PHASE TRANSITIONS AND SYMMETRY IN PDES

JEAN DOLBEAULT

Phase transitions are essential phenomena in various areas of physics. They are usually classified according to a certain degree of regularity when parameters vary and here we are interested in transitions which are usually classified as *transitions of second order*.

A very nice illustration of our purpose has been studied by Xingyu Li in his PhD thesis[1], which deals with model for the flocking of birds, who have an optimal flight speed but no preferred direction and try to align with the average velocity of the other birds. Since there is some noise (because, for instance, birds make mistakes in computing the average velocity), one can distinguish two regimes, corresponding to low and high noise. In the high noise regime (disordered phase), birds cannot align and the average velocity of a stationary solution is zero, producing a stationary symmetric solution (speeds follow a distribution which is isotropic). At low noise (ordered phase), on the contrary, birds agree on a common direction and speeds align, thus producing stationary solution which are polarized. Since the equations are invariant under rotations, there is a continuum of solutions generated by the rotation of the average direction. In this model, the average velocity is the *order parameter*, which determines the *phase*, that is the most stable stationary configuration: there is a critical intensity of the noise which determines the phase transition between the ordered phase and the disordered phase. However, the analysis can be pushed much further. It is not a big surprise that the isotropic stationary solution also exists in the low noise regime, but it is linearly unstable and the critical intensity of the noise determines the sign change of the lowest eigenvalue of the linearized evolution operator. In the ordered phase, the effect of the average velocity is to tilt the potential in such a way that the minimal energy is not anymore achieved by the isotropic stationary solution. Since the model involves a mean field term (the average velocity of the solution), an interesting point is to consider an adapted scalar product with a non-local term (actually, the average velocity) and compute the eigenvalues in this functional framework. The model is based on an evolution equation which is not limited to stationary solutions. The beauty of the adapted scalar product is that the eigenvalues are also a sharp measure of the exponential rate of convergence of non-stationary solutions towards the stationary solutions, at least in some appropriate bassin of attraction. This provides a beautiful link of the dynamical properties of the solutions in the non-linear regime of evolution with the stability properties of the solutions in the stationary regime and the the phase transition analysis.

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[1] X. Li. Flocking: Phase transition and asymptotic behaviour. arXiv e-prints:1906.07517, Jun 2019.

The fact that eigenvalues play an important role in symmetry results is of course not a surprise and even if it is sometimes hidden in classical methods like the *moving planes method*, this has already been investigated (see below in the reference list).

Of course, symmetry is not to be expected for all solutions and we focus here on *ground states* in the sense of physics, that is, solutions which minimize an energy. The question at stake is the question of the symmetry of the solutions of a PDE which is invariant under a transformation. This is very close of some of the ideas of the *Principe de symétrie de Pierre Curie* stated at the end of the 19^{th} century. If a PDE is invariant under rotations, when do we know that solutions, for instance the solutions which minimize the energy, inherit of radial symmetry ? If a nonlinearity tends to concentrate the solution and if an external potential pushes in the same direction (in the sense that symmetrization decreases the potential energy), there is no issue and the ground state (or at least one ground state) is symmetric if it exists. However, when the external potential tends to decenter the solution, the answer is more subtle. If the symmetric solution is linearly unstable, it is clear that the ground state cannot be symmetric. A natural question is to ask whether the converse is true: if the symmetric solution is linearly stable, is it a ground state ? The generic answer is of course *no*, because linear stability is a local property while the symmetry of the ground state is a global property, but an essential question in many applications is to identify models for which this is true. The same question can be asked when the external potential is replaced by some geometric constraint (problems on the sphere, for instance) or by isotropic weights which compete with the concentration.

A beautiful tool is provided by the *carré du champ method* developed by D. Bakry and M. Emery in the mid-80^{ies} and later extended to nonlinear settings: a well chosen flow allows to relate an arbitrary initial datum through an evolution equation to a stationary or self-similar state and enlights the role of the eigenvalues (in the appropriate functional framework) as the key tool for recovering the symmetry properties of any possible ground state. This can applied to interpolation inequalities on the sphere or to Caffarelli-Kohn-Nirenberg inequalities (see the link on *Symmetry and symmetry breaking* below). By the *carré du champ method*, one can also prove an inequality which had been found originally by P. Exner, E. Harrel and M. Loss and applies to *magnetic rings*. Magnetic rings estimates are useful for proving magnetic Hardy inequalities in dimension 2 and consider nonlinear magnetic Schrödinger equations obtained as the *ground states* of interpolation inequalities with kinetic energies involving a magnetic field. For constant magnetic fields, symmetry issues are essentially open. However, in the special case of a Bohm-Aharonov magnetic fields in dimension two, with invariance under rotation, it is possible to reduce the problem to some Caffarelli-Kohn-Nirenberg type inequalities and obtain a symmetry result (see the link on *Magnetic fields* below) which is almost complementary of a symmetry breaking result. This establishes the existence of a phase transition whose exact nature is still to be elucidated.

This extended abstract comes as an explanation for the lecture given at the 2019 Flacam conference whose slides can be found at

https://www.ceremade.dauphine.fr/~dolbeaul/Lectures/files/Flacam2019.pdf

and there is a significant overlap with the presentations of *Symmetry and symmetry breaking* issues that the interested reader could find at

https://www.ceremade.dauphine.fr/~dolbeaul/News/Symmetry.pdf

and of results on *Magnetic fields* available at

https://www.ceremade.dauphine.fr/~dolbeaul/News/Magnetic.pdf

The published version of the papers are available using the links below with

login: Dolbeault
password: Jean

▷ J. DOLBEAULT AND P. FELMER, *Symétrie des solutions d'équations semi-linéaires elliptiques*, C. R. Acad. Sci. Paris Sér. I Math., 329 (1999), pp. 677–682.

This note is devoted to symmetry and monotonicity results corresponding to two cases where the classical theorem by B. Gidas, W.-M. Ni and L. Nirenberg does not apply. When the equation is non-autonomous, we introduce a comparison with a linear problem. When the nonlinearity is non-Lipschitz, we use local moving hyperplanes, local inversion and unique continuation methods. The usual notion of symmetry is then replaced by a notion of local radial symmetry.

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▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs.*, Proc. Int. Cong. of Math. 2018, Rio de Janeiro, 3 (2018), pp. 2279–2304.

The issue of symmetry and symmetry breaking is fundamental in all areas of science. Symmetry is often assimilated to order and beauty while symmetry breaking is the source of many interesting phenomena such as phase transitions, instabilities, segregation, self-organization, *etc.* In this contribution we review a series of sharp results of symmetry of nonnegative solutions of nonlinear elliptic differential equation associated with minimization problems on Euclidean spaces or manifolds. Nonnegative solutions of those equations are unique, a property that can also be interpreted as a rigidity result. The method relies on linear and nonlinear flows which reveal deep and robust properties of a large class of variational problems. Local results on linear instability leading to symmetry breaking and the bifurcation of non-symmetric branches of solutions are reinterpreted in a larger, global, variational picture in which our flows characterize directions of descent.

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▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*, Journal of elliptic and parabolic equations, 2 (2016), pp. 267–295.

This paper is devoted to the computation of the asymptotic boundary terms in entropy methods applied to a fast diffusion equation with weights associated with Caffarelli-Kohn-Nirenberg interpolation inequalities. So far, only elliptic equations have been considered and our goal is to justify, at least partially, an extension of the *carré du champ* / Bakry-Emery / Rényi entropy methods to parabolic equations. This makes sense because evolution equations are at the core of the heuristics of the method even when only elliptic equations are considered, but this also raises difficult questions on the regularity and on the growth of the solutions in presence of weights.
We also investigate the relations between the optimal constant in the entropy – entropy production inequality, the optimal constant in the information – information production inequality, the asymptotic growth rate of generalized Rényi entropy powers under the action of the evolution equation and the optimal range of parameters for symmetry breaking issues in Caffarelli-Kohn-Nirenberg inequalities, under the assumption that the weights do not introduce singular boundary terms at $x = 0$. These considerations are new even in the case without weights. For instance, we establish the equivalence of carré du champ and Rényi entropy methods and explain why entropy methods produce optimal constants in entropy – entropy production and Gagliardo-Nirenberg inequalities in absence of weights, or optimal symmetry ranges when weights are present.

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▷ D. BONHEURE, J. DOLBEAULT, M. J. ESTEBAN, A. LAPTEV, AND M. LOSS, *Symmetry results in two-dimensional inequalities for Aharonov-Bohm magnetic fields*, Communications in Mathematical Physics, (2019).

This paper is devoted to interpolation inequalities of Gagliardo-Nirenberg type associated with Schrödinger operators involving Aharonov-Bohm magnetic potentials and related magnetic Hardy inequalities in dimensions 2 and 3. The focus is on symmetry properties of the optimal functions, with explicit ranges of symmetry and symmetry breaking in terms of the intensity of the magnetic potential.

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▷ J. DOLBEAULT, M. J. ESTEBAN, A. LAPTEV, AND M. LOSS, *Magnetic rings*, Journal of Mathematical Physics, 59 (2018), p. 051504.

We study functional and spectral properties of perturbations of the operator $-(\partial_s + i\,a)^2$ in $\mathrm{L}^2(\mathbb{S}^1)$. This operator appears when considering the restriction to the unit circle of a two dimensional Schrödinger operator with the Bohm-Aharonov vector potential. We prove a Hardy-type inequality on $\mathbb{R}^2$ and, on $\mathbb{S}^1$, a sharp interpolation inequality and a sharp Keller-Lieb-Thirring inequality.

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▷ J. DOLBEAULT, M. J. ESTEBAN, A. LAPTEV, AND M. LOSS, *Interpolation inequalities and spectral estimates for magnetic operators*, Annales Henri Poincaré, 19 (2018), pp. 1439–1463.

We prove magnetic interpolation inequalities and Keller-Lieb-Thirring estimates for the principal eigenvalue of magnetic Schrödinger operators. We establish explicit upper and lower bounds for the best constants and show by numerical methods that our theoretical estimates are accurate.

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J. DOLBEAULT: CEREMADE (CNRS UMR N° 7534), PSL UNIVERSITY, UNIVERSITÉ PARIS-DAUPHINE, PLACE DE LATTRE DE TASSIGNY, 75775 PARIS 16, FRANCE

E-mail address: dolbeaul@ceremade.dauphine.fr