New results on entropy methods for (non) linear diffusions

Jean DOLBEAULT

Ceremade, Université Paris IX-Dauphine,
Place de Lattre de Tassigny,
75775 Paris Cédex 16, France

E-mail: dolbeaul@ceremade.dauphine.fr

http://www.ceremade.dauphine.fr/~dolbeaul/

Vienna, November 4, 2003

I — Entropy methods for linear diffusions

The logarithmic Sobolev inequality

Convex Sobolev inequalities

I-A. Entropy method for getting the intermediate asymptotics of the heat equation

Consider the heat equation:

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \quad (1)$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$. What is the (optimal) rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t}\,, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n\,,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \, \log\left(\frac{v}{v_\infty}\right) \, dx$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 \, dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\boxed{\int_{\mathbf{R}^n} v \log v \, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \le \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v} \, dx}$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} \, e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \le \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \le e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \, \log\left(\frac{u_0}{v_\infty}\right) \, dx$$

• Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v\,dx = \int_{\mathbf{R}^n} \bar{v}\,dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log\left(\frac{v}{\bar{v}}\right)\,dx \geq \frac{1}{4\,M}\|v-\bar{v}\|^2_{L^1(\mathbf{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbf{R}^n)} \leq 4M\Sigma[u_0]\,e^{-2\tau}$$

$\tau(t) = \log\sqrt{1+2t}$

$$\|u(\cdot,t) - u_\infty(\cdot,t)\|^2_{L^1(\mathbf{R}^n)} \leq \frac{4}{1+2t}\,\Sigma[u_0]$$

$$u_\infty(x,t) = \frac{1}{R^n(t)}\,v_\infty\left(\frac{x}{R(t)},\tau(t)\right)$$

The proof of the Csiszár-Kullback inequality is given by a Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension[Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M\, v_\infty}}$ and optimize for $w_\lambda(x) = \lambda^{n/2} w(\lambda\, x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2\, dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2\, dx$$
$$= \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2\, dx - \int_{\mathbb{R}^n} w^2 \log w^2\, dx - n \log \lambda \int_{\mathbb{R}^n} w^2\, dx$$

$\square$

Entropy-entropy production method: a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)]\right) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a\, \frac{w_i w_j}{w} + b\, w\, \delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \geq 0$ for $\tau > 0$

I-B. Entropy-entropy production method: improvements of convex Sobolev inequalities

goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \mathrm{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] = \mathrm{div}[e^{-A}\nabla(ve^{A})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^n \\ v(x,t=0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \qquad (3)$$

$A(x)\ldots$ given 'potential'
$v_\infty(x) = e^{-A(x)} \in L^1\ldots$ (unique) steady state
mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ? [Bakry-Emery '84, Gross '75, Toscani '96, AMTU...]
[Anton Arnold, J.D.]

Entropy-entropy production method

[Bakry, Emery, 84]

[Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condtion: $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''\left(\frac{v}{v_\infty}\right) |\underbrace{\nabla\left(\frac{v}{v_\infty}\right)}_{=:u}|^2 v_\infty\, dx \le 0$$

Assume: $D \equiv 1, \quad \underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \ge \lambda_1 Id, \ \lambda_1 > 0 \quad$ ($A(x)\ldots$ unif. convex)

entropy production rate:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u\, v_\infty\, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, v_\infty\, dx}_{\ge 0} \\ &\ge -2\lambda_1 I \end{aligned}$$

with

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad |I(t)| \leq e^{-2\lambda_1 t} \, |I(t=0)| \qquad t > 0$$

$$\forall v_0 \text{ with } |I(v_0|v_\infty)| < \infty$$

Exponential decay of relative entropy

$$\begin{array}{llcl} \text{known:} & I' & \geq & -2\lambda_1 \underbrace{I}_{=\Sigma'} \qquad , \qquad \int_t^\infty ...dt \\ \Rightarrow & \Sigma' = I & \leq & -2\lambda_1 \Sigma \end{array} \qquad (4)$$

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id$ (*"Bakry–Emery condition"*), $\Sigma[v_0|v_\infty]<\infty$

$$\Rightarrow \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty]\, e^{-2\lambda_1 t}, \qquad t > 0$$

$\|v(t) - v_\infty\|_{L^1}^2 \leq C\,\Sigma[v(t)|v_\infty] \ldots$ Csiszár-Kullback

Convex Sobolev inequalities

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \le \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln\left(\frac{v}{v_\infty}\right) dx \le \frac{1}{2\lambda_1} \int v \left|\nabla \ln\left(\frac{v}{v_\infty}\right)\right|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^n),\ \int v\,dx = \int v_\infty\,dx = 1$$

logarithmic Sobolev inequality – "entropy version"

Set $f^2 = \frac{v}{v_\infty} \Rightarrow$

$$\int f^2 \ln f^2 dv_\infty \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty),\ \int f^2 dv_\infty = 1$$

logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

Example 2: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \le 2$

$$(B_p)\ \ \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^p\right] \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \le 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$
\begin{aligned}
I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0} \\
&\geq -2\lambda_1 I
\end{aligned}
$$

[Arnold, J.D.]: Observation for $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$:

$$
X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0
$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{v}{v_\infty}\right)|\nabla \frac{v}{v_\infty}|^2 dv_\infty$

"refined convex Sobolev inequality" with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$
$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty)$$

"refined Beckner inequality" [Arnold, J.D. '00]

$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

I-C. An example of application: the flashing ratchet. Long time behavior and dynamical systems interpretation

[M. Chipot, D. Heath, D. Kinderlehrer, M. Kowalczyk, N. Walkington,...]

[J.D., David Kinderlehrer, Michał Kowalczyk]

Flashing ratchet: a simple model for a molecular motor (Brownian motors, molecular ratchets, or Brownian ratchets)

Diffusion tends to spread and dissipate density / transport concentrates density at specific sites determined by the energy landscape: unidirectional transport of mass.

Fokker-Planck type problem

$$\begin{array}{ll} u_t = (u_x + \psi_x u)_x & (x,t) \in \Omega \times (0,\infty) \\ u_x + \psi_x u = 0 & (x,t) \in \partial\Omega \times (0,\infty) \\ u(x,0) = u_0(x) & x \in \Omega \end{array} \qquad (5)$$

$u_0 > 0$, $\int_\Omega u_0 = 1$, $\psi = \psi(x,t)$

PERIODIC STATE AND ASYMPTOTIC BEHAVIOUR

Theorem 1 *Let $\psi \in L^\infty([0,T)\times\Omega)$ be a T-periodic potential and assume that there exists a finite partition of $[0,T)$ into intervals $[T_i, T_{i+1})$, $i = 0, ..., n$ with $T_0 = 0$, $T_n = T$ such that $\psi_{[T_i,T_{i+1})} \in L^\infty([T_i, T_{i+1}), W^{1,\infty}(\Omega))$. Then there exists a unique nonnegative T-periodic solution U to (5) such that $\int_\Omega U(x,t)\,dx = 1$ for any $t \in [0,T)$.*

Entropy and entropy production : $\sigma_q(u) = \begin{cases} \frac{u^q - 1}{q-1} & \text{if } q > 1, \\ u \ln u & \text{if } q = 1. \end{cases}$

$$\Sigma_q[u|v] = \int_\Omega \left[\sigma_q\left(\frac{u}{v}\right) - \sigma_q'(1)\left(\frac{u}{v} - 1\right)\right] v\, dx$$

$$I_q[u|v] = \int_\Omega \sigma_q''\left(\frac{u}{v}\right)\left|\nabla\left(\frac{u}{v}\right)\right|^2 v\, dx,$$

Theorem 2 *Let* u_1, u_2 *be any two solutions to (5).*

$$\Sigma_q[u_1(t)|u_2(t)] \leq e^{-C_q t}\,\Sigma_q[u_1(0)|u_2(0)]$$

Proposition 3 Ω *is a bounded domain in* $\mathbb{R}^d$ *with* C^1 *boundary. Let* u *and* v *be two nonnegative functions in* $L^1 \cap L^q(\Omega)$ *if* $q \in (1,2]$ *and in* $L^1(\Omega)$ *with* $u \log u$ *and* $u \log v$ *in* $L^1(\Omega)$ *(*$q=1$*).*

$$\Sigma_q[u|v] \geq 2^{-2/q} q \left[\max\left(\|u\|^{2-q}_{L^q(\Omega)}, \|v\|^{2-q}_{L^q(\Omega)}\right)\right]^{-1} \|u-v\|^2_{L^q(\Omega)}$$

Corollary 4 *Let* $q \in [1,2]$*. Any solution of (5) with initial data* $u_0 \in L^1 \cap L^q(0,1)$ $u_0 \log u_0 \in L^1(0,1)$ *if* $q = 1$*, converges to* $\|u_0\|_{L^1} U(x,t)$*, (periodic solution):*

$$\|u(x,t) - \|u_0\|_{L^1}\, U(x,t)\|_{L^q(0,1;dx)} \leq K\, e^{-C_{q,\psi}\, t} \quad \forall\, t \geq 0$$

Let $u_\psi := \|u_0\|_{L^1} \frac{e^{-\psi}}{\int_\Omega e^{-\psi}\,dx}$.

$$\begin{aligned}\frac{d}{dt}\Sigma_1[u|u_\psi] &= \int_\Omega \left[1 + \log\left(\frac{u}{u_\psi}\right)\right] u_t\,dx - \int_\Omega \frac{u}{u_\psi}\,u_{\psi,t}\,dx \\ &= -I_1[u|u_\psi] - \int_\Omega \frac{u}{u_\psi}\,u_{\psi,t}\,dx\end{aligned}$$

Lemma 5 *Let $u \geq 0$ be a solution to (5) such that $\int_\Omega u\,dx = 1$. With the above notations, the following estimate holds:*

$$\frac{d}{dt}\Sigma_1[u|u_\psi] \leq -C_\psi\,\Sigma_1[u|u_\psi] + K_\psi .$$

Fixed-point for the map $\mathcal{T}(u(\cdot,0)) = u(\cdot,T)$ in

$\mathcal{Y} = \{u \in H^1(\Omega) \mid u \geq 0\,,\ \|u\|_{L^1(\Omega)} = 1\,,\ \Sigma_1[u|u_0(\cdot,0)] \leq K_\psi/C_\psi\}$.

Flashing potentials: same on each time interval.

Let $\mathcal{X}$ be the set of bounded nonnegative functions u in $L^1 \cap L^q(\Omega)$ (resp. in $L^1(\Omega)$ with $u \log u$ in in $L^1(\Omega)$) if $q \in (1,2]$ (resp. if $q = 1$) such that $\int_\Omega u\, dx = 1$.

Theorem 6 *Assume that* $v \in \mathcal{X}$ *with* $0 < m := \inf_\Omega v \leq v \leq \sup_\Omega v =: M < \infty$*. For any* $q \in [1,2]$

$$\mathcal{I} = \frac{q}{q-1} \inf_{\substack{u \in \mathcal{X} \\ u \neq v \, a.e.}} \frac{I_q[u|v]}{\Sigma_q[u|v]} \quad \text{if } q > 1, \quad \mathcal{I} = \inf_{\substack{u \in \mathcal{X} \\ u \neq v \, a.e.}} \frac{I_1[u|v]}{\Sigma_1[u|v]} \quad \text{if } q = 1 \tag{6}$$

can be estimated by

$$\mathcal{I} \geq 4\, \lambda_1(\Omega)\, \frac{m}{M} \tag{7}$$

where $\lambda_1(\Omega)$ *is Poincaré's constant of* Ω *(with weight* 1*).*

Relation between entropy and entropy production: exponential decay of the relative entropy.

Dissipation principle

[Jordan, Kinderlehrer, Otto], [Chipot, Kinderlehrer, Kowalczyk]

Wasserstein distance between Borel probability measures μ, μ^*:

$$d(\mu,\mu^*)^2 = \inf_{p\in\mathcal{P}(\mu,\mu^*)} \int_{\Omega\times\Omega} |x-\xi|^2\, p(dx d\xi),$$

$\phi : \Omega \to \Omega$, $\phi(0) = 0$, $\phi(1) = 1$, strictly increasing continuous

$$\int_\Omega \zeta\, f\, d\xi = \int_\Omega \zeta(\phi(x))\, f^*(x)\, dx, \qquad \text{for any } \zeta \in C^0(\Omega).$$

$f = F'$ is the push forward of $f^* = (F^*)'$, ϕ is the transfer function. In particular if $\zeta = \chi_{[0,x]}$, then

$$\int_0^{\phi(x)} f(\xi)\, d\xi = F(\phi(x)) = \int_0^x f^*(x')\, dx' = F^*(x) \Longrightarrow \phi = F^{-1}\circ F^* .$$

Wasserstein distance: $d(f,f^*)^2 = \int_\Omega |x-\phi(x)|^2 f^*(x)\, dx$.

[Benamou and Brenier]: convex duality. Differentiating with respect to t and x yields

$$f_\xi(\phi(x,t),t)\,\frac{\partial\phi}{\partial t}\frac{\partial\phi}{\partial x} + f_t(\phi(x,t),t)\,\frac{\partial\phi}{\partial x} + f(\phi(x,t),t)\,\frac{\partial^2\phi}{\partial x\partial t} = 0.$$

We implicitly define a velocity ν by $\nu(\phi,t) = \frac{\partial\phi}{\partial t}$. Using $\frac{\partial^2\phi}{\partial x\partial t} = \nu_\xi(\phi,t)\frac{\partial\phi}{\partial x}$ we find a continuity equation for $f(x,t)$:

$$f_t + (\nu f)_x = 0, \qquad \text{in } \Omega\times(0,\tau).$$

$$d(f^{**},f^*)^2 = \tau \min_\nu \int_0^\tau \int_\Omega \nu(x,t)^2 f(x,t)\,dxdt,$$

where the minimum is taken over all velocities ν such that

$$\begin{array}{ll} f_t + (\nu f)_x = 0, & \text{in } \Omega\times(0,\tau), \\ f(x,0) = f^*, \quad f(x,\tau) = f^{**}(x) & x\in\Omega. \end{array}$$

Free energy functional:

$$F(u) = \int_\Omega (\psi\, u + \sigma\, u \log u)\, dx.$$

[Kinderlehrer, Otto, Jordan]: Determine $u^{(k)}$ such that

$$\frac{1}{2} d(u^{(k-1)}, u^{(k)})^2 + \tau F(u^{(k)}) = \min_u \left[\frac{1}{2} d(u^{(k-1)}, u)^2 + \tau F(u)\right]. \tag{8}$$

Then $u_\tau(x,t) := u^{(k)}(x) \quad \text{if } t \in [k\,\tau, (k+1)\,\tau),\ x \in \Omega.$

(1) There exists a unique solution to the above scheme.

(2) As $\tau \to 0$, u_τ converges strongly in $L^1((0,t) \times \Omega)$ to the unique solution to (5).

Observe that in the limit $\tau \to 0$, $\nu(x,t) = -(\sigma \log u + \psi)_x$.

After [Kinderlehrer and Walkington], a new numerical scheme, based on the spatial discretization of

$$U_t = u(\log u + \psi)_x.$$

I-D. Heat equation with a source term

[J.D., G. Karch]

I-E. A model for traffic flow

[J.D., Reinhard Illner] $f = f(t,v)$ is an homogeneous distribution function, with velocities ranging in $(0,1)$:

$$f_t = (-B(t,v)\, f + D(t,v)\, f')' \,, \quad (t,v) \in \mathbb{R}^+ \times (0,1)$$

where $f_t = \partial f/\partial t$, $f' = \partial f/\partial v$. Let $C(t,v) := -\int_0^v \frac{B(t,w)}{D(t,w)}\, dw$

$$g(t,v) = \rho\, \frac{e^{-C(t,v)}}{\int_0^1 e^{-C(t,w)}\, dw} \quad \text{is a } \textit{local equilibrium}$$

Zero flux: $-B(t,v)\, g + D(t,v)\, g' = 0$ but $g_t \equiv 0$ is not granted.
Relative entropy:

$$e[t,f] := \int_0^1 \left(f \log f - g \log g + C(t,v)(f-g)\right) dv - \int\int_{(0,1)\times(0,t)} C_t(s,v)(f-g)(s,v)\, dv\, ds$$

Then $\frac{d}{dt} e[t,f(t,.)] = -\int_0^1 D(t,v)\, f \left|\frac{f'}{f} - \frac{g'}{g}\right|^2\, dv$ ... but we dont have a lower bound for $e[t,f(t,.)]$.

Density: $\rho = \int_0^1 f(t,v)\,dv$ does not depend on t
Mean velocity: $u(t) = \frac{1}{\rho}\int_0^1 v\,f(t,v)\,dv$
Braking term:

$$B(t,v) = \begin{cases} -C_B\,|v-u(t)|^2\rho\left(1-\left|\frac{v-u(t)}{1-u(t)}\right|^\delta\right) & \text{if } v > u(t) \\ \\ C_A\,|v-u(t)|^2(1-\rho) & \text{if } v \le u(t) \end{cases}$$

Diffusion term: $D(t,v) = \sigma\,m_1(\rho)\,m_2(u(t))\,|v-u(t)|^\gamma$

Proposition 7 [Illner-Klar-Materne02] *Any stationary solution is uniquely determined by* ρ *and its average velocity* u. *The set* $(\rho, u[\rho])$ *is in general multivalued. For any* $\rho \in (0,1]$.

Example. *The Maxwellian case.*

Convex entropies

Relative entropy of f *w.r.t.* g by $E[\,f\,|\,g\,] = \int_0^1 \Phi\left(\frac{f}{g}\right) g\,dv$

"Standard" example: $\Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$ for some $\alpha > 1$, $\Phi(x) = x\log x$ if "$\alpha = 1$"

$$\begin{cases} f_t = \left[D(t,v)\,f\left(\frac{f'}{f} - \frac{g'}{g}\right)\right]' = \left[D(t,v)\,g\left(\frac{f}{g}\right)'\right]' & \forall\,(t,v) \in \mathbb{R}^+ \times (0,1) \\ \left(\frac{f}{g}\right)'(t,v) = 0 & \forall\,t \in \mathbb{R}^+,\ v = 0,\ 1 \end{cases}$$

$$g(t,v) := \kappa(t)\,e^{-C(t,v)} \text{ for some } \kappa(t) \neq 0.$$

$$\frac{d}{dt}E[\,f(t,\cdot)\,|\,g(t,\cdot)\,] = \int_0^1 \Phi'\left(\frac{f}{g}\right) f_t\,dv + \underbrace{\int_0^1 \left[\Phi\left(\frac{f}{g}\right) - \frac{f}{g}\,\Phi'\left(\frac{f}{g}\right)\right] g_t\,dv}_{}$$

$$= 0 \quad \text{if} \quad \dot{\kappa} = \kappa\,\frac{\int_0^1 \Psi(\frac{f}{g})\,g\,C_t(t,v)\,dv}{\int_0^1 \Psi(\frac{f}{g})\,g\,dv}, \quad \kappa(0) = 1$$

with $\Psi(x) := \Phi(x) - x\Phi'(x) < 0$

Convergence to a stationary solution

$$\limsup_{t\to+\infty} \kappa(t) < +\infty \, .$$

Theorem 8 *Let* $\Phi = \Phi_\alpha(x) = (x^\alpha - x)/(\alpha - 1)$, f *be a smooth global in time solution and assume that* $E[\, f \,|\, g\,]$ *is well defined and* C^1 *in* t. *If* $\exists\, \varepsilon \in (0, \frac{1}{2})$ *s.t.* $\varepsilon < u(t) = \frac{1}{\rho} \int_0^1 v\, f(t,v)\, dv < 1 - \varepsilon$ $\forall\, t > 0$, *then, as* $t \to +\infty$, $f(t, \cdot)$ *converges a.e. to a stationary solution* f_∞.

I-F. Navier-Stokes in dimension 2

[J.D., T. Gallay, Wayne, C. Villani, A. Munnier]

II – Entropy methods for (non)linear diffusions

The logarithmic Sobolev inequality in $W^{1,p}$

[coll. Manuel del Pino (Universidad de Chile), Ivan Gentil (Ceremade)]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 9 $1<p<n$, $1<a \le \frac{p(n-1)}{n-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \textit{if } a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \textit{if } a<p$$

Equality if $w(x) = A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a>p$: $\theta = \frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a<p$: $\theta = \frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

Proof based on [Serrin, Tang]

Nonhomogeneous version – Gagliardo-Nirenberg ineq.

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p \, dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q \, dx$$

$\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$

Corollary 3 $n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall\ v$ *s.t.* $\|v\|_{L^1} = \|v_\infty\|_{L^1}$

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

The optimal L^p-Euclidean logarithmic Sobolev inequality (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 10 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx \right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that

$$\inf_{\mu>0}\left[\frac{n}{p}\log\left(\frac{n}{p\mu}\right)+\mu\,\frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right]=n\,\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right)+\frac{n}{p}\,.$$

Corollary 11 *For any* $w\in W^{1,p}(\mathbb{R}^n)$, $w\neq 0$, *for any* $\mu>0$,

$$p\int|w|^p\log\left(\frac{|w|}{\|w\|_p}\right)\,dx+\frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int|w|^p\,dx\leq\mu\int|\nabla w|^p\,dx\,.$$

Consequences

II-A. Existence and uniqueness: [M. Del Pino, J.D., I. Gentil]
Cauchy problem for $u_t = \Delta_p(u^{1/(p-1)}) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$

II-B. Applications to nonlinear diffusions: [M. Del Pino, J.D.] intermediate asymptotics for $u_t = \Delta_p u^m$

II-C. Hypercontractivity, Ultracontractivity, Large deviations: [M. Del Pino, J.D., I. Gentil] Connections with $u_t + |\nabla v|^p = 0$

II-A. Existence and uniqueness

[Manuel Del Pino, J.D., Ivan Gentil] Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \tag{9}$$

$\Delta_p u^m = \text{div}\left(|\nabla u^m|^{p-2} \nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.

Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 12 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}^+_t, L^1)$ of (9) with initial data f, such that $u^{1/p} \in L^1_{\text{loc}}(\mathbb{R}^+_t, W^{1,p}_{\text{loc}})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]
[Bernis, 88], [Ishige, 96]

The *a priori* estimate on the *entropy term* $\int u \log u \, dx$ plays a crucial role in the proof.

(9) is 1-homogenous: we assume that $\int f\,dx = 1$. u is a solution of (9) if and only if v is a solution of

$$\begin{cases} v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases} \tag{10}$$

provided u and v are related by the transformation

$$u(x,t) = \frac{1}{R(t)^n}\, v(\xi,\tau)\,,\ \xi = \frac{x}{R(t)}\,,\ \tau(t) = \log R(t)\,,\ R(t) = (1 + p\,t)^{1/p}$$

[DelPino, J.D., 01]. Let

$$v_\infty(\xi) = \pi^{-\frac{n}{2}} (\frac{p}{\sigma})^{n/p^*} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{p^*}+1)} \exp(-\frac{p}{\sigma}\,|x|^{p^*})\,, \quad \sigma = (p^*)^2$$

$\forall \mu > 0$, $\mu\, v_\infty$ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) = 0$$

In the original variables, t and x: consider $u_\infty = \frac{1}{R(t)^n}\, v_\infty(\frac{x}{R(t)}, \log R(t))$.

$$\int u \log\left(\frac{u}{u_\infty}\right) dx = \int u \log u \, dx + (p-1)(R(t))^{-p^*} \int |x|^{p^*} u \, dx + \sigma(t) \int u \, dx$$

Note that:

$$\frac{d}{dt} \int u \log u \, dx = -\frac{1}{p-1} \int \left| p^* \nabla u^{1/p} \right|^p \, dx \, .$$

Lemma 13 [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87]
On the space $\{u \in L^1(\mathbb{R}^n) \; : \; u^{1/p} \in W^{1,p}(\mathbb{R}^n)\}$*, the functional* $F[u] := \int |\nabla u^\alpha|^p \, dx$ *is convex for any* $p > 1$*,* $\alpha \in [\frac{1}{p}, 1]$*.*

From $(p-1)\nabla u^{1/(p-1)} = p\, u^{1/(p(p-1))} \nabla u^{1/p}$, we get by Hölder's inequality (with Hölder exponents p and p^*)

$$\|\nabla u^{1/(p-1)}\|_{p-1} \leq p^* \|u\|_1^{1/(p(p-1))} \, \|\nabla u^{1/p}\|_p$$

Remark 14 *The entropy decays exponentially since* $\frac{d}{dt}\int u \log\big(\frac{u}{\int u\,dx}\big)\,dx = -\frac{1}{p-1}\int |p^*\,\nabla u^{1/p}|^p\,dx$*, and*

For any $w \in W^{1,p}(\mathbb{R}^n)$*,* $w \neq 0$*, for any* $\mu > 0$*,*

$$p\int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right)\,dx + \frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int |w|^p\,dx \le \mu\int |\nabla w|^p\,dx .$$

applied with $w = u^{1/p}$*,* $\mu = \frac{n\,\mathcal{L}_p}{p\,e}$*, gives*

$$\frac{d}{dt}\int u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx \le -\frac{(p*)^{p+1}e}{n\,\mathcal{L}_p}\int u \log\Big(\frac{u}{\int u\,dx}\Big)\,dx\ .$$

Uniqueness. Consider two solutions u_1 and u_2 of (9).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1 + \log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \,.$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1} \|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \leq \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \leq \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

II-B. Optimal constants, Optimal rates

[Manuel Del Pino, J.D.]

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x\,v)$$

Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$.

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\; dx$$

$\sigma(s) = \frac{s^q - s}{q-1}$ if $q \neq 1$

$\sigma(s) = s \log s$ if $q = 1$ $(p = 2)$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 15 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\text{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-\left(\frac{\alpha}{2}+n\left(1-\frac{1}{q}\right)\right)}$$

$$\text{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii):* $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = \left(1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}}\right)\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$

$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$

$v_\infty(x) = \left(C - \frac{p-1}{mp}(q-1)\,|x|^{\frac{p}{p-1}}\right)_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$

$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla \cdot (x\, v)$

$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

Case $q \neq 1$: apply one of the optimal Gagliardo-Nirenberg inequalities.
Case $q = 1$: apply the optimal L^p-Euclidean logarithmic Sobolev inequality.

$$\frac{d\Sigma}{dt} \leq -C\,\Sigma\,.$$

Csiszár-Kullback inequality: an extension [Cáceres-Carrillo-JD]

Lemma 16 *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in $\mathbb{R}^n$. Assume that $q \in (1,2]$. Then*

$$\int_\Omega [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g} - 1)]\, g^q\, dx \geq \frac{q}{2} \max(\|f\|_{L^q(\Omega)}^{q-2}, \|g\|_{L^q(\Omega)}^{q-2})\, \|f-g\|_{L^q(\Omega)}^2$$

II-C. Hypercontractivity, Ultracontractivity, Large deviations

[Manuel Del Pino, J.D., Ivan Gentil]

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 17 *Let* α*,* $\beta \in [1, +\infty]$ *with* $\beta \geq \alpha$*. Under the same assumptions as in the existence Theorem, if moreover* $f \in L^\alpha(\mathbb{R}^n)$*, any solution with initial data* f *satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t > 0$$

with $A(n,p,\alpha,\beta) = (\mathcal{C}_1\,(\beta-\alpha))^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}$,

$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}$. *Case* $p = 2$, $\mathcal{L}_2 = \frac{2}{\pi\,n\,e}$, *[Gross 75]*

As a special case of Theorem 17, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 18 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot,t)\|_\infty \leq \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Proof. Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq} \int u^q \, dx = \int u^q \log u \; dx \; . \tag{11}$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1, +\infty)$,

$$\frac{d}{dt} \int u^q \, dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p \, dx \; . \tag{12}$$

Assume that q depends on t and let $F(t) = \|u(\cdot, t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (11) and (12) gives

$$\frac{F'}{F} = \frac{q'}{q^2} \left[\int \frac{u^q}{F^q} \log \left(\frac{u^q}{F^q}\right) dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}} \frac{1}{F^q} \int u^{q-p} |\nabla u|^p \, dx \right] \; .$$

Since $\int u^{q-p}|\nabla u|^p\,dx = (\frac{p}{q})^p \int |\nabla u^{q/p}|^p\,dx$, Corollary 11 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } \; A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)ds$$

$$\text{and } \; \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 \Longleftrightarrow q(t) = \frac{1}{a\,t + b}\,.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$.

Large deviations and Hamilton-Jacobi equations

Consider a solution of

$$\begin{cases} v_t + \frac{1}{p}\,|\nabla v|^p = \frac{1}{p-1}\,p^{\frac{2-p}{p-1}}\,\varepsilon^{p^*}\Delta_p v & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, t=0) = g \end{cases} \tag{13}$$

Lemma 19 *Let $\varepsilon > 0$. Then v is a C^2 solution of (13) iff*

$$u = e^{-\frac{1}{\lambda \varepsilon^{p^*}}\,v} \quad \textit{with } \lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$$

is a C^2 positive solution of

$$u_t = \varepsilon^p\,\Delta_p(u^{1/(p-1)})$$

with initial data $f = e^{-\frac{1}{\lambda \varepsilon^{p^*}}\,g}$.

Conclusion: The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$