\mathbb{R}^n) : u^{1/p} \in W^{1,p}(\mathbb{R}^n)\}$, *the functional* $F[u] := \int |\nabla u^\alpha|^p \, dx$ *is convex for any* $p > 1$, $\alpha \in [\frac{1}{p}, 1]$.

From $(p-1)\nabla u^{1/(p-1)} = p u^{1/(p(p-1))} \nabla u^{1/p}$, we get by Hölder’s inequality (with Hölder exponents $p$ and $p^*$)

$$\|\nabla u^{1/(p-1)}\|_{p-1} \leq p^* \|u\|_1^{1/(p(p-1))} \|\nabla u^{1/p}\|_p$$
Remark 14  The entropy decays exponentially since
\[
\frac{d}{dt} \int u \log \left( \frac{u}{\int u \, dx} \right) \, dx = -\frac{1}{p-1} \int |p^* \nabla u^{1/p}|^p \, dx, \quad \text{and}
\]

For any \( w \in W^{1,p}(\mathbb{R}^n) \), \( w \neq 0 \), for any \( \mu > 0 \),
\[
p \int |w|^p \log \left( \frac{|w|}{\|w\|_p} \right) \, dx + \frac{n}{p} \log \left( \frac{p \mu e}{n \mathcal{L}_p} \right) \int |w|^p \, dx \leq \mu \int |\nabla w|^p \, dx.
\]

applied with \( w = u^{1/p}, \mu = \frac{n \mathcal{L}_p}{p e} \), gives
\[
\frac{d}{dt} \int u \log \left( \frac{u}{\int u \, dx} \right) \, dx \leq -\left( p^* \right)^{p+1} e \frac{1}{n \mathcal{L}_p} \int u \log \left( \frac{u}{\int u \, dx} \right) \, dx.
\]
**Uniqueness.** Consider two solutions $u_1$ and $u_2$ of (9).

\[
\frac{d}{dt} \int u_1 \log \left( \frac{u_1}{u_2} \right) \, dx \\
= \int \left( 1 + \log \left( \frac{u_1}{u_2} \right) \right) (u_1)_t \, dx - \int \left( \frac{u_1}{u_2} \right) (u_2)_t \, dx \\
= -(p-1)^{-}(p-1) \int u_1 \left[ \frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2} \right] \cdot \left[ \left| \frac{\nabla u_1}{u_1} \right|^{p-2} \frac{\nabla u_1}{u_1} - \left| \frac{\nabla u_2}{u_2} \right|^{p-2} \frac{\nabla u_2}{u_2} \right] \, dx.
\]

It is then straightforward to check that two solutions with same initial data $f$ have to be equal since

\[
\frac{1}{4} \| u_1(\cdot,t) - u_2(\cdot,t) \|_1^2 \leq \int u_1(\cdot,t) \log \left( \frac{u_1(\cdot,t)}{u_2(\cdot,t)} \right) \, dx \leq \int f \log \left( \frac{f}{f} \right) \, dx = 0
\]

by the Csiszár-Kullback inequality.
II-B. Optimal constants, Optimal rates
[Manuel Del Pino, J.D.]

Intermediate asymptotics for:

\[ u_t = \Delta_p u^m \]

Convergence to a stationary solution for:

\[ v_t = \Delta_p v^m + \nabla(x v) \]

Let \( q = 1 + m - (p - 1)^{-1} \). Whether \( q \) is bigger or smaller than 1 determines two different regimes like for \( m = 1 \).

For \( q > 0 \), define the entropy by

\[ \Sigma[v] = \int \left[ \sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty) \right] \, dx \]

\( \sigma(s) = \frac{s^q - s}{q - 1} \) if \( q \neq 1 \)

\( \sigma(s) = s \log s \) if \( q = 1 \) \( (p = 2) \)
Intermediate asymptotics of $u_t = \Delta_p u^m$

**Theorem 15**  $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ and $q = 1 + m - \frac{1}{p-1}$

\[
\begin{align*}
(i) \quad & \|u(t, \cdot) - u_\infty(t, \cdot)\|_q \leq K R^{-\left(\frac{\alpha}{2} + n\left(1-\frac{1}{q}\right)\right)} \\
(ii) \quad & \|u^q(t, \cdot) - u^q_\infty(t, \cdot)\|_{1/q} \leq K R^{-\frac{\alpha}{2}}
\end{align*}
\]

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$  
(ii): $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p} (p-1) \frac{p-1}{p}) \frac{p}{p-1}$,  
$R = (1 + \gamma t)^{1/\gamma}$,  
$\gamma = (mn + 1)(p - 1) - (n - 1)$

$u_\infty(t, x) = \frac{1}{R^n} v_\infty(\log R, \frac{x}{R})$

$v_\infty(x) = (C - \frac{p-1}{mp} (q-1) |x|^{\frac{p}{p-1}})^{1/(q-1)}$ if $m \neq \frac{1}{p-1}$

$v_\infty(x) = C e^{-(p-1)^2 |x|^{p/(p-1)/p}}$ if $m = (p - 1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla \cdot (x v)$

$w = v^{(mp+q-(m+1))/p}$,  
$a = b q = p \frac{m(p-1)+p-2}{mp(p-1)-1}$. 
Case $q \neq 1$: apply one of the optimal Gagliardo-Nirenberg inequalities.

Case $q = 1$: apply the optimal $L^p$-Euclidean logarithmic Sobolev inequality.

$$\frac{d\Sigma}{dt} \leq -C\Sigma .$$

Csiszár-Kullback inequality: an extension [Cáceres-Carrillo-JD]

**Lemma 16** Let $f$ and $g$ be two nonnegative functions in $L^q(\Omega)$ for a given domain $\Omega$ in $\mathbb{R}^n$. Assume that $q \in (1, 2]$. Then

$$\int_\Omega [\sigma\left(\frac{f}{g}\right) - \sigma'(1)\left(\frac{f}{g} - 1\right)] g^q \, dx \geq \frac{q}{2} \max(\|f\|_{L^q(\Omega)}^{q-2}, \|g\|_{L^q(\Omega)}^{q-2}) \|f - g\|_{L^q(\Omega)}^2$$
II-C. Hypercontractivity, Ultracontractivity, Large deviations
[Manuel Del Pino, J.D., Ivan Gentil]

Understanding the regularizing properties of

\[ u_t = \Delta_p u^{1/(p-1)} \]

**Theorem 17** Let \( \alpha, \beta \in [1, +\infty] \) with \( \beta \geq \alpha \). Under the same assumptions as in the existence Theorem, if moreover \( f \in L^\alpha(\mathbb{R}^n) \), any solution with initial data \( f \) satisfies the estimate

\[ \|u(\cdot, t)\|_\beta \leq \|f\|_\alpha A(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha \beta}} \quad \forall t > 0 \]

with \( A(n, p, \alpha, \beta) = (\mathcal{C}_1 (\beta - \alpha))^{\frac{n}{p} \frac{\beta-\alpha}{\alpha \beta}} \mathcal{C}_2^n \), \( \mathcal{C}_1 = n \mathcal{L}_p e^{p-1} \frac{(p-1)^{p-1}}{p^{p+1}} \), \( \mathcal{C}_2 = (\beta-1) \frac{1-\beta}{1-\alpha} \frac{1-p}{\beta} \frac{1-p}{\alpha} \frac{1}{\beta+1} \). **Case** \( p = 2 \), \( \mathcal{L}_2 = \frac{2}{\pi n e} \), [Gross 75]
As a special case of Theorem 17, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

**Corollary 18** *Consider a solution $u$ with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$
\|u(\cdot, t)\|_{\infty} \leq \|f\|_1 \left( \frac{C_1}{t} \right)^{\frac{n}{p}}.
$$

*Case $p = 2$, [Varopoulos 85]*
Proof. Take a nonnegative function \( u \in L^q(\mathbb{R}^n) \) with \( u^q \log u \) in \( L^1(\mathbb{R}^n) \). It is straightforward that

\[
\frac{d}{dq} \int u^q \, dx = \int u^q \log u \, dx.
\] (11)

Consider now a solution \( u_t = \Delta_p u^{1/(p-1)} \). For a given \( q \in [1, +\infty) \),

\[
\frac{d}{dt} \int u^q \, dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p \, dx.
\] (12)

Assume that \( q \) depends on \( t \) and let \( F(t) = \|u(\cdot, t)\|_{q(t)} \). Let \( s = \frac{d}{dt} \). A combination of (11) and (12) gives

\[
\frac{F'}{F} = \frac{q'}{q^2} \left[ \int \frac{u^q}{F_q} \log \left( \frac{u^q}{F_q} \right) \, dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}} \frac{1}{F_q} \int u^{q-p} |\nabla u|^p \, dx \right].
\]
Since $\int u^{q-p}|\nabla u|^p \, dx = \left(\frac{p}{q}\right)^p \int |\nabla u^{q/p}|^p \, dx$, Corollary 11 applied with $w = u^{q/p}$,

$$
\mu = \frac{(q-1)p^p}{q' q^{p-2} (p-1)^{p-1}}
$$
gives for any $t \geq 0$

$$
F(t) \leq F(0) e^{A(t)} \quad \text{with } A(t) = \frac{n}{p} \int_0^t \frac{q'}{q^2} \log \left( K_p \frac{q^{p-2} q'}{q - 1} \right) \, ds
$$

and $K_p = \frac{n \mathcal{L}_p (p-1)^{p-1}}{e} \frac{(p-1)^{p-1}}{p^{p+1}}$.

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$
q'' q = 2 q'^2 \iff q(t) = \frac{1}{a t + b}.
$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha - \beta}{\alpha \beta}$ and $b = \frac{1}{\alpha}$.
Consider a solution of

\[
\begin{cases}
    v_t + \frac{1}{p} |\nabla v|^p = \frac{1}{p-1} p^{2-p} \varepsilon^{p^*} \Delta_p v & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\
v(\cdot, t = 0) = g
\end{cases}
\]  

(13)

Lemma 19 Let \( \varepsilon > 0 \). Then \( v \) is a \( C^2 \) solution of (13) iff

\[ u = e^{-\frac{1}{\lambda \varepsilon^{p^*}}} v \quad \text{with} \quad \lambda = \frac{1}{p-1} \]

is a \( C^2 \) positive solution of

\[ u_t = \varepsilon^p \Delta_p (u^{1/(p-1)}) \]

with initial data \( f = e^{-\frac{1}{\lambda \varepsilon^{p^*}}} g \).
Conclusion: The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[ \mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let $P^p_t$ be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P^p_t f\|_{\beta} \leq \|f\|_\alpha A(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let $Q^p_t$ be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q^p_t g}\|_{\beta} \leq \|e^g\|_\alpha B(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$