ENTROPY METHODS: FLOWS OR MULTIPLIERS ?

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ABSTRACT. *Entropy methods* are by now standard tools for the study of simple functional inequalities. Optimal decay rates of (some) evolution problems are given by equality cases in related functional inequalities. Reciprocally, it turns out that some well chosen evolution equations, or *flows*, can be used to characterize the critical points associated with the functional inequalities. In other words, the problem has a *parabolic* formulation in terms of flows and an *elliptic* formulation in a variational framework. Up to adapted changes of variables, the critical points can be seen as stationary solutions of the flow. Testing a critical point by the flow amounts to testing the elliptic equation by a *mutiplier*. One can therefore wonder if there is any advantage in introducing a time dependence. The answer is subtle and the two formulation have their own interest. Where we list what can be learned from the two points of view.

Entropy methods are deeply rooted in evolution equations. For a large community of applied mathematicians interested in kinetic equations, entropy estimates were for a long time almost the only *a priori* estimate available for the study of global in time Cauchy problems and of the large time asymptotics of the solutions. At the turn of the century, links between kinetic and parabolic equations were studied by the so-called *diffusion limits* and raised interest in applying entropy methods to parabolic problems. It was soon clear that, in simple situations, optimal rates of convergence were characterized as optimal constants in related functional inequalities of entropy – entropy production type. This was of course not a surprise in view of the *carré du champ methods* and probabilistic methods for Markov processes, but the breakthrough came from the use of nonlinear diffusions equations like the porous medium equation.

The deep analogy of the computations in *carré du champ methods* as introduced by D. Bakry and M. Emery and elliptic rigidity estimates of B. Gidas & J. Spruck, later improved by M.-F. Bidaut-Véron & L. Véron, is known for many years and was exploited by J. Demange in his thesis. The main clarification came from the use of nonlinear flows of porous medium / fast diffusion type, which also produce similar results in absence of external potentials or curvature, but just because of the nonlinearities. In the Euclidean case, this has been fully exploited in the framework of generalized Rényi entropies.

One of the main interests of parabolic flows is that the key monotonicity property in the *carré du champ* computations can be reduced to the positivity of a quadratic form with constant coefficients. This quadratic form is the same in the initial, nonlinear regime of

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the flow and in the asymptotic regime as $t \to +\infty$: the positivity can be interpreted as a spectral gap of the linearized problem. Moreover, this allows to identify the optimal constant with a simple spectral problem and explain why sharp results can be expected.

• The *parabolic point of view* has several advantages:
1) the global convexity, as a function of time, of entropy related quantities explain why global results and not only local ones can be achieved. It is also possible in some simple cases to reinterpret the results in a gradient flow framework, which emphasizes the convexity properties of the problem (with the appropriate topology).
2) the strategy of entropy methods relates nonlinear quantities with their linear counterparts, and nonlinear interpolation inequalities with spectral gap inequalities.
3) beyond monotonicity properties, *carré du champ methods* provide remainder terms that can be exploited to establish quantitative stability results in functional inequalities, which are so far out of reach by elliptic methods.

However, there are also some drawbacks, as the justification of integrations by parts requires lots of care on unbounded domains (for intance in the case of Gagliardo-Nirenberg inequalities), and non-trivial regularity estimates (for instance in presence of weights) have to be done to manipulate terms which involve high order derivatives: some painful approximations are sometimes unavoidable.

• In the *elliptic point of view*, one has to deal only with critical points or minimizers of an energy and convexity (which results in, for instance, rigidity / uniqueness results and explains why properties are global) is hidden. Testing a special solution by a direction corresponding to the nonlinear flow amounts to introduce the right multiplier, which can be guessed directly in some simple cases (although specialists know that in the approach of B. Gidas & J. Spruck, computations are not at all straightforward). However, the major advantage is that solutions of the elliptic equation have nice regularity and decay properties, which allow us to justify, for instance, integrations by parts. In the case of Caffarelli-Kohn-Nirenberg inequalities, not enough is known so far to give a complete proof of symmetry in the parabolic setting, despite of various attempts.

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Here is a short description of some papers on optimal functional inequalities and improvements relying on *parabolic* and *elliptic* equations:

▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Nonlinear flows and rigidity results on compact manifolds*, Journal of Functional Analysis, 267 (5): 1338 – 1363, 2014.

This paper is devoted to rigidity results for some elliptic PDEs and to optimal constants in related interpolation inequalities of Sobolev type on smooth compact connected Riemannian manifolds without boundaries. Rigidity means that the PDE has no other solution than the constant one at least when a parameter is in a certain range. The largest value of this parameter provides an estimate for the optimal constant in the corresponding interpolation inequality. Our approach relies on a nonlinear flow of porous medium / fast diffusion type which gives a clear-cut interpretation of technical choices of exponents done in earlier works on rigidity. We also establish two integral criteria for rigidity that improve upon known, pointwise conditions, and hold for general manifolds without positivity conditions on the curvature. Using the flow, we are also able to discuss the optimality of the corresponding constants in the interpolation inequalities.

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▷ J. DOLBEAULT AND G. TOSCANI, *Nonlinear diffusions: Extremal properties of Barenblatt profiles, best matching and delays*, Nonlinear Analysis, 138: 31–43, Jun 2016.

In this paper, we consider functionals based on moments and nonlinear entropies which have a linear growth in time in case of source-type solutions to the fast di usion or porous medium equations, that are also known as Barenblatt solutions. As functions of time, these functionals have convexity properties for generic solutions, so that their asymptotic slopes are extremal for Barenblatt pro les. The method relies on scaling properties of the evolution equations and provides a simple and direct proof of sharp Gagliardo–Nirenberg–Sobolev inequalities in scale invariant form. The method also gives re ned estimates of the growth of the second moment and, as a consequence, establishes the monotonicity of the delay corresponding to the best matching Barenblatt solution compared to the Barenblatt solution with same initial second moment. Here the notion of best matching is de ned in terms of a relative entropy.

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▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*, Journal of elliptic and parabolic equations, 2: 267–295, 2016.

This paper is devoted to the computation of the asymptotic boundary terms in entropy methods applied to a fast diffusion equation with weights associated with Caffarelli-Kohn-Nirenberg interpolation inequalities. So far, only elliptic equations have been considered and our goal is to justify, at least partially, an extension of the *carré du champ* / Bakry-Emery / Rényi entropy methods to parabolic equations. This makes sense because evolution equations are at the core of the heuristics of the method even when only elliptic equations are considered, but this also raises difficult questions on the regularity and on the growth of the solutions in presence of weights.

We also investigate the relations between the optimal constant in the entropy – entropy production inequality, the optimal constant in the information – information production inequality, the asymptotic growth rate of generalized Rényi entropy powers under the action of the evolution equation and the optimal range of parameters for symmetry breaking issues in Caffarelli-Kohn-Nirenberg inequalities, under the assumption that the weights do not introduce singular boundary terms at $x = 0$. These considerations are new even in the case without weights. For instance, we establish the equivalence of carré du champ and Rényi entropy methods and explain why entropy methods produce optimal constants in entropy – entropy production and Gagliardo-Nirenberg inequalities in absence of weights, or optimal symmetry ranges when weights are present.

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▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, Invent. Math., 206 (2): 397–440, 2016.

This paper is motivated by the characterization of the optimal symmetry breaking region in Caffarelli-Kohn-Nirenberg inequalities. As a consequence, optimal functions and sharp constants are computed in the symmetry region. The result solves a longstanding conjecture on the optimal symmetry range.
As a byproduct of our method we obtain sharp estimates for the principal eigenvalue of Schrödinger operators on some non-flat non-compact manifolds, which to the best of our knowledge are new.
The method relies on generalized entropy functionals for nonlinear diffusion equations. It opens a new area of research for approaches related to *carré du champ* methods on non-compact manifolds. However, key estimates depend as much on curvature properties as on purely nonlinear effects. The method is well adapted to functional inequalities involving simple weights and also applies to general cylinders. Beyond results on symmetry and symmetry breaking, and on optimal constants in functional inequalities, rigidity theorems for nonlinear elliptic equations can be deduced in rather general settings.

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▷ J. DOLBEAULT AND M. J. ESTEBAN, *Improved interpolation inequalities and stability*. Preprint hal-02266625 & arXiv: 1908.08235

For exponents in the subcritical range, we revisit some optimal interpolation inequalities on the sphere with *carré du champ* methods and use the remainder terms to produce improved inequalities. The method provides us with lower estimates of the optimal constants in the symmetry breaking range and stability estimates for the optimal functions. Some of these results can be reformulated in the Euclidean space using the stereographic projection.

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▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities on the sphere: linear vs. nonlinear flows*, Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2): 351–379, 2017.

This paper is devoted to sharp interpolation inequalities on the sphere and their proof using flows. The method explains some rigidity results and proves uniqueness in related semilinear elliptic equations. Nonlinear flows allow to cover the interval of exponents ranging from Poincaré to Sobolev inequality, while an intriguing limitation (an upper bound on the exponent) appears in the *carré du champ* method based on the heat flow. We investigate this limitation, describe a counter-example for exponents which are above the bound, and obtain improvements below.

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