Radial symmetry and symmetry breaking for some interpolation inequalities - Dolbeault, Esteban, Tarantello, Tertikas

Caffarelli-Kohn-Nirenberg interpolation inequalities: how to determine a region for symmetry ? Figure 1

We numerically solve the equality case for the two conditions: a priori estimates and monotonicity condition for applying Schwarz' symmetrization method.

Definitions

$$\mathtt{Theta}[p\_, d\_] := d\,\frac{p-2}{2\,p}$$

$$\mathtt{S}[d\_] := \frac{2\,\pi^{\frac{d}{2}}}{\mathtt{Gamma}\left[\frac{d}{2}\right]}$$

$$\mathtt{K}[\theta\_, p\_] := \left(\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right)^{\frac{p-2}{2\,p}}\left(\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right)^{\theta}\left(\frac{4}{p+2}\right)^{\frac{6-p}{2\,p}}\left(\frac{\mathtt{Gamma}\left[\frac{2}{p-2}+\frac{1}{2}\right]}{\sqrt{\pi}\,\mathtt{Gamma}\left[\frac{2}{p-2}\right]}\right)^{\frac{p-2}{p}}$$

$$\mathtt{Cstar}[\theta\_, p\_, d\_] := \mathtt{S}[d]^{\frac{2}{p}-1}\,\mathtt{K}[\theta, p]$$

$$\mathtt{ac}[d\_] := \frac{d-2}{2}$$

Solving the system by eliminating t

$$\mathtt{R}[a\_, \theta\_, p\_, d\_] :=$$
$$\mathtt{Evaluate}\Big[\left(t+(a-\mathtt{ac}[d])^2\right)^{\theta} - \frac{\left(\mathtt{Cstar}\left[1, \frac{2\,d}{d-2}, d\right]\mathtt{ac}[d]^{-2\,\frac{d-1}{d}}\right)^{\mathtt{Theta}[p,d]}}{\mathtt{Cstar}[\theta, p, d]}\left(t+\mathtt{ac}[d]^2\right)^{\mathtt{Theta}[p,d]}$$
$$(\mathtt{ac}[d]-a)^{2\,\theta-\frac{2}{d}\mathtt{Theta}[p,d]}\ \mathtt{/.}\ t \to \frac{\theta\,\mathtt{ac}[d]^2-(\mathtt{ac}[d]-a)^2}{1-\theta}\Big]$$

```
Iter[θ_, p_, d_, a_, h_, ϵ_, Nmax_, n_] :=
 Module[{M = Evaluate[R[a, θ, p, d]]}, If[n > Nmax || Abs[M] < ϵ, {M, a, h, n}, If[M > ϵ,
    Iter[θ, p, d, a + h, h, ϵ, Nmax, n + 1], Iter[θ, p, d, a - h / 2, h / 2, ϵ, Nmax, n + 1]]]]
```

```
F[θ_, p_, d_, h_, ϵ_, Nmax_] := Iter[θ, p, d, ac[d] - h, -h, ϵ, Nmax, 0]
```

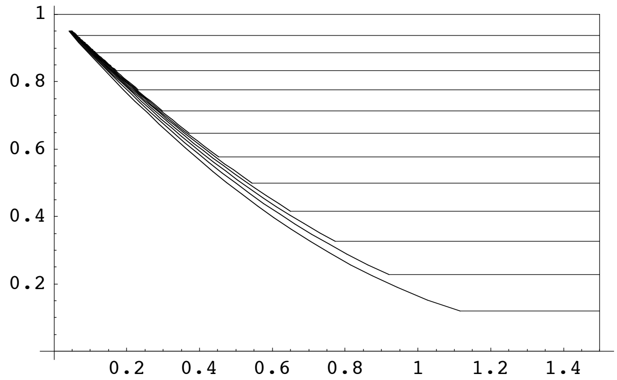
```
L[p_, d_, h_, Nmax_] := Module[{M = F[Theta[p, d], p, d, h, eps, Nmax][[2]]},
  ListPlot[{{M, Theta[p, d]}, {ac[d], Theta[p, d]}},
   PlotJoined → True, DisplayFunction → Identity]]

G[d_, p_, DF_, Marge_, h_, Nmax_] := ParametricPlot[
  {F[s, p, d, h, eps, Nmax][[2]], s}, {s, Theta[p, d], 1 - Marge}, DisplayFunction → DF ]
```

Results: dimension d=5

```
Off[Greater::"nord"]
Off[ParametricPlot::"pptr"]
Off[Graphics::"gptn"]
dim = 5;
eps = 10^-8;
Orgn = {0, 0};
LL = ListPlot[{{Orgn[[2]], 0}, {ac[dim], 0}, {ac[dim], 1}, {Orgn[[2]], 1}},
   PlotJoined → True, DisplayFunction → Identity];

Show[Table[G[dim, p, Identity, 0.05, 0.01, 200], {p, 2.1, 3.2, 0.1}],
  Table[L[p, dim, 0.01, 200], {p, 2.1, 3.2, 0.1}], LL, PlotRange → All,
  DisplayFunction → $DisplayFunction, AxesOrigin → {Orgn[[1]], Orgn[[2]]}];
```



Comparison with a gaussian test function : Symmetry breaking for Caffarelli-Kohn-Nirenberg interpolation inequalities Figure 2

The method is based on an estimate of the optimal constant in a certain Gagliardo-Nirenberg inequality using a gaussian test function. By comparing with 1 the quotient of this estimate with the optimal constant of the Caffarelli-Kohn-Nirenberg interpolation inequality (restricted to radial functions), we prove a symmetry breaking phenomenon away from the region found by the method of Felli and Schneider, when the exponent p is sufficiently close to 2. A proof is given in the second part of this section.

Plots

$$\mathbf{K}[\theta\_, \mathbf{p}\_, \sigma\_] := \left(\frac{\sigma^2\ (\mathbf{p}-2)^2}{2+(2\,\theta-1)\ \mathbf{p}}\right)^{\frac{p-2}{2p}} \left(\frac{2+(2\,\theta-1)\ \mathbf{p}}{2\,\mathbf{p}\,\theta\,\sigma^2}\right)^{\theta} \left(\frac{4}{\mathbf{p}+2}\right)^{\frac{6-p}{2p}} \left(\frac{\mathrm{Gamma}\left[\frac{2}{p-2}+\frac{1}{2}\right]}{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{2}{p-2}\right]}\right)^{\frac{p-2}{p}}$$

$$\mathbf{u}[\mathbf{r}\_] := e^{-\frac{r^2}{4}}$$

```
Integrate[r^(d-1) u[r]^p, {r, 0, ∞},
 Assumptions -> Re[d] > 0 && Re[p] > 0]
```

$$\mathrm{Nrm}[\mathbf{p}\_, \mathbf{d}\_] := 2^{-1+d}\ p^{-d/2}\ \mathrm{Gamma}\left[\frac{d}{2}\right]$$

$$2^{-1+d}\ p^{-d/2}\ \mathrm{Gamma}\left[\frac{d}{2}\right]$$

```
Integrate[r^(d-1) u'[r]^2,
 {r, 0, ∞}, Assumptions -> Re[d] > -2]
```

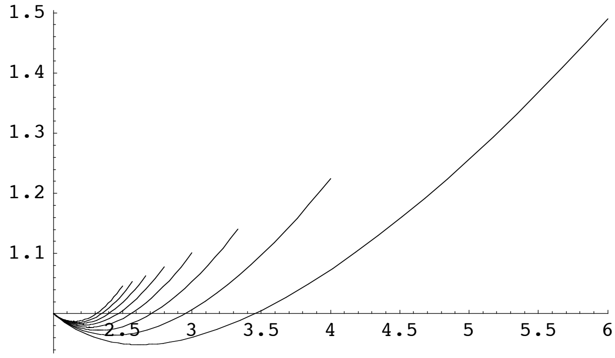
$$\mathrm{J}[\mathbf{d}\_] := 2^{-2+\frac{d}{2}}\ \mathrm{Gamma}\left[1+\frac{d}{2}\right]$$

$$2^{-2+\frac{d}{2}}\ \mathrm{Gamma}\left[1+\frac{d}{2}\right]$$

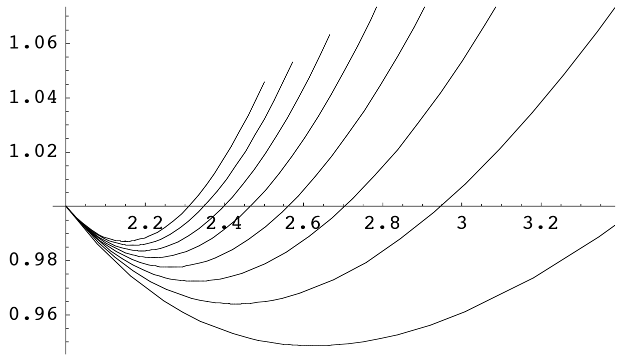
$$\mathrm{Val}[\mathbf{p}\_, \mathbf{d}\_] := \mathrm{K}\left[d\ \frac{p-2}{2\,p},\ p,\ 2\ \frac{d-1}{p+2}\right] \frac{\mathrm{J}[d]^{d\ \frac{p-2}{2p}}\ \mathrm{Nrm}[2, d]^{1-d\ \frac{p-2}{2p}}}{\mathrm{Nrm}[p, d]^{\frac{2}{p}}}$$

```
Off[General::"ovfl"]
Off[Plot::"plnr"]
```

$$\texttt{Show}\left[\texttt{Table}\left[\texttt{Plot}\left[\texttt{Val[p, d]}, \left\{\texttt{p, 2, }\frac{\texttt{2 d}}{\texttt{d - 2}}\right\}, \texttt{DisplayFunction} \to \texttt{Identity}\right], \texttt{\{d, 3, 10\}}\right], \texttt{DisplayFunction} \to \texttt{\$DisplayFunction}, \texttt{PlotRange} \to \texttt{\{\{2, 6\}, All\}}\right];$$



$$\texttt{Show}\left[\texttt{Table}\left[\texttt{Plot}\left[\texttt{Val[p, d]}, \left\{\texttt{p, 2, }\frac{\texttt{2 d}}{\texttt{d - 2}}\right\}, \texttt{DisplayFunction} \to \texttt{Identity}\right], \texttt{\{d, 3, 10\}}\right], \texttt{DisplayFunction} \to \texttt{\$DisplayFunction}\right];$$



Computation of the derivative at p=2

$$\texttt{FullSimplify[Val[p, d]];}$$

$$\texttt{FullSimplify}\left[\texttt{Limit}\left[\frac{\texttt{\% - 1}}{\texttt{p - 2}}, \texttt{p} \to \texttt{2}\right]\right]$$

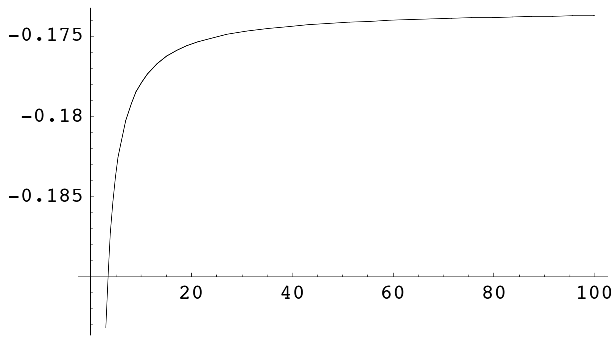
$$\frac{1}{4}\left(-1+d+d\,\mathrm{Log}[4]+d\,\mathrm{Log}\left[\frac{1}{(-1+d)\,d}\right]+\mathrm{Log}\left[\frac{-1+d}{2\,\pi}\right]+d\,\mathrm{Log}\left[2^{-2+\frac{d}{2}}\,\mathrm{Gamma}\left[1+\frac{d}{2}\right]\right]-(-2+d)\,\mathrm{Log}\left[2^{-1+\frac{d}{2}}\,\mathrm{Gamma}\left[\frac{d}{2}\right]\right]\right)$$

$$g[d\_] := \frac{1}{4}\left(-1 + d + d\,\mathrm{Log}[4] + d\,\mathrm{Log}\left[\frac{1}{(-1+d)\,d}\right] + \mathrm{Log}\left[\frac{-1+d}{2\,\pi}\right] - (-2+d)\,\mathrm{Log}\left[2^{-1+\frac{d}{2}}\,\mathrm{Gamma}\left[\frac{d}{2}\right]\right] + d\,\mathrm{Log}\left[2^{-2+\frac{d}{2}}\,\mathrm{Gamma}\left[\frac{2+d}{2}\right]\right]\right)$$

$$\mathrm{Limit}[g[d],\ d \to \infty]$$

$$-\frac{\mathrm{Log}[2]}{4}$$

$$\mathrm{Plot}[g[d],\ \{d,\ 3,\ 100\}];$$



$$\mathrm{FullSimplify}[D[g[d],\ \{d,\ 2\}]]$$

$$\frac{1}{8}\left(-\frac{2}{-1+d} + \mathrm{PolyGamma}\left[1,\ \frac{d}{2}\right]\right)$$

Logarithmic Hardy inequality: symmetry breaking - Figure 3

We use the optimal constant of the logarithmic Sobolev inequality in the euclidean space (in Weissler's form) and compare it with the optimal constant for the logarithmic Hardy inequality restricted to radial functions in the limiting regime corresponding to γ=d/4. When the quotient is less than 1, for the value corresponding to the threshold of the domain found by the method of Felli and Schneider: a =1/2, symmetry breaking is achieved away from the region found by the method of Felli and Schneider. There is no restriction on the dimension d≥3.

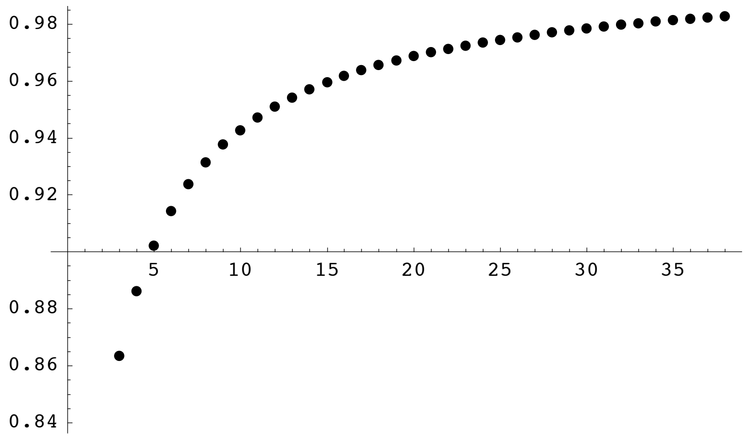
$$\mathrm{CWLH}[d\_,\ a\_] := \frac{4}{d}\,\frac{\mathrm{Gamma}\left[\frac{d}{2}\right]^{\frac{2}{d}}}{\left(8\,\pi^{d+1}\,e\right)^{\frac{1}{d}}}\left(\frac{d-1}{(d-2-2\,a)^{2}}\right)^{\frac{d-1}{d}}$$

$$\mathbf{FullSimplify}\left[\frac{1}{2}\,\pi\,\mathbf{d}\,e\,\mathbf{CWLH}\left[\mathbf{d},\ -\frac{1}{2}\right]\right]$$

$$2^{\frac{-3+d}{d}}\left(\frac{1}{-1+d}\right)^{\frac{-1+d}{d}} e^{1-\frac{1}{d}}\,\pi\,(\pi^{1+d})^{-1/d}\,\text{Gamma}\left[\frac{d}{2}\right]^{2/d}$$

$$\mathbf{Quo[d\_]} := 2^{\frac{-3+d}{d}}\left(\frac{1}{-1+d}\right)^{\frac{-1+d}{d}} e^{1-\frac{1}{d}}\,\pi\,(\pi^{1+d})^{-1/d}\,\mathbf{Gamma}\left[\frac{d}{2}\right]^{2/d}$$

```
ListPlot[Table[Quo[d], {d, 3, 40}], PlotStyle → PointSize[0.015]];
```



Logarithmic Hardy inequality: symmetry breaking - Figure 4

We compare the range of symmetry breaking given by the method of Catrina-Wang/Felli-Schneider with the new method based on the comparison with the best constant for the logarithmic Sobolev inequality

```
Off[Solve::"ifun"]
```

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\lambda\ /.\ \mathbf{Solve}\left[\frac{1}{4\gamma}\,\frac{\mathbf{Gamma}\left[\frac{d}{2}\right]^{\frac{1}{2\gamma}}}{(2\,\pi^{d+1}\,e)^{\frac{1}{4\gamma}}}\left(\frac{4\gamma-1}{\lambda}\right)^{\frac{4\gamma-1}{4\gamma}} == \frac{2}{\pi\,d\,e},\ \lambda\right]\right][[1]]\right]$$

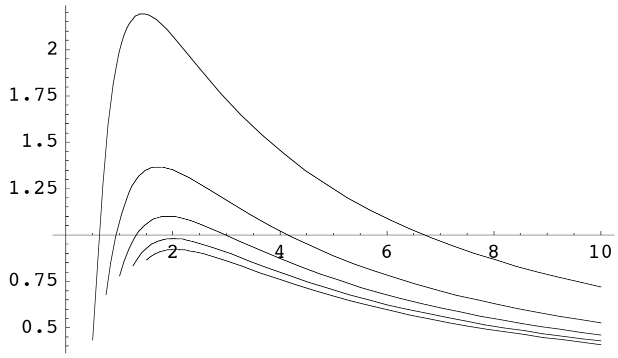
$$2^{-3+\frac{4}{1-4\gamma}}\,d^{1+\frac{1}{-1+4\gamma}}\,e\,\pi^{1+\frac{d}{1-4\gamma}}\,\gamma^{\frac{4\gamma}{1-4\gamma}}\,(-1+4\gamma)\,\text{Gamma}\left[\frac{d}{2}\right]^{\frac{2}{-1+4\gamma}}$$

$$\mathbf{Lambda}[\gamma\_,\ d\_] := 2^{-3+\frac{4}{1-4\gamma}}\,d^{1+\frac{1}{-1+4\gamma}}\,e\,\pi^{1+\frac{d}{1-4\gamma}}\,\gamma^{\frac{4\gamma}{1-4\gamma}}\,(-1+4\gamma)\,\mathbf{Gamma}\left[\frac{d}{2}\right]^{\frac{2}{-1+4\gamma}}$$

$$\mathbf{H}[\gamma\_,\ d\_] := \mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\frac{\mathbf{Lambda}[\gamma,\ d]}{\frac{1}{4}\,(4\gamma-1)\,(d-1)}\right]\right]$$

```
R[d_] := {
d, γ /. FindRoot[H[γ, d] - 1 == 0, {γ, d / 4, 2}], γ /. FindRoot[H[γ, d] - 1 == 0, {γ, 2, 10}]}
```

```
Show[Table[Plot[H[γ, d], {γ, d/4, 10}, DisplayFunction → Identity], {d, 2, 6}],
  DisplayFunction → $DisplayFunction];
R[2]
R[3]
R[4]
```




```
{2, 0.621414, 6.69625}

{3, 0.937725, 4.14851}

{4, 1.31303, 2.98835}
```