A logarithmic Hardy inequality

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Symmetry breaking by linearization

Definition of the spectrum

$$\lambda = \frac{1}{4}\,(d-2-2a)\,(p-2)\,\sqrt{\frac{p+2}{2\,p\,\theta-(p-2)}}$$

$$\frac{1}{4}\,(-2-2a+d)\,(-2+p)\,\sqrt{\frac{2+p}{2-p+2\,p\,\theta}}$$

$$\kappa = \text{Simplify}\left[\frac{p-1}{\theta}\,\frac{\lambda^2\,\frac{4}{p^2-4}+\frac{(d-2-2a)^2}{4}}{\frac{4}{p+2}}\right]$$

$$\frac{(-2-2a+d)^2\,(-1+p)\,p\,(2+p)}{8\,(2+p\,(-1+2\,\theta))}$$

$$\mu = \text{Simplify}\left[\frac{(d-2-2a)^2}{4}+\frac{1-\theta}{\theta}\left(\lambda^2\,\frac{4}{p^2-4}+\frac{(d-2-2a)^2}{4}\right)\right]$$

$$\frac{(-2-2a+d)^2\,(2+p)}{8+p\,(-4+8\,\theta)}$$

$$\text{L[i\_, j\_]} := \mu + i\,(d+i-2) - \frac{\lambda^2}{4}\left(\sqrt{1+\frac{4\,\kappa}{\lambda^2}}-(1+2\,j)\right)^2$$

$$\theta 1[\eta\_] := 1-\eta$$

Discussion of the sign of the eigenvalue $\lambda(1,0)$

```
Simplify[L[1, 0], Assumptions → p > 2]
```

$$\frac{1}{64}\left(-64+64\,d-\frac{4\,(-2-2a+d)^2\,p^2\,(2+p)}{2+p\,(-1+2\,\theta)}+\frac{64\,(-2-2a+d)^2\,(2+p)}{8+p\,(-4+8\,\theta)}\right)$$

- ### We want the following expression to be negative

```
Simplify[Expand[64 (8 + p (-4 + 8 θ)) L[1, 0]], Assumptions → p > 2]
```

$$-16\,(4\,a^2\,(-2+p)\,(2+p)^2-4\,a\,(-2+d)\,(-2+p)\,(2+p)^2+d^2\,(-2+p)\,(2+p)^2+4\,p\,(-8+2\,p+p^2+8\,\theta)-4\,d\,p\,(-8+2\,p+p^2+8\,\theta))$$

$$f[\theta\_, p\_] := -16\,(4\,a^2\,(-2+p)\,(2+p)^2-4\,a\,(-2+d)\,(-2+p)\,(2+p)^2+d^2\,(-2+p)\,(2+p)^2+4\,p\,(-8+2\,p+p^2+8\,\theta)-4\,d\,p\,(-8+2\,p+p^2+8\,\theta))$$

▪ Sign changes in terms of θ ?

```
Simplify[Simplify[θ /. Solve[f[θ, p] == 0, θ]][[1]] /. p -> 2 d/(-2 + d + 2 η)]
```

$$-((-1+\eta)\,(4 a^2 (-1+d+\eta)^2 - 4 a (-2+d) (-1+d+\eta)^2 + d (d^3 + 4 (-1+\eta) + d^2 (-5+2\eta) + d (8 - 6\eta + \eta^2))))\,/\,((-1+d)\, d\, (-2+d+2\eta)^2)$$

```
θ2[a_, d_, η_] := -((-1 + η) (4 a^2 (-1 + d + η)^2 - 4 a (-2 + d) (-1 + d + η)^2 +
      d (d^3 + 4 (-1 + η) + d^2 (-5 + 2 η) + d (8 - 6 η + η^2)))) / ((-1 + d) d (-2 + d + 2 η)^2)
```

▪ Sign changes in terms of η ?

```
Simplify[(-2 + d + 2 η)^3/1024 f[θ, p] /. p -> 2 d/(-2 + d + 2 η)]
```

$$4 a^2 (-1+\eta) (-1+d+\eta)^2 - 4 a (-2+d) (-1+\eta) (-1+d+\eta)^2 + d (-4 (-1+\eta)^2 (-1+\theta) + d^3 (-1+\eta+\theta) + d (2 - 3\eta + \eta^2) (-4+\eta+4\theta) + d^2 (5 + 2\eta^2 - 5\theta + \eta (-7 + 4\theta)))$$

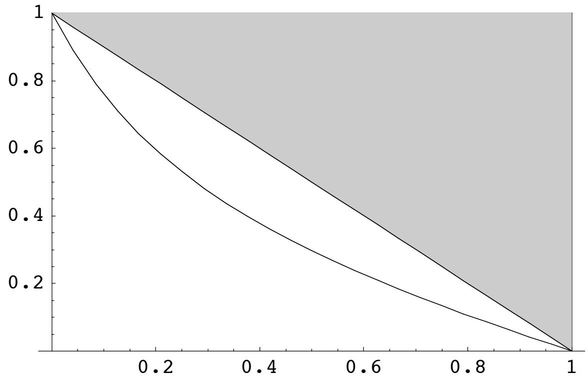
```
g[a_, θ_, d_] := η /. Solve[4 a^2 (-1 + η) (-1 + d + η)^2 -
      4 a (-2 + d) (-1 + η) (-1 + d + η)^2 + d (-4 (-1 + η)^2 (-1 + θ) + d^3 (-1 + η + θ) +
        d (2 - 3 η + η^2) (-4 + η + 4 θ) + d^2 (5 + 2 η^2 - 5 θ + η (-7 + 4 θ))) == 0, η][[3]]
```

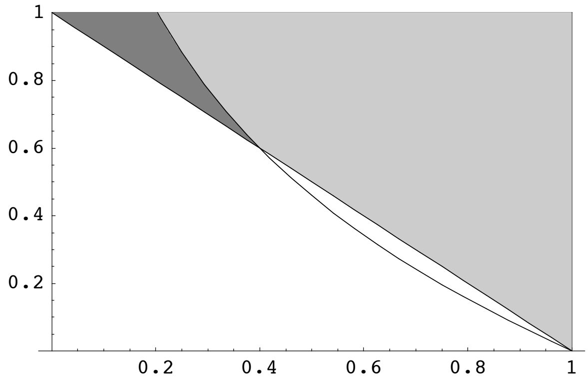
▪ Parameters and definitions for the plots

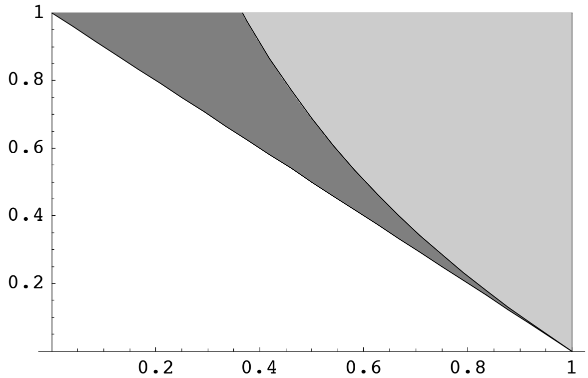
```
Nbre = 400;

amin = -4;

Gcrit[θ_, d_] := Chop[a /. FindRoot[g[a, θ, d] == 1 - θ, {a, -1}]]

Low[a_, d_] := If[a >= 0, 0, If[a <= -0.5, 1, η /. Solve[θ2[a, d, η] == 1 - η, η][[2]]]]
High[a_, d_] := η /. Solve[θ2[a, d, η] == 1, η][[3]]

Breaking[a_, d_] :=
 Show[Graphics[{GrayLevel[0.5], Polygon[Module[{M = {Low[a, d], High[a, d]}},
      Join[{{0, 1}}, Table[{M[[2]] + k (M[[1]] - M[[2]]) / Nbre,
         θ2[a, d, M[[2]] + k (M[[1]] - M[[2]]) / Nbre]}, {k, 0, Nbre}], {{0, 1}}]]]}],
   DisplayFunction → Identity, Axes → True]]

Symmetry[a_, d_] :=
 Show[Graphics[{GrayLevel[0.8], Polygon[Module[{M = {Low[a, d], High[a, d]}},
      Join[{{1, 1}}, Table[{M[[2]] + k (M[[1]] - M[[2]]) / Nbre,
         θ2[a, d, M[[2]] + k (M[[1]] - M[[2]]) / Nbre]}, {k, 0, Nbre}],
        {{1, 0}, {1, 1}, {1, 0}}]]]}], DisplayFunction → Identity, Axes → True]]

Breaking[-0.25, 3];

h[a_, d_] := 2 a (1 - d)/(2 a + d)
```

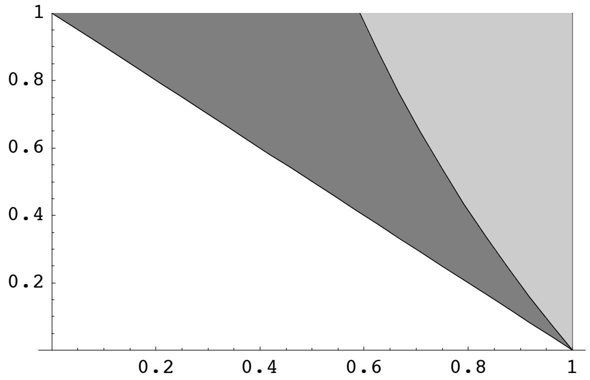
Symmetry breaking: sections for fixed values of a and d

```
Pl[a_, d_] := Show[ListPlot[{{1, 0}, {1, 1}, {0, 1}},
    DisplayFunction → Identity, PlotJoined → True], Breaking[a, d], Symmetry[a, d],
   Plot[{θ1[η], θ2[a, d, η]}, {η, 0, 1}, DisplayFunction → Identity],
   DisplayFunction -> $DisplayFunction, PlotRange → {All, {0, 1}}];
ValuesOfa = {0, -0.25, -0.5, -1};
Table[Pl[ValuesOfa[[k]], 3], {k, 1, Length[ValuesOfa]}];
```



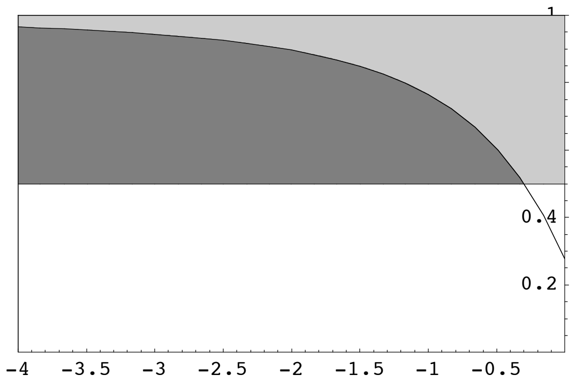






Symmetry breaking: sections for fixed values of θ and d

```
Off[FindRoot::"lstol"]

AdmissibleZoneUp[θ_, d_, GL_, DF_] :=
  Show[Graphics[{GrayLevel[GL], Polygon[Module[{M = Gcrit[θ, 3]}, Join[
        Table[{amin + k (M - amin) / Nbre, g[amin + k (M - amin) / Nbre, θ, 3]}, {k, 0, Nbre}],
        {{M, g[M, θ, 3]}, {0, g[M, θ, 3]}, {0, 1}, {amin, 1}, {amin, g[amin, θ, 3]}}]]]},
    DisplayFunction → Identity], ListPlot[{{amin, 0}, {amin, 1}, {0, 1}},
    PlotJoined → True, DisplayFunction → Identity], DisplayFunction → DF, Axes → True];

AdmissibleZoneDown[θ_, d_, GL_, DF_] :=
  Show[Graphics[{GrayLevel[GL], Polygon[Module[{M = Gcrit[θ, 3]},
       Join[Table[{amin + k (M - amin) / Nbre, g[amin + k (M - amin) / Nbre, θ, 3]},
          {k, 0, Nbre}], {{M, g[M, θ, 3]}, {amin, g[M, θ, 3]}, {amin, g[amin, θ, 3]}}]]]},
    DisplayFunction → Identity], ListPlot[{{amin, 0}, {amin, 1}, {0, 1}},
    PlotJoined → True, DisplayFunction → Identity], DisplayFunction → DF, Axes → True];

Show[AdmissibleZoneUp[0.5, 3, 0.8, Identity],
  AdmissibleZoneDown[0.5, 3, 0.5, Identity], ListPlot[
   {{amin, 0}, {amin, 1}, {0, 1}}, PlotJoined → True, DisplayFunction → Identity],
  Plot[{g[a, 0.5, 3], 1 - 0.5}, {a, amin, 0}, DisplayFunction → Identity],
  DisplayFunction -> $DisplayFunction, PlotRange → {{amin, 0}, {0, 1}}];
```



```
ValuesOfTheta = {1, 0.75, 0.5, 0.3, 0.2, 0.1, 0.05, 0.02};
Show[ListPlot[{{amin, 0}, {amin, 1}, {0, 1}},
  PlotJoined → True, DisplayFunction → Identity],
 Table[AdmissibleZoneDown[ValuesOfTheta[[k]], 3, 0.9 - 0.1 * k, Identity],
  {k, 1, Length[ValuesOfTheta]}],
 Table[Plot[g[a, ValuesOfTheta[[k]], 3], {a, amin, Gcrit[ValuesOfTheta[[k]], 3]},
   DisplayFunction → Identity], {k, 1, Length[ValuesOfTheta]}],
 Table[ListPlot[{{amin, 1 - ValuesOfTheta[[k]]}, {0, 1 - ValuesOfTheta[[k]]}},
   PlotJoined → True, DisplayFunction → Identity], {k, 1, Length[ValuesOfTheta]}],
 Plot[h[a, 3], {a, -0.5, 0}, DisplayFunction → Identity],
 DisplayFunction -> $DisplayFunction, PlotRange → {{amin, 0}, {0, 1}}];
```