Workgroup on
Asymptotics of nonlinear diffusion, related functional inequalities, and quantitative stability

Banff, August 30 - September 4, 2026

Abstract

Optimal rates of convergence in nonlinear diffusion equations can be identified with best constants in some simple functional inequalities, for instance on the Euclidean space and on the sphere.

$\triangleright$ Under which conditions is this also true for more general diffusions or functional inequalities?

$\triangleright$ What is the role of second moments?

$\triangleright$ What happens in presence of weights for which symmetry breaking occurs ?

Stability in functional inequalities means that we control the distance to the set of optimal functions, if possible in strong norms, by the deficit, *i.e.*, by the difference of the two sides of the inequality written with optimal constant.

$\triangleright$ How can we use nonlinear diffusions to prove such results?

$\triangleright$ Can we extend such methods to other equations, *e.g.*, kinetic equations involving nonlinear (degenerate) diffusion operators acting on the velocity variable ?

Stability in Sobolev type inequalities

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$, $2^* = \frac{2\,d}{d-2}$ and sharp constant S_d

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d \, \|f\|_{2^*}^2 \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) = \mathscr{D}^{1,2}(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g_{a,b,c}(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

[Bianchi, Egnell, 1991] there is some non-explicit $c_{\mathrm{BE}} > 0$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq c_{\mathrm{BE}} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2^2$$

Theorem (JD, Esteban, Figalli, Frank, Loss)

There is a constant $\beta > 0$ which does not depend on d such that

$$c_{\mathrm{BE}} \geq \beta/d \quad \forall\, d \geq 3$$

▷ Can we give simpler characterization of the optimal c_{BE} as $d \to +\infty$?
▷ How good is the estimate β ?

Stability in logarithmic Sobolev type inequalities

Let $d\gamma = e^{-\pi\,|x|^2}\,dx$ be the Gaussian measure

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta > 0$ as in the result for the Sobolev inequality, for any $u \in \mathrm{H}^1(d\gamma)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \pi \int_{\mathbb{R}^d} u^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}}\right) d\gamma \geq \frac{\beta\,\pi}{2} \inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}} \int_{\mathbb{R}^d} |u - c\,e^{a\cdot x}|^2\,d\gamma$$

▷ The optimal stability constant is monotone non-increasing w.r.t. d
Does it depend on d ?

▷ Role of moments in the Euclidean logarithmic Sobolev inequality ?

▷ What is the sharp condition for stability in $\mathrm{H}^1(d\gamma)$?

Nonlinear diffusion and entropy

Fast diffusion equation ($m<1$) on $\mathbb{R}^d$ in ***self-similar variables***

$$\frac{\partial v}{\partial t}+\nabla\cdot\Big[v\,\big(\nabla v^{m-1}-2\,x\big)\Big]=0 \qquad \text{(FDE)}$$

Relative entropy $\mathcal{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\big(v^m-\mathcal{B}^m-m\,\mathcal{B}^{m-1}\,(v-\mathcal{B})\big)\,dx$

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\mathcal{I}[v(t,\cdot)]$$

where $\mathcal{B}(x):=\big(1+|x|^2\big)^{\frac{1}{m-1}}$ is the stationary Barenblatt solution
If $m=1-1/d$, then Sobolev's inequality (with optimal constant) is equivalent to

$$\mathcal{I}[v]\ge 4\,\mathcal{F}[v]$$

so that $\mathcal{F}[v(t,\cdot)]\le\mathcal{F}[v_0]\,e^{-\,4\,t}$ is the optimal decay rate of $\mathcal{F}$

- Asymptotic regime and spectral gap
  $\Longrightarrow$ improved spectral gap under orthogonality conditions
- Far away regime: improved global entropy – entropy production inequality $\mathcal{I}[v]\ge 4\,\varphi(\mathcal{F}[v])$ with $\varphi''>0$, $\varphi(0)=\varphi'(0)-1=0$
- [BDNS 2025] Glueing (of the two regimes) provides a stability result under a tail condition: bad constant and an overkill !

▷ Can we better exploit the (regularity) properties of the flow ?

More open questions

▷ Regularity in $\mathrm{H}^1(\mathbb{R}^d)$: can we get rid of tail conditions and still justify the asymptotic expansion without tail conditions on the initial datum ?

▷ Which functional inequalities ? Which nonlinear flows ?

▷ Manifolds: the sphere, the weighted sphere (non-uniform probability measure on $\mathbb{S}^d$) ?

▷ Phase transitions and symmetry breaking: quantitative estimates for entropy – entropy production inequalities in the symmetry breaking regime for Caffarelli-Kohn-Nirenberg inequalities (role of the moments) ?

Linear kinetic equations: hypocoercivity

Vlasov-Fokker-Planck equation with external potential ψ

$$\partial_t f + \underbrace{v \cdot \nabla_x f - \nabla_x \psi \cdot \nabla_v f}_{\mathsf{T} f} = \underbrace{\Delta_v f + \nabla_v (v\, f)}_{\mathsf{L} f}$$

▷ ***Homogeneous case:*** no dependence in x: Fokker-Planck equation, exponential rate of convergence to a stationary solution

$$\mathscr{M}(v) = (2\pi)^{-d/2}\, e^{-\frac{1}{2}|v|^2}$$

▷ ***Inhomogeneous case:*** rate of convergence to a stationary solution ? rate of decay if there is no stationary solution ?

- $\mathrm{H}^1(\mathbb{R}^d \times \mathbb{R}^d)$ framework [Villani] [Li, JD]
- $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$ framework with *twisted* norm (DMS) [JD, Mouhot, Schmeiser], [Hérau]
- $\mathrm{L}^2\left(\mathbb{R}^d, \mathrm{H}^{-1}\left(\mathbb{R}^d, \mathscr{M}^{-1}\, dx\, dv\right)\right)$ framework [Albritton, Armstrong, Mourrat]

Towards nonlinear kinetic equations ?

$\mathrm{H}^1(\mathbb{R}^d \times \mathbb{R}^d)$ framework: let $\psi(x) = |x|^2/2$ and $p \in [1,2]$, $d\mu := \mathfrak{M}\, dx\, dv$

The function $h = (f/\mathscr{M})^{p/2}$ solves

$$\frac{\partial h}{\partial t} + \mathsf{T} h = \mathsf{L}\, h + \frac{2-p}{p}\, \frac{|\nabla_v h|^2}{h}$$

$$\mathcal{J}[h] = \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_v h|^2\, d\mu + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_x h|^2\, d\mu + \iint_{\mathbb{R}^d \times \mathbb{R}^d} |\nabla_x h + \nabla_v h|^2\, d\mu$$

- *Carré du champ* $\frac{d}{dt}\mathcal{J}[h(t,\cdot)] \le -\mathcal{J}[h(t,\cdot)]$

$$\mathcal{J}[h(t,\cdot,\cdot)] \le e^{-t}\mathcal{J}[h_0] \quad \forall\, t \ge 0$$

- There is a constant $\mathcal{C} > 0$ such that $\big\| (f/\mathscr{M}) - 1 \big\|_{\mathrm{L}^p(d\mu)} \le \mathcal{C}\, e^{-\lambda t}$
- For some (very limited range of) m, the method applies to

$$\partial_t f + \mathsf{T} f = \mathsf{L} f^m \tag{NKE}$$

[Bouin, JD, Schmeiser]

▷ How can we extend the *twisted* $\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$ framework to (NKE) ?
▷ Nonlinear entropy ? Twist ? Functional inequalities ?
▷ Rates / best constants ?

Practical aspects (to be discussed)

- You are interested: send me an e-mail at dolbeaul@ceremade.dauphine.fr with *subject* starting with WG
- No recording, no on-line participation: feel free to say stupid things
- If you work on somebody's open questions, please stay in touch with the speaker (and eventually collaborate)
- Blackboard explanations preferred. Questions and interruptions or discussions welcome !
- First session: establish a list of problems and ”speakers” (please volunteer)
- The spirit of focused research groups !