SYMMETRY AND SYMMETRY BREAKING

JEAN DOLBEAULT

The study of symmetry breaking in ferro-magnetic materials by Pierre Curie is at the origin of the topic. When they are at rest, many system of physics minimize an anergy. In simple and often fundamental cases, laws of physics are isotropic in the sense that there is no *a priori* preferred direction, and the corresponding partial differential equations, when the system can be conveniently described with such tools, are for instance invariant under rotation. This raises the question to know if the solutions then inherit of the property, *i.e.*, if they are spherically symmetric.

The notion of solution has of course to be made precise, as out of equilibrium solutions or excited states generically have less symmetry than the equations they solved. What we have in mind are *ground state solutions* in the sense of global minimizers of an energy, or when it comes to the study of the underlying functional inequalities, optimal functions for the inequality written in sharp form, with optimal constants.

A very standard case is obtained by considering energies which involve a focusing nonlinearity, which tend to concentrate the solution into a radially symmetric, bubble-like profile, and a radial potential which tries to move away the optimal solution from the center. If the coupling with the potential is strong enough, the minimizing solution cannot anymore be radially symmetric. This scenario is typical of the *symmetry breaking* mechanism, which produces a continuum of highly non-trivial solutions which are of importance in various areas of physics.

There are several classical methods for proving the symmetry breaking, which range from non-radial perturbations of radially symmetric solutions to direct comparison of energies or rely on well chosen test functions. On the contrary, when a potential competes with a nonlinearity in such a way that there is no simple reduction to equations for which standard methods for proving symmetry apply (like symmetrization, use of reflections, or moving plane methods), proving symmetry turns out to be much more difficult and beyond uniqueness or convexity methods, there are very few available tools. The *carré du champ method* is one of such tools. Only recently it has been realized how powerful the method is and it has been applied to various problems in which a non-symmetric branch of solutions bifurcates from the symmetric ones.

As an introduction to the topic, the reader is invited to refer to the slides of a lecture on *Symmetry and uniqueness by nonlinear flow methods* that can be found at

Date: February 26, 2020.

https://www.ceremade.dauphine.fr/~dolbeaul/Lectures/files/Monastir-8-2-2019.pdf

The published version of the papers are available using the links below with

login: Dolbeault
password: Jean

Here is a short description of some papers on *symmetry* and *symmetry breaking*:

▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs.*, Proc. Int. Cong. of Math. 2018, Rio de Janeiro, 3 (2018), pp. 2279–2304.

The issue of symmetry and symmetry breaking is fundamental in all areas of science. Symmetry is often assimilated to order and beauty while symmetry breaking is the source of many interesting phenomena such as phase transitions, instabilities, segregation, self-organization, *etc.* In this contribution we review a series of sharp results of symmetry of nonnegative solutions of nonlinear elliptic differential equation associated with minimization problems on Euclidean spaces or manifolds. Nonnegative solutions of those equations are unique, a property that can also be interpreted as a rigidity result. The method relies on linear and nonlinear flows which reveal deep and robust properties of a large class of variational problems. Local results on linear instability leading to symmetry breaking and the bifurcation of non-symmetric branches of solutions are reinterpreted in a larger, global, variational picture in which our flows characterize directions of descent.

Pdf file

▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization*, Journal of elliptic and parabolic equations, 2 (2016), pp. 267–295.

This paper is devoted to the computation of the asymptotic boundary terms in entropy methods applied to a fast diffusion equation with weights associated with Caffarelli-Kohn-Nirenberg interpolation inequalities. So far, only elliptic equations have been considered and our goal is to justify, at least partially, an extension of the *carré du champ* / Bakry-Emery / Rényi entropy methods to parabolic equations. This makes sense because evolution equations are at the core of the heuristics of the method even when only elliptic equations are considered, but this also raises difficult questions on the regularity and on the growth of the solutions in presence of weights.
We also investigate the relations between the optimal constant in the entropy – entropy production inequality, the optimal constant in the information – information production inequality, the asymptotic growth rate of generalized Rényi entropy powers under the action of the evolution equation and the optimal range of parameters for symmetry breaking issues in Caffarelli-Kohn-Nirenberg inequalities, under the assumption that the weights do not introduce singular boundary terms at $x = 0$. These considerations are

new even in the case without weights. For instance, we establish the equivalence of carré du champ and Rényi entropy methods and explain why entropy methods produce optimal constants in entropy – entropy production and Gagliardo-Nirenberg inequalities in absence of weights, or optimal symmetry ranges when weights are present.

Pdf file

▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Interpolation inequalities on the sphere: linear vs. nonlinear flows (inégalités d'interpolation sur la sphère : flots non-linéaires vs. flots linéaires)*, Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2017), pp. 351–379.

This paper is devoted to sharp interpolation inequalities on the sphere and their proof using flows. The method explains some rigidity results and proves uniqueness in related semilinear elliptic equations. Nonlinear flows allow to cover the interval of exponents ranging from Poincaré to Sobolev inequality, while an intriguing limitation (an upper bound on the exponent) appears in the carré du champ method based on the heat flow. We investigate this limitation, describe a counter-example for exponents which are above the bound, and obtain improvements below.

Pdf file

▷ J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and euclidean spaces*, Inventiones mathematicae, 206 (2016), pp. 397–440.

This paper is motivated by the characterization of the optimal symmetry breaking region in Caffarelli-Kohn-Nirenberg inequalities. As a consequence, optimal functions and sharp constants are computed in the symmetry region. The result solves a longstanding conjecture on the optimal symmetry range. As a byproduct of our method we obtain sharp estimates for the principal eigenvalue of Schrödinger operators on some non-flat non-compact manifolds, which to the best of our knowledge are new. The method relies on generalized entropy functionals for nonlinear diffusion equations. It opens a new area of research for approaches related to carré du champ methods on non-compact manifolds. However, key estimates depend as much on curvature properties as on purely nonlinear effects. The method is well adapted to functional inequalities involving simple weights and also applies to general cylinders. Beyond results on symmetry and symmetry breaking, and on optimal constants in functional inequalities, rigidity theorems for nonlinear elliptic equations can be deduced in rather general settings.

Pdf file

▷ J. DOLBEAULT, M. J. ESTEBAN, M. LOSS, AND M. MURATORI, *Symmetry for extremal functions in subcritical Caffarelli–Kohn–Nirenberg inequalities*, Comptes Rendus Mathématique, 355 (2017), pp. 133 – 154.

We use the formalism of the Rényi entropies to establish the symmetry range of extremal functions in a family of subcritical Caffarelli–Kohn–Nirenberg inequalities. By extremal functions we mean functions that realize the equality case in the inequalities, written with optimal constants. The method extends recent results on critical Caffarelli–Kohn-Nirenberg inequalities. Using heuristics given by a nonlinear diffusion equation, we give a variational proof of a symmetry result, by establishing a rigidity theorem: in the symmetry region, all positive critical points have radial symmetry and are therefore equal to the unique positive, radial critical point, up to scalings and multiplications. This result is sharp. The condition on the parameters is indeed complementary of the condition that determines the region in which symmetry breaking holds as a consequence of the linear instability of radial optimal functions. Compared to the critical case, the subcritical range requires new tools. The Fisher information has to be replaced by Rényi entropy powers, and since some invariances are lost, the estimates based on the Emden–Fowler transformation have to be modified.

Pdf file

▷ J. DOLBEAULT, AND M. J. ESTEBAN, *Improved interpolation inequalities and stability*, Preprint.

For exponents in the subcritical range, we revisit some optimal interpolation inequalities on the sphere with carré du champ methods and use the remainder terms to produce improved inequalities. The method provides us with lower estimates of the optimal constants in the symmetry breaking range and stability estimates for the optimal functions. Some of these results can be reformulated in the Euclidean space using the stereographic projection.

Pdf file

J. DOLBEAULT: CEREMADE (CNRS UMR N° 7534), PSL UNIVERSITY, UNIVERSITÉ PARIS-DAUPHINE, PLACE DE LATTRE DE TASSIGNY, 75775 PARIS 16, FRANCE

E-mail address: dolbeaul@ceremade.dauphine.fr