HYPOCOERCIVITY

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Hypocoercivity is a method intended to construct a norm which is equivalent to some well-know norm and has exponential decay properties when evolved along an evolution equation, typically a non-homogenous kinetic equation, which relies on an anti-symmetric transport operator and a linear collision operator. The typical example of collision operators are Fokker-Planck operators acting on the velocity variable (but which are degenerate with respect to the collision variable) or linear scattering operators conserving mass. One can roughly distinguish two types of hypocoercivity methods:
1) H^1 methods popularised by C. Villani which are based on *twisted* Fisher information functionals and an algebraic structure which is reminiscent of Hörmander's hypoelliptic theory. Entropy type estimates are controlled by logarithmic Sobolev and related inequalities.
2) L^2 methods which are inspired by diffusion limits. The standard case is obtained when the transport operator contains a force term induced by a confining potential which has sufficient growth properties to guarantee that the invariant measure induces a Poincaré inequality.

More recently, the case without confinement at all has been studied on the basis of a mode-by-mode analysis in Fourier variables. At macroscopic level, the Poincaré inequality can be replaced by a Nash inequality and the decay rate of the heat equation is obtained in the appropriate norms. Such a result raises the question of decay or convergence rates when the external potential is either confining but not strong enough to provide us with a Poincaré inequality, or even non-confining. At diffusive level, it is known that, in the first case, the Poincaré inequality can be replaced by a weak Poincaré inequality considered by Röckner and Wang, to the price of an *a priori* estimate in a strong norm. As an alternative approach, which applies to the potential $V(x) = |x|^\alpha$ for $\alpha \in (0,1)$, it is possible to rely on moments, which provides as $\alpha \to +1$ a unified framework with Poincaré inequalities valid for $\alpha \geq 1$. In the non-confining case of a potential $V(x) \sim \gamma \log|x|$ as $|x| \to +\infty$ with $\gamma < \infty$, moments can also be combined with Caffarelli-Kohn-Nirenberg inequalities of Nash type to produce various decay rates which, however, depend on the functional space, the range of γ and the assumptions on the initial datum. The corresponding weighted Nash and Caffarelli-Kohn-Nirenberg inequalities are by themselves interesting inequalities which have been studied for their own interest.

Date: February 26, 2020.

The decay rates that are obtained at the diffusive level are then extended to the kinetic level using adapted hypocoercivity methods.

An interesting application to a nonlinear case is the case corresponding to a Poisson coupling. The linearized equations, which a non-local coupling, as well as the full nonlinear case in dimension $d = 1$ have been studied. Notice that in dimension $d = 2$, *a priori* estimates based on generalized logarithmic Hardy-Littlewood-Sobolev inequality are needed to prove that the free energy is bounded from below.

As an introduction to the topic, the reader is invited to refer to the slides of a lecture on *Hypocoercivity* that can be found at

https://www.ceremade.dauphine.fr/~dolbeaul/Lectures/files/Flacam2019-PDE.pdf

The published version of the papers are available using the links below with

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▷ J. Dolbeault and X. Li, Φ-*Entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2637–2666.

This paper is devoted to φ-entropies applied to Fokker-Planck and kinetic Fokker-Planck equations in the whole space, with confinement. The so-called φ-entropies are Lyapunov functionals which typically interpolate between Gibbs entropies and L^2 estimates. We review some of their properties in the case of diffusion equations of Fokker-Planck type, give new and simplified proofs, and then adapt these methods to a kinetic Fokker-Planck equation acting on a phase space with positions and velocities. At kinetic level, since the diffusion only acts on the velocity variable, the transport operator plays an essential role in the relaxation process. Here we adopt the H^1 point of view and establish a sharp decay rate. Rather than giving general but quantitatively vague estimates, our goal here is to consider simple cases, benchmark available methods and obtain sharp estimates on a key example. Some φ-entropies give rise to improved entropy – entropy production inequalities and, as a consequence, to faster decay rates for entropy estimates of solutions to non-degenerate diffusion equations. We prove that faster entropy decay also holds at kinetic level away from equilibrium and that optimal decay rates are achieved only in asymptotic regimes.

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▷ J. Dolbeault, C. Mouhot, and C. Schmeiser, *Hypocoercivity for kinetic equations with linear relaxation terms*, Comptes Rendus Mathématique, 347 (2009), pp. 511 – 516.

This Note is devoted to a simple method for proving the hypocoercivity associated to a kinetic equation involving a linear time relaxation operator. It is based on the construction of an adapted Lyapunov functional satisfying a Gronwall-type inequality. The method clearly distinguishes the coercivity at microscopic level, which directly arises from the properties of the relaxation operator, and a spectral gap inequality at the macroscopic level for the spatial density, which is connected to the diffusion limit. It improves on previously known results. Our approach is illustrated by the linear BGK model and a relaxation operator which corresponds at macroscopic level to the linearized fast diffusion.

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▷ J. DOLBEAULT, C. MOUHOT, AND C. SCHMEISER, *Hypocoercivity for linear kinetic equations conserving mass*, Trans. Amer. Math. Soc., 367 (2015), pp. 3807–3828.

We develop a new method for proving hypocoercivity for a large class of linear kinetic equations with only one conservation law. Local mass conservation is assumed at the level of the collision kernel, while transport involves a confining potential, so that the solution relaxes towards a unique equilibrium state. Our goal is to evaluate in an appropriately weighted L^2 norm the exponential rate of convergence to the equilibrium. The method covers various models, ranging from diffusive kinetic equations like Vlasov-Fokker-Planck equations, to scattering models or models with time relaxation collision kernels corresponding to polytropic Gibbs equilibria, including the case of the linear Boltzmann model. In this last case and in the case of Vlasov-Fokker-Planck equations, any linear or superlinear growth of the potential is allowed.

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▷ J. DOLBEAULT, A. KLAR, C. MOUHOT, AND C. SCHMEISER, *Exponential rate of convergence to equilibrium for a model describing fiber lay-down processes*, Applied Mathematics Research eXpress, (2012).

This paper is devoted to the adaptation of the method developed with C. Mouhot and C. Schmeiser to a Fokker-Planck equation for fiber lay-down which has been studied by L. L. Bonilla, T. Götz, A. Klar, N. Marheineke, and R. Wegener. Exponential convergence towards a unique stationary state is proved in a norm which is equivalent to a weighted L^2 norm. The method is based on a micro / macro decomposition which is well adapted to the diffusion limit regime.

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▷ EMERIC BOUIN, JEAN DOLBEAULT, STEPHANE MISCHLER, CLÉMENT MOUHOT, CHRISTIAN SCHMEISER, *Hypocoercivity without confinement*, Preprint, 2017.

In this paper, hypocoercivity methods are applied to linear kinetic equations with mass conservation and without confinement, in order to prove that the solutions have an algebraic decay rate in the long-time range, which the same as the rate of the heat equation. Two alternative approaches are developed: an analysis based on decoupled Fourier modes and a direct approach where, instead of the Poincaré inequality for the Dirichlet form, Nash's inequality is employed. The first approach is also used to provide a simple proof of exponential decay to equilibrium on the flat torus. The results are obtained on a space with exponential weights and then extended to larger function spaces by a factorization method. The optimality of the rates is discussed. Algebraic rates of decay on the whole space are improved when the initial datum has moment cancellations.

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▷ EMERIC BOUIN, JEAN DOLBEAULT, CHRISTIAN SCHMEISER, *Diffusion with very weak confinement*, Preprint, to appear in Kinetic and Related Models.

This paper is devoted to Fokker-Planck and linear kinetic equations with very weak confinement corresponding to a potential with an at most logarithmic growth and no integrable stationary state. Our goal is to understand how to measure the decay rates when the diffusion wins over the confinement although the potential diverges at infinity. When there is no confinement potential, it is possible to rely on Fourier analysis and mode-by-mode estimates for the kinetic equations. Here we develop an alternative approach based on moment estimates and Caffarelli-Kohn-Nirenberg inequalities of Nash type for diffusion and kinetic equations.

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▷ E. BOUIN, J. DOLBEAULT, AND C. SCHMEISER, *A variational proof of Nash's inequality*. Preprint https://hal.archives-ouvertes.fr/hal-01940110 and https://arxiv.org/abs/1811.12770, to appear in Atti della Accademia Nazionale dei Lincei. Rendiconti Lincei. Matematica e Applicazioni, 2019.

This paper is intended to give a characterization of the optimality case in Nash's inequality, based on methods of nonlinear analysis for elliptic equations and techniques of the calculus of variations. By embedding the problem into a family of Gagliardo-Nirenberg inequalities, this approach reveals why optimal functions have compact support and also why optimal constants are determined by a simple spectral problem.

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▷ E. BOUIN, J. DOLBEAULT, L. LAFLECHE, AND C. SCHMEISER, *Hypocoercivity and sub-exponential local equilibria*. In preparation, 2019.

Hypocoercivity methods are applied to linear kinetic equations without any space confinement, when local equilibria have a sub-exponential decay. By Nash type estimates, global rates of decay are obtained, which reflect the behavior of the heat equation obtained in the diffusion limit. The method applies to Fokker-Planck and scattering collision operators. The main tools are a weighted Poincaré inequality (in the Fokker-Planck case) and norms with various weights. The advantage of weighted Poincaré inequalities compared to the more classical weak Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential and exponential local equilibria to sub-exponential local equilibria.

▷ E. BOUIN, J. DOLBEAULT, AND C. SCHMEISER, *Sharp constants in weighted Nash inequalities*. In preparation, 2019.

The goal of this paper is to identify the optimal constants and the functions which realize the equality cases in some weighted Nash inequalities. The method is based on a generalization of the proof of the equality case in the usual Nash inequality, without weights, by E. Carlen and M. Loss. It relies on symmetrization results, a change of variable which amounts to a change of the dimension, at least from the point of view of scalings, and the rewriting of the inequality as a Poincaré type inequality, on a ball. A perturbation method, which is classical in the study of symmetry in Caffarelli-Kohn-Nirenberg inequalities, provides us with a neighborhood in the set of parameters, in which the equality case is achieved only by radial functions. The inequality determines optimal decay rates in linear diffusions with homogeneous weights.

▷ JEAN DOLBEAULT AND XINGYU LI, *Generalized logarithmic Hardy-Littlewood-Sobolev inequality*, Preprint, 2019, to appear in International Math. Res. Notices.

This paper is devoted to logarithmic Hardy-Littlewood-Sobolev inequalities in the two-dimensional Euclidean space, in presence of an external potential with logarithmic growth. The coupling with the potential introduces a new parameter, with two regimes. The attractive regime reflects the standard logarithmic Hardy-Littlewood-Sobolev inequality. The second regime corresponds to a reverse inequality, with the opposite sign in the convolution term, that allows us to bound the free energy of a drift-diffusion-Poisson system from below. Our method is based on an extension of an entropy method proposed by E. Carlen, J. Carrillo and M. Loss, and on a nonlinear diffusion equation.

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▷ LANOIR ADDALA, JEAN DOLBEAULT, XINGYU LI, MOHAMED LAZHAR TAYEB, *L2-Hypocoercivity and large time asymptotics of the linearized Vlasov-Poisson-Fokker-Planck system*, Preprint, 2019.

This paper is devoted to the linearized Vlasov-Poisson-Fokker-Planck system in presence of an external potential of confinement. We investigate the large time behaviour of the solutions using hypocoercivity methods and a notion of scalar product adapted to the presence of a Poisson coupling. Our framework provides estimates which are uniform in the diffusion limit. As an application, we study the one-dimensional case and prove the exponential convergence of the nonlinear Vlasov-Poisson-Fokker-Planck system without any small mass assumption.

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