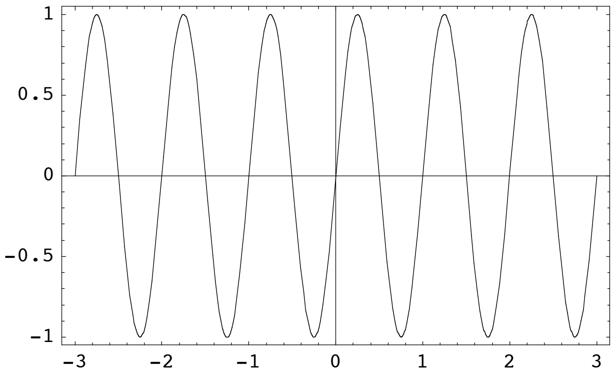
Travelling fronts in stochastic Stokes' drifts (1 / 3)

Adrien Blanchet, Jean Dolbeault, and Michal Kowalczyk

```
Off[NIntegrate::"slwcon"]
Off[General::"spell1"]
Off[NIntegrate::"ncvb"]
Off[NIntegrate::"ploss"]
```

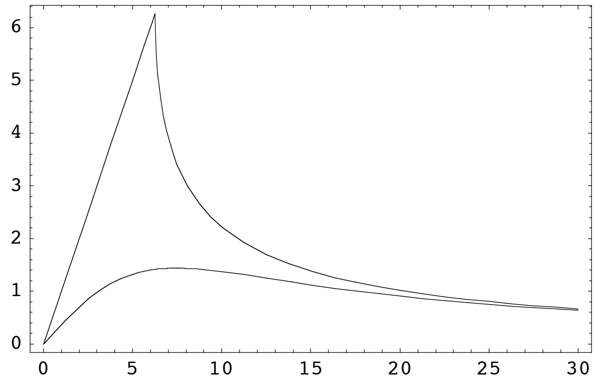
The case of the sinusoidal function

```
ψ[x_] := Sin[2 π x]
Plot[ψ[x], {x, -3, 3}, Frame → True];
```



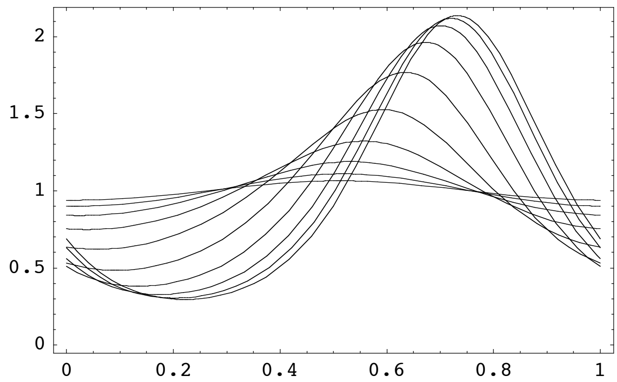
Speed of the center of mass

```
T[ω_] := NIntegrate[1/(ω + ψ'[x]), {x, 0, 1}]
Speed[ω_] := If[ω < 2 π, ω, ω - 1 / T[ω]]
a[ω_] := ⅇ^ω - 1
b[ω_] := NIntegrate[Exp[ψ[y] + ω * y], {y, 0, 1}]
c[ω_] := NIntegrate[Exp[-ψ[y] - ω * y], {y, 0, 1}]
d[ω_] := NIntegrate[x * Exp[ψ[t * x] + ω * t * x - ψ[x] - ω * x], {x, 0, 1}, {t, 0, 1}]
A[ω_] := a[ω] / (b[ω] c[ω] + a[ω] d[ω])
B[ω_] := b[ω] / (b[ω] c[ω] + a[ω] d[ω])
Velocity[ω_] := ω - A[ω]
Plot[{Speed[ω], Velocity[ω]}, {ω, 0, 30}, PlotRange → All, Frame → True];
```



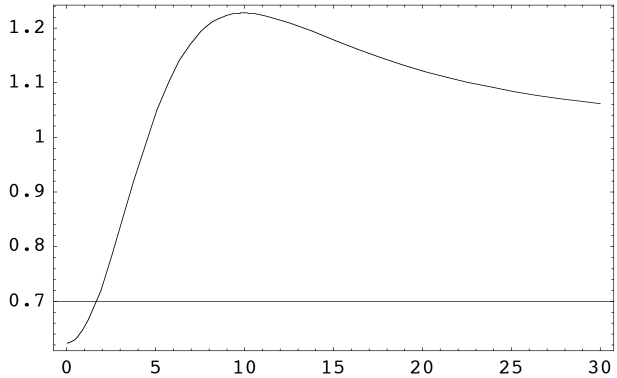
The solution of the doubly periodic problem (tilted Brownian ratchet)

```
g[z_, ω_] := Module[{M = {A[ω], B[ω]}},
  g[z] /.
   NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x], g[0] == M[[2]] Exp[-ψ[0]]}, g, {x, 0, 1}]]
Show[Table[Plot[g[y, Exp[Log[100] k / 9]], {y, 0, 1}, DisplayFunction -> Identity],
  {k, 0, 9}], DisplayFunction → $DisplayFunction, Frame → True];
```



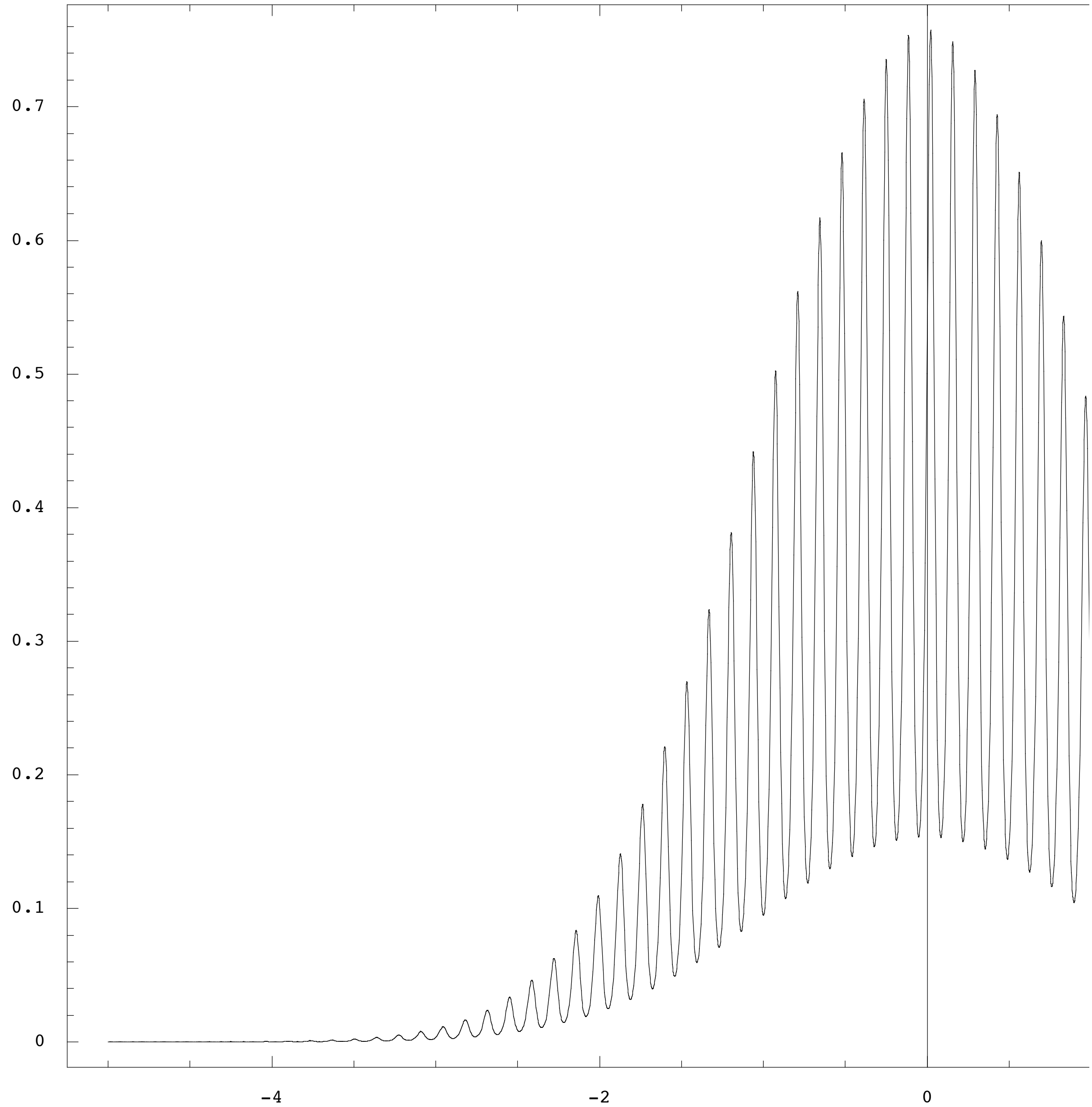
The effective diffusion coefficient

```
a1[ω_] := Module[{M = {A[ω], B[ω], Velocity[ω]}},
   h1[1]/(ⅇ^ω - 1) /. NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x],
      g[0] == M[[2]] Exp[-ψ[0]], h0'[x] == (ψ'[x] + M[[3]]) g[x], h0[0] == 0,
      h1'[x] == -(2 g[x] + h0[x]) Exp[ω x + ψ[x]], h1[0] == 0}, {g, h0, h1}, {x, 0, 1}]][[1]]
ϰ[ω_] := Module[{M = {A[ω], B[ω], Velocity[ω], a1[ω]}},
   h2[1] /. NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x], g[0] == M[[2]] Exp[-ψ[0]],
      h0'[x] == (ψ'[x] + M[[3]]) g[x], h0[0] == 0, h1'[x] == -(2 g[x] + h0[x]) Exp[ω x + ψ[x]],
      h1[0] == 0, g1'[x] == -(2 g[x] + h0[x] + (ω + ψ'[x]) g1[x]), g1[0] == M[[4]] Exp[-ψ[0]],
      h2'[x] == (ψ'[x] + M[[3]]) g1[x], h2[0] == 1}, {g, h0, h1, g1, h2}, {x, 0, 1}]][[1]]
Plot[ϰ[ω], {ω, 0.01, 30}, Frame → True];
```

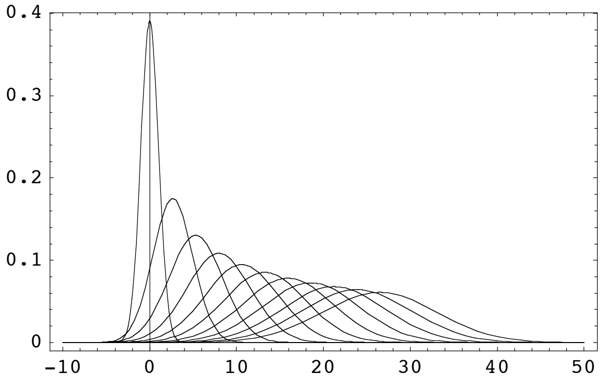


Intermediate asymptotics: the traveling front

```
Frac[x_] := 1 + x - Ceiling[x]
IA[t_, ω_] := Module[{M = {κ[ω], g[Frac[e^t x + (e^(2 t) - 1) A[ω] / 2], ω]}}, Plot[
   M[[2]] e^(-x^2/(2 M[[1]])) / Sqrt[2 π M[[1]]], {x, -5, 5}, PlotRange → All, Frame → True, PlotPoints → 250]]
IA[
  2,
  5];
```

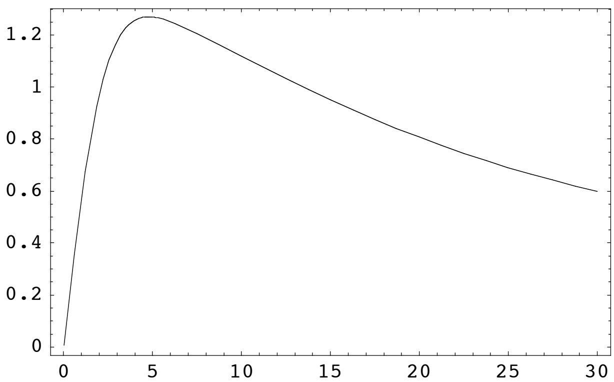


```
Profile[t_, ω_] := Module[{M = {κ[ω], Velocity[ω]}}, Plot[E^(-(x - M[[2]] t)^2/(2 M[[1]] (1 + 2 t)))/Sqrt[2 π M[[1]] (1 + 2 t)],
    {x, -10, 50}, PlotRange → All, Frame → True, DisplayFunction -> Identity]]
Show[Table[Profile[n, 5], {n, 0, 20, 2}], DisplayFunction -> $DisplayFunction];
```



Computation of the Péclet number

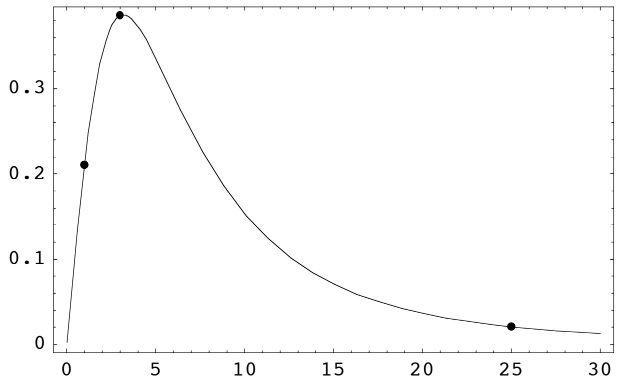
```
Plot[Velocity[ω]/κ[ω], {ω, 0.01, 30}, Frame → True];
```



Computation of the Efficiency

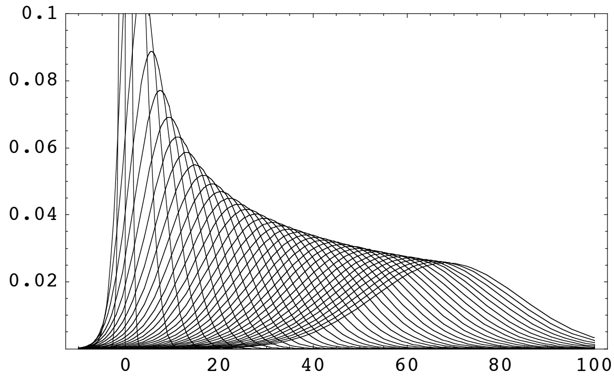
```
Eff[ω_] := Module[{M = ϰ[ω]}, Evaluate[(Velocity[ω])^2 / (ω M)]]
Values = {1, 3, 25};
Table[
 {Values[[k]], Velocity[Values[[k]]], ϰ[Values[[k]]], Eff[Values[[k]]]}, {k, 1, 3}]
EffPoints = Table[{Values[[k]], Eff[Values[[k]]]}, {k, 1, 3}];
Show[Plot[Eff[ω], {ω, 0.01, 30}, Frame → True, DisplayFunction → Identity],
  ListPlot[EffPoints, PlotStyle → PointSize[0.015], DisplayFunction → Identity],
  DisplayFunction → $DisplayFunction];
```

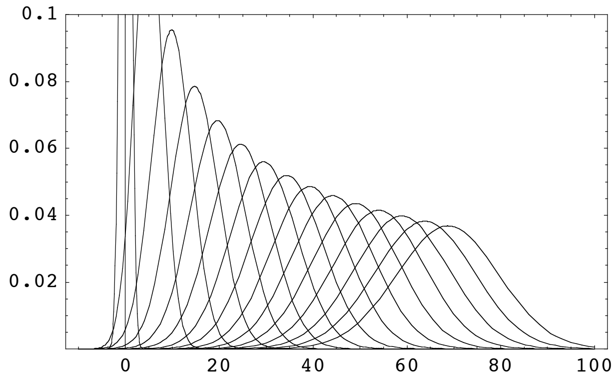
```
{{1, 0.370027, 0.651926, 0.210024},
 {3, 0.981553, 0.83311, 0.385482}, {25, 0.748605, 1.08446, 0.0206705}}
```

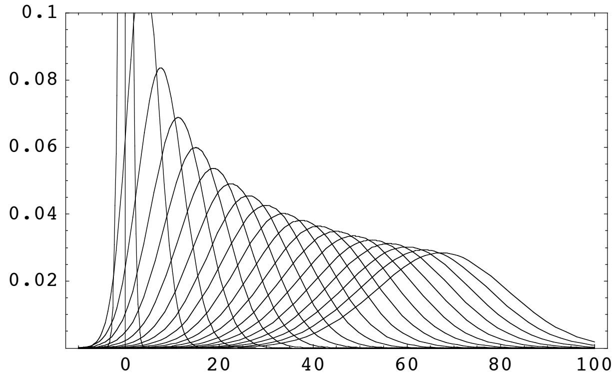



```
Profile[t_, ω_] := Module[{M = {ϰ[ω], Velocity[ω]}},
  Plot[e^(-(x-M[[2]] t)^2/(2 M[[1]] (1+2 t))) / Sqrt[2 π M[[1]] (1 + 2 t)], {x, -10, 100}, Frame → True, DisplayFunction -> Identity]]

Front[ω_] := Show[Table[Profile[n, ω], {n, 0, 70/Velocity[ω], 5}],
  DisplayFunction → $DisplayFunction, PlotRange → {All, {0, 0.1}}]
Show[Table[Front[Values[[k]]], {k, 1, 3}], DisplayFunction → Identity];
```

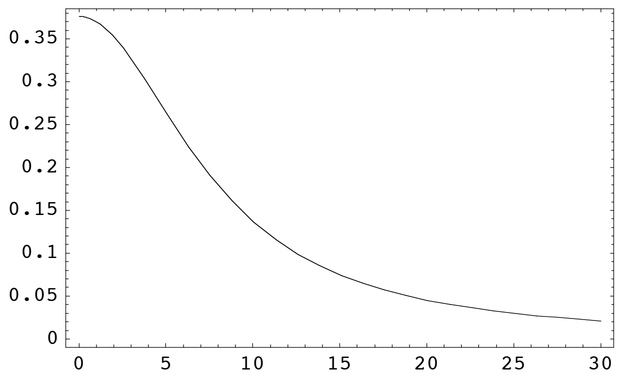




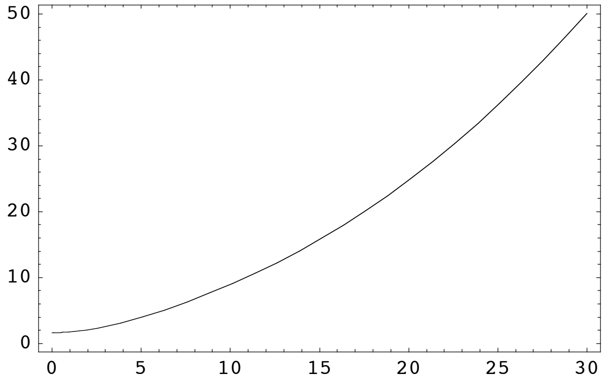


Computation of the mobility and violation of Einstein's relation

```
μ[ω_] := Velocity[ω] / ω
Plot[μ[ω], {ω, 0.01, 30}, Frame → True];
```



```
Off[Part::"partd"]
Einstein[ω_] := Module[{M = {κ[ω], μ[ω]}}, Evaluate[M[[1]]/M[[2]]]]
Plot[Einstein[ω], {ω, 0.01, 30}, Frame → True];
```



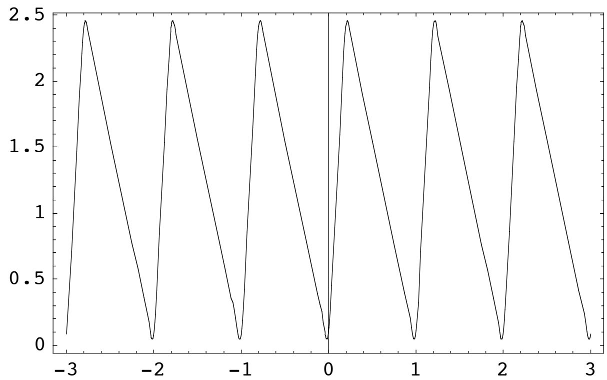
Travelling fronts in stochastic Stokes' drifts (2 / 3)

Adrien Blanchet, Jean Dolbeault, and Michal Kowalczyk

```
Off[NIntegrate::"slwcon"]
Off[General::"spell1"]
Off[NIntegrate::"ncvb"]
Off[NIntegrate::"ploss"]
```

The case of the smooth sawtooth potential

```
Intensity = 5; ϵ = 0.2; Nbre = 8;
Potential[x_] := If[x < ϵ, x/ϵ, (1 - x)/(1 - ϵ)]
Acoef = Table[NIntegrate[Potential[x] Sin[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
Bcoef = Table[NIntegrate[Potential[x] Cos[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
B0 = NIntegrate[Potential[x], {x, 0, 1}];
ψ[x_] := Intensity (Sum[Acoef[[k]] Sin[2 π k x], {k, 1, Nbre}] +
    B0/2 + Sum[Bcoef[[k]] Cos[2 π k x], {k, 1, Nbre}])
Plot[ψ[x], {x, -3, 3}, Frame → True];
```



Speed of the center of mass for positive values of ω

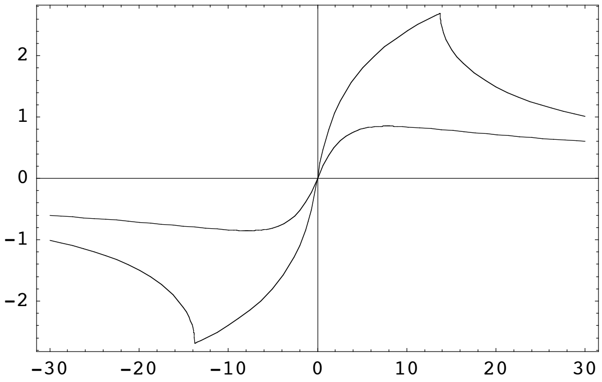
```
x /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
ω0 = ψ'[x] /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
T[ω_] := NIntegrate[1/(ω - ψ'[x]), {x, 0, 1}]
Speed[ω_] := If[ω < ω0, ω, ω - 1 / T[ω]]
a[ω_] := e^ω - 1
b[ω_] := NIntegrate[Exp[ψ[y] + ω * y], {y, 0, 1}]
c[ω_] := NIntegrate[Exp[-ψ[y] - ω * y], {y, 0, 1}]
d[ω_] := NIntegrate[x * Exp[ψ[t * x] + ω * t * x - ψ[x] - ω * x], {x, 0, 1}, {t, 0, 1}]
A[ω_] := a[ω] / (b[ω] c[ω] + a[ω] d[ω])
B[ω_] := b[ω] / (b[ω] c[ω] + a[ω] d[ω])
Velocity[ω_] := ω - A[ω]
Plog1 = Plot[{Log[1 + Speed[ω]], Log[1 + Velocity[ω]]},
   {ω, 0, 30}, PlotRange → All, DisplayFunction → Identity];
```

Speed of the center of mass for negative values of ω

```
Intensity = 5; ϵ = 0.8; Nbre = 8;
Potential[x_] := If[x < ϵ, x/ϵ, (1 - x)/(1 - ϵ)]
Acoef = Table[NIntegrate[Potential[x] Sin[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
Bcoef = Table[NIntegrate[Potential[x] Cos[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
B0 = NIntegrate[Potential[x], {x, 0, 1}];
ψ[x_] := Intensity (Sum[Acoef[[k]] Sin[2 π k x], {k, 1, Nbre}] +
     B0/2 + Sum[Bcoef[[k]] Cos[2 π k x], {k, 1, Nbre}])
x /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
ω0 = ψ'[x] /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
T[ω_] := NIntegrate[1/(ω - ψ'[x]), {x, 0, 1}]
Speed[ω_] := If[ω < ω0, ω, ω - 1 / T[ω]]
a[ω_] := e^ω - 1
b[ω_] := NIntegrate[Exp[ψ[y] + ω * y], {y, 0, 1}]
c[ω_] := NIntegrate[Exp[-ψ[y] - ω * y], {y, 0, 1}]
d[ω_] := NIntegrate[x * Exp[ψ[t * x] + ω * t * x - ψ[x] - ω * x], {x, 0, 1}, {t, 0, 1}]
A[ω_] := a[ω] / (b[ω] c[ω] + a[ω] d[ω])
B[ω_] := b[ω] / (b[ω] c[ω] + a[ω] d[ω])
Velocity[ω_] := ω - A[ω]
Plog2 = Plot[{-Log[1 + Speed[-ω]], -Log[1 + Velocity[-ω]]},
   {ω, -30, 0}, PlotRange → All, DisplayFunction → Identity];
```

Plot of the speed of the center of mass for positive and negative values of ω

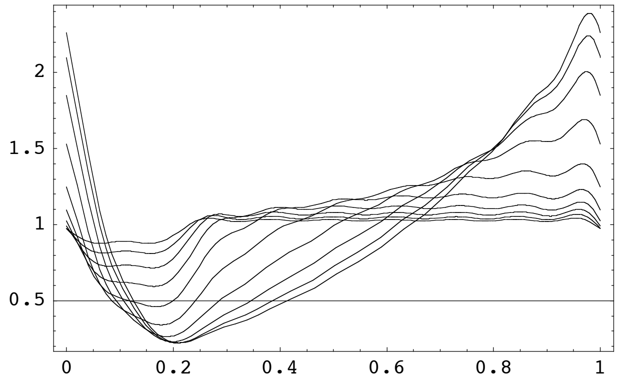
```
Show[Plog1, Plog2, Frame → True, DisplayFunction → $DisplayFunction];
```



The solution of the doubly periodic problem (tilted Brownian ratchet)

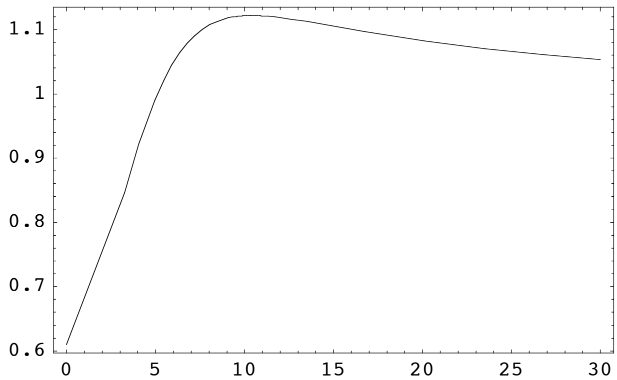
```
Intensity = 5; ϵ = 0.2; Nbre = 8;
Potential[x_] := If[x < ϵ, x/ϵ, (1 - x)/(1 - ϵ)]
Acoef = Table[NIntegrate[Potential[x] Sin[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
Bcoef = Table[NIntegrate[Potential[x] Cos[2 π k x], {x, 0, 1}], {k, 1, Nbre}];
B0 = NIntegrate[Potential[x], {x, 0, 1}];
ψ[x_] := Intensity (Sum[Acoef[[k]] Sin[2 π k x], {k, 1, Nbre}] +
     B0/2 + Sum[Bcoef[[k]] Cos[2 π k x], {k, 1, Nbre}])
x /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
ω0 = ψ'[x] /. FindRoot[ψ''[x] == 0, {x, 0.02, 0.1}];
T[ω_] := NIntegrate[1/(ω - ψ'[x]), {x, 0, 1}]
Speed[ω_] := If[ω < ω0, ω, ω - 1 / T[ω]]
a[ω_] := e^ω - 1
b[ω_] := NIntegrate[Exp[ψ[y] + ω * y], {y, 0, 1}]
c[ω_] := NIntegrate[Exp[-ψ[y] - ω * y], {y, 0, 1}]
d[ω_] := NIntegrate[x * Exp[ψ[t * x] + ω * t * x - ψ[x] - ω * x], {x, 0, 1}, {t, 0, 1}]
A[ω_] := a[ω] / (b[ω] c[ω] + a[ω] d[ω])
B[ω_] := b[ω] / (b[ω] c[ω] + a[ω] d[ω])
Velocity[ω_] := ω - A[ω]
```

```
Pltg[ω_] := Module[{M = {A[ω], B[ω]}}, Plot[
   Evaluate[g[z] /. NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x], g[0] == M[[2]] Exp[-ψ[0]]},
      g, {x, 0, 1}], {z, 0, 1}], DisplayFunction → Identity]]
Show[Table[Pltg[Exp[Log[100] k / 9]], {k, 0, 9}], DisplayFunction → $DisplayFunction,
  Frame → True, PlotRange → All];
```



The effective diffusion coefficient

```
a1[ω_] := Module[{M = {A[ω], B[ω], Velocity[ω]}},
   h1[1] / (e^ω - 1) /. NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x],
      g[0] == M[[2]] Exp[-ψ[0]], h0'[x] == (ψ'[x] + M[[3]]) g[x], h0[0] == 0,
      h1'[x] == -(2 g[x] + h0[x]) Exp[ω x + ψ[x]], h1[0] == 0}, {g, h0, h1}, {x, 0, 1}]][[1]]
κ[ω_] := Module[{M = {A[ω], B[ω], Velocity[ω], a1[ω]}},
   h2[1] /. NDSolve[{g'[x] == M[[1]] - (ω + ψ'[x]) g[x], g[0] == M[[2]] Exp[-ψ[0]],
      h0'[x] == (ψ'[x] + M[[3]]) g[x], h0[0] == 0, h1'[x] == -(2 g[x] + h0[x]) Exp[ω x + ψ[x]],
      h1[0] == 0, g1'[x] == -(2 g[x] + h0[x] + (ω + ψ'[x]) g1[x]), g1[0] == M[[4]] Exp[-ψ[0]],
      h2'[x] == (ψ'[x] + M[[3]]) g1[x], h2[0] == 1}, {g, h0, h1, g1, h2}, {x, 0, 1}]][[1]]
Plot[κ[ω], {ω, 0.01, 30}, Frame → True, PlotPoints → 10, PlotRange → All];
```



Travelling fronts in stochastic Stokes' drifts (3 / 3)

Adrien Blanchet, Jean Dolbeault, and Michal Kowalczyk

Error function and definition of the efficiency

```
Integrate[E^(-x^2/2)/Sqrt[2 π], {x, -∞, -1}]
N[%]
```

$\frac{1}{2}\,\text{Erfc}\left[\frac{1}{\sqrt{2}}\right]$

0.158655

```
x0 = 3;
Pa =
  Plot[{E^(-x^2/2)/Sqrt[2 π], E^(-(x-x0)^2/(2 x0^2))/(Sqrt[2 π] x0)}, {x, -7, 12}, PlotRange → All, DisplayFunction → Identity];
Pb = ListPlot[{{x0, -0.07}, {x0, 0.15}}, PlotJoined → True, DisplayFunction → Identity];
TableErf = Join[{{0, 0}, {-7, 0}}, Table[{x, E^(-(x-x0)^2/(2 x0^2))/(Sqrt[2 π] x0)}, {x, -7, 0, 0.04}]];
Pc = Show[Graphics[{GrayLevel[0.5], Polygon[TableErf]}], DisplayFunction → Identity];
Show[Pa, Pb, Pc, DisplayFunction → $DisplayFunction, Ticks → None];
```