Symmetry of extremals of functional inequalities via spectral estimates for linear operators

Preliminaries

```
b /. Solve[p == 2 d/(d - 2 + 2 (b - a)), b][[1]]
b[a_, p_, d_] := (2 d + 2 p + 2 a p - d p)/(2 p)
```

$$\frac{2d+2p+2ap-dp}{2p}$$

Best constant in the radial case ($\theta = 1$)

```
ac = (d - 2)/2;
Theta[p_, d_] := d (p - 2)/(2 p)
S[d_] := 2 π^(d/2)/Gamma[d/2]
K[θ_, p_] := ((p - 2)^2/(2 + (2 θ - 1) p))^((p-2)/(2 p)) ((2 + (2 θ - 1) p)/(2 p θ))^θ (4/(p + 2))^((6-p)/(2 p)) (Gamma[2/(p-2) + 1/2]/(Sqrt[π] Gamma[2/(p-2)]))^((p-2)/p)
Cstar[θ_, p_, d_] := S[d]^(2/p-1) K[θ, p]
FullSimplify[PowerExpand[1/K[1, p] ((a - ac)^2 (p - 2)^2/(p + 2))^((p-2)/(2 p)) (p + 2)/(2 p (a - ac)^2) (4/(p + 2))^((6-p)/(2 p)) (Gamma[2/(p-2) + 1/2]/(Sqrt[π] Gamma[2/(p-2)]))^((p-2)/p)]]
```

$$\left(1+a-\frac{d}{2}\right)^{-\frac{2+p}{p}}$$

$$\text{Simplify}\left[\frac{2\,d}{d-2+2\,(b-a)}\ /.\ a\to -\frac{1}{2}\right];$$

$$\text{Simplify}[\%\ /.\ b\to 0];$$

$$\text{FullSimplify}\Big[\text{PowerExpand}\Big[S[d]^{\frac{p-2}{p}}\left(\frac{(ac-a)^2\,(p-2)^2}{p+2}\right)^{\frac{p-2}{2p}}\frac{p+2}{2\,p\,(a-ac)^2}\left(\frac{4}{p+2}\right)^{\frac{6-p}{2p}}\left(\frac{\text{Gamma}\left[\frac{2}{p-2}+\frac{1}{2}\right]}{\sqrt{\pi}\ \text{Gamma}\left[\frac{2}{p-2}\right]}\right)^{\frac{p-2}{p}}\ /.\ p\to\%\Big],$$

$$\text{Assumptions}\to d>1\Big];$$

$$\text{C0}=\text{FullSimplify}\left[\%\ /.\ a\to -\frac{1}{2}\right];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{1}{\text{C0}}\ \frac{4}{d\,(d-1)}\left(\frac{\pi^{\frac{d-1}{2}}\ \text{Gamma}\left[d+\frac{1}{2}\right]}{\text{Gamma}\left[\frac{d}{2}\right]\ \text{Gamma}[d]}\right)^{\frac{1}{d}}\right]\right]$$

1

The curve of Felli and Schneider : (b - a) as a function of a

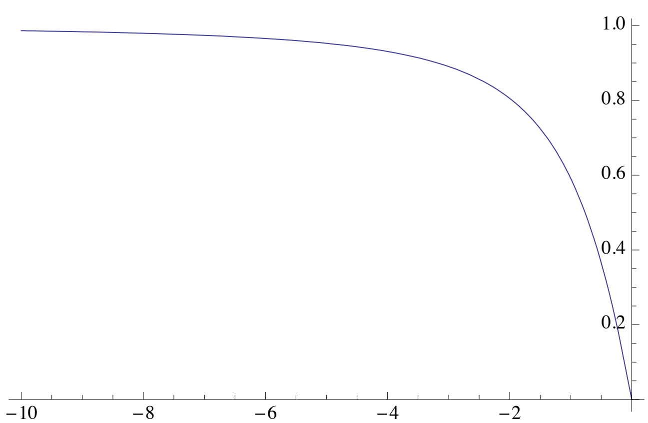
$$\text{Solve}\left[\frac{d-2}{2}-2\sqrt{\frac{d-1}{p^2-4}}==a,\ p\right][[2]]$$

$$\text{pFS}[a\_,\ d\_]:=\frac{2\sqrt{8\,a+4\,a^2-4\,a\,d+d^2}}{\sqrt{4+8\,a+4\,a^2-4\,d-4\,a\,d+d^2}}$$

```
FS[a_, d_] := b[a, pFS[a, d], d] - a
```

$$\left\{p\to\frac{2\sqrt{8\,a+4\,a^2-4\,a\,d+d^2}}{\sqrt{4+8\,a+4\,a^2-4\,d-4\,a\,d+d^2}}\right\}$$

```
Plot[FS[a, 3], {a, -10, 0}, PlotRange → All]
```



The new curve : (b - a) as a function of a

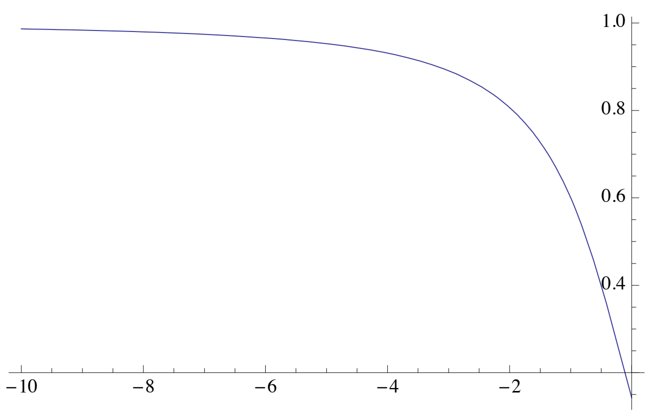
```
Solve[(d - 2)/2 - Sqrt[((d - 1) (6 - p))/(4 (p - 2))] == a, p][[1]]
pLT[a_, d_] := (2 (1 + 8 a + 4 a^2 - d - 4 a d + d^2))/(3 + 8 a + 4 a^2 - 3 d - 4 a d + d^2)
LT[a_, d_] := b[a, pLT[a, d], d] - a

Factor[FullSimplify[LT[-1/2, d] - 1/2]]
```

$$\left\{p \to \frac{2\left(1 + 8\,a + 4\,a^2 - d - 4\,a\,d + d^2\right)}{3 + 8\,a + 4\,a^2 - 3\,d - 4\,a\,d + d^2}\right\}$$

$$-\frac{-2 + d}{2\,(2 + d)}$$

```
Plot[LT[a, 3], {a, -10, 0}, PlotRange → All]
Simplify[LT[0, d]]
N[% /. d → 3]
```



$$\frac{1}{1 - d + d^2}$$

0.142857

Plots for d=3, 5, 7, 9, 11

```
P[d_] := Plot[{FS[a, d], LT[a, d]}, {a, -6, 0}, PlotRange → All]
Table[P[d], {d, 2, 12, 2}]
```

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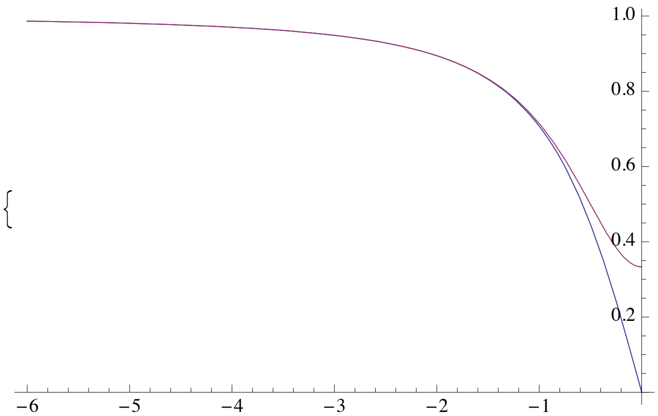
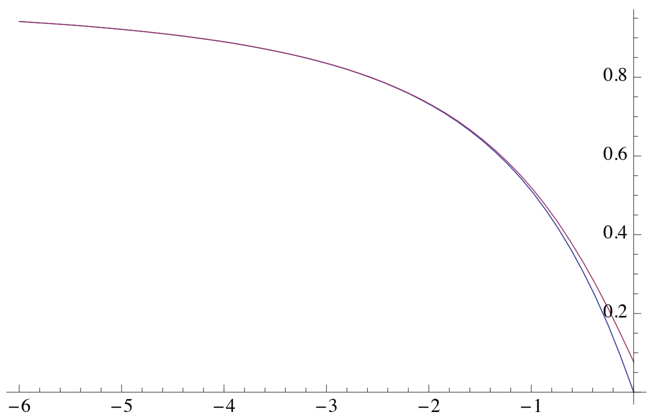

1.0
0.8
0.6
0.4
0.2
−6
−5
−4
−3
−2
−1

,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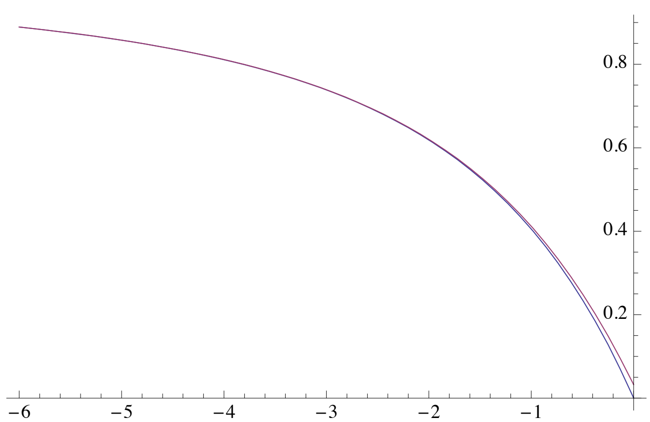
0.8
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0.4
0.2
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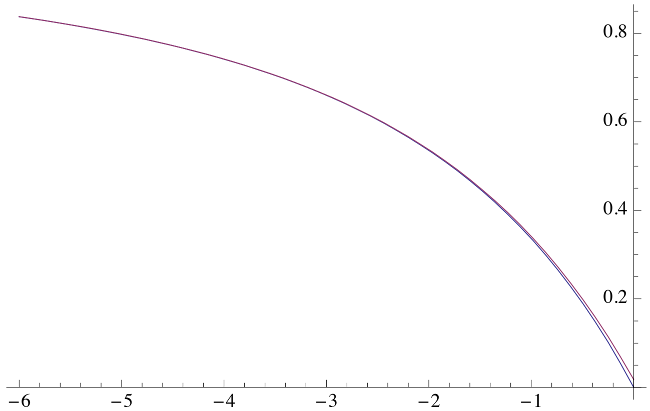
−2
−1

,



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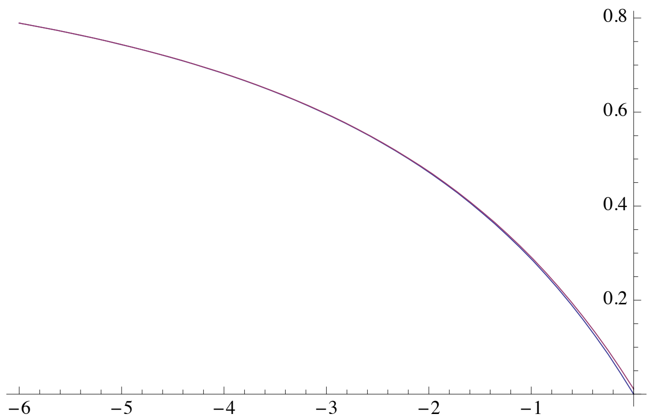

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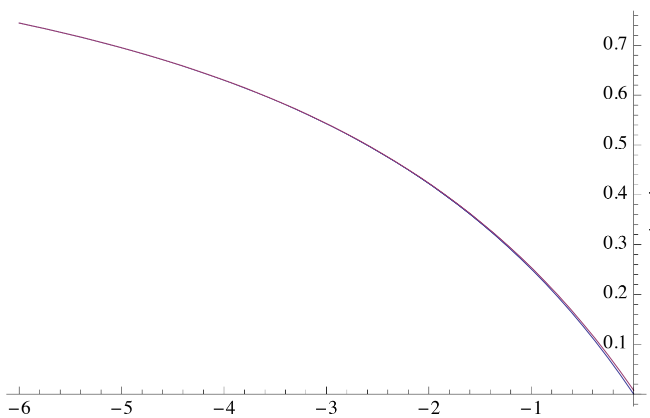


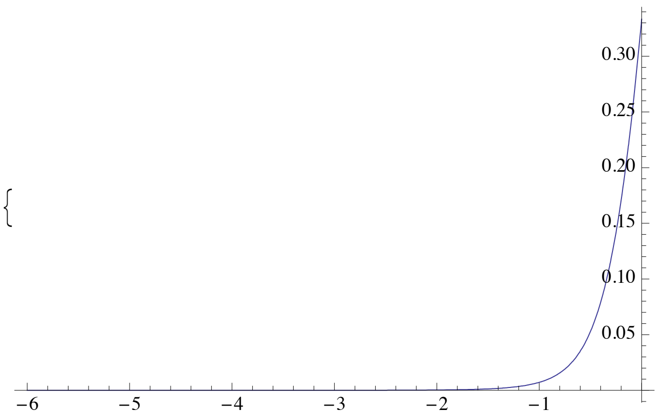
}

```
FullSimplify[LT[a, d] - FS[a, d], Assumptions → d - 2 - 2 a > 0]
```

$$\frac{1}{2} d \left(1 + \frac{2}{\sqrt{4 a (2 + a) - 4 a d + d^2}} + \frac{2 a}{\sqrt{4 a (2 + a) - 4 a d + d^2}} - \frac{d}{\sqrt{4 a (2 + a) - 4 a d + d^2}} + \frac{2 - 2 d}{1 + 4 a (2 + a) - d - 4 a d + d^2}\right)$$

```
PLTta[d_] := Plot[LT[a, d] - FS[a, d], {a, -6, 0}, PlotRange → All]
Table[PLTta[d], {d, 2, 12, 2}]
```

{


0.30
0.25
0.20
0.15
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0.05
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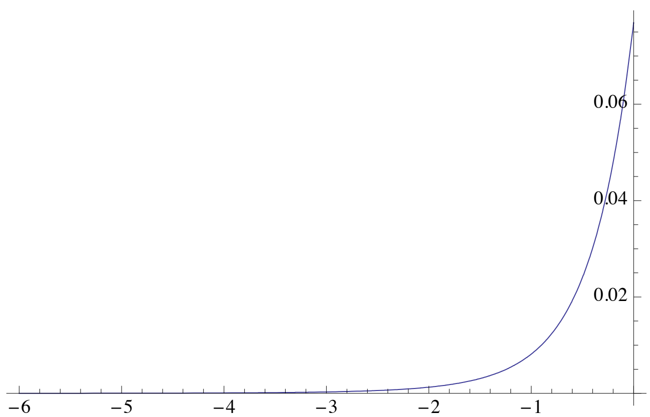
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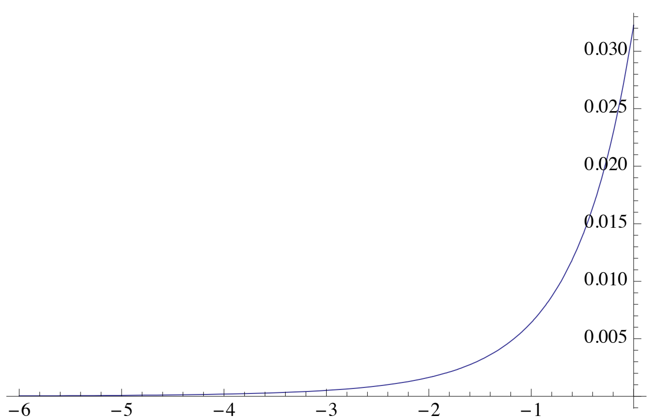
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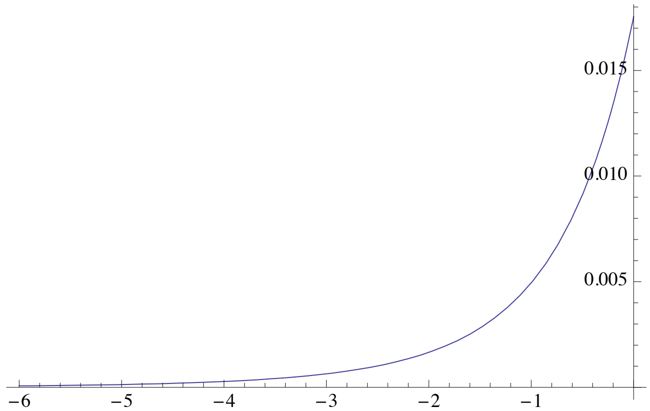
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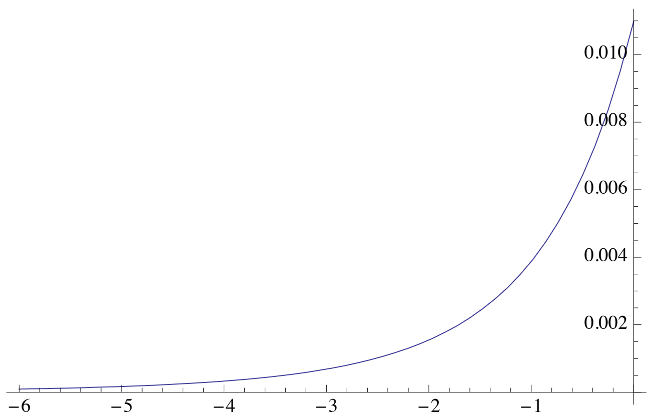
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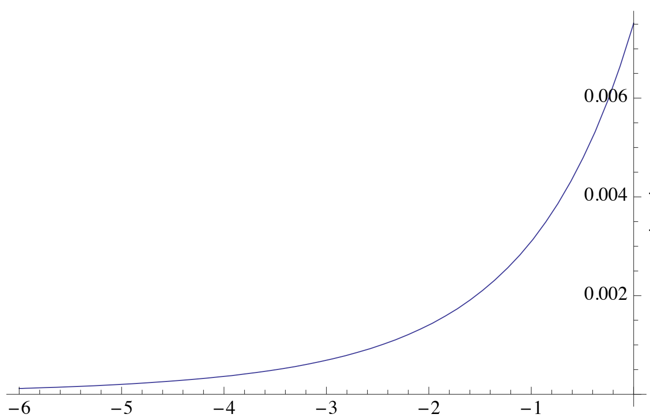
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}

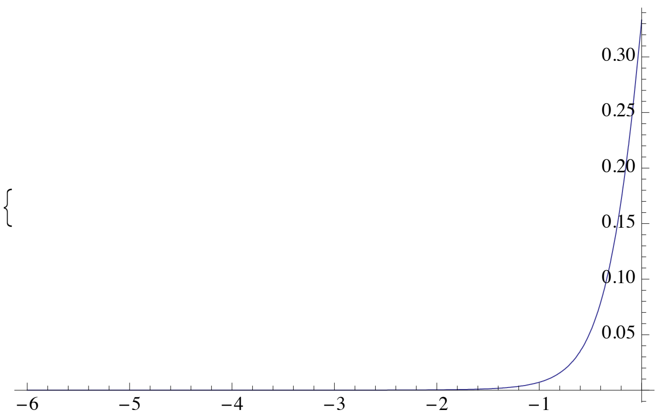
$$\text{FDelta[a\_, d\_]} := \frac{d}{2}\left(1 - \frac{ac - a}{\sqrt{d - 1 + (a - ac)^2}} - \frac{2\,(d - 1)}{4\,(a - ac)^2 + 3\,(d - 1)}\right)$$

$$\text{Simplify}\left[\frac{1}{2}\,d\left(1 + \frac{2}{\sqrt{4\,a\,(2 + a) - 4\,a\,d + d^2}} + \frac{2\,a}{\sqrt{4\,a\,(2 + a) - 4\,a\,d + d^2}} - \frac{d}{\sqrt{4\,a\,(2 + a) - 4\,a\,d + d^2}} + \frac{2 - 2\,d}{1 + 4\,a\,(2 + a) - d - 4\,a\,d + d^2}\right) - \text{FDelta[a, d]} \;/.\; \sqrt{4\,a^2 - 4\,a\,(-2 + d) + d^2} \to x\right]$$

```
Table[Plot[FDelta[a, d], {a, -6, 0}, PlotRange → All], {d, 2, 12, 2}]
```

0

{



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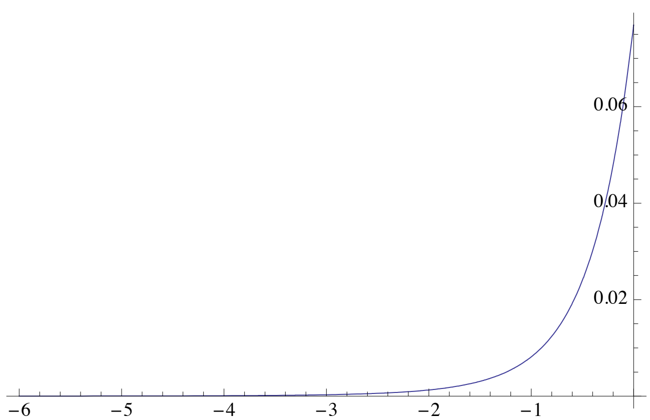

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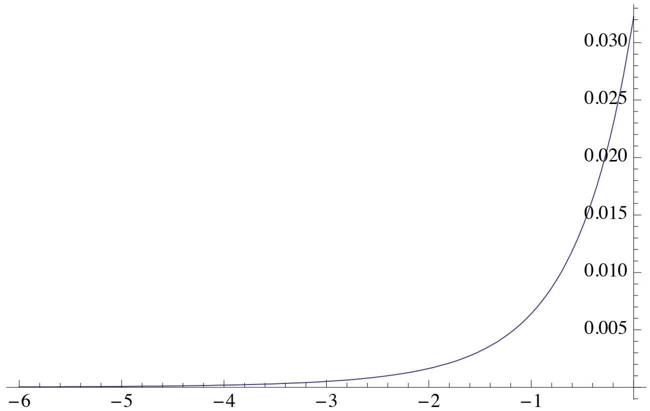
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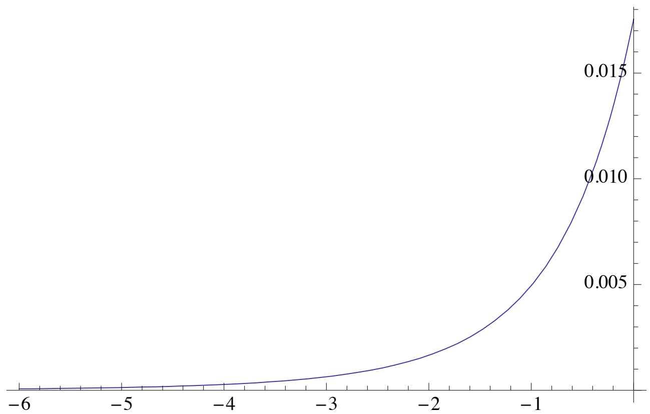
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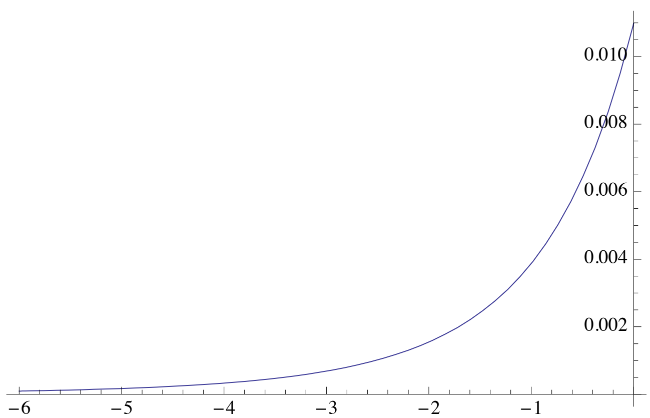
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0.008
0.006
0.004
0.002
−6
−5
−4
−3
−2
−1

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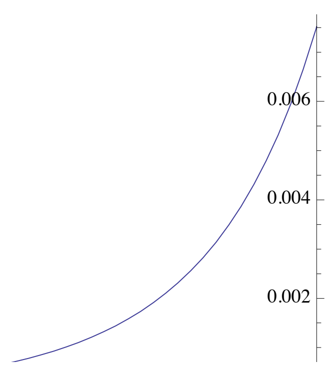
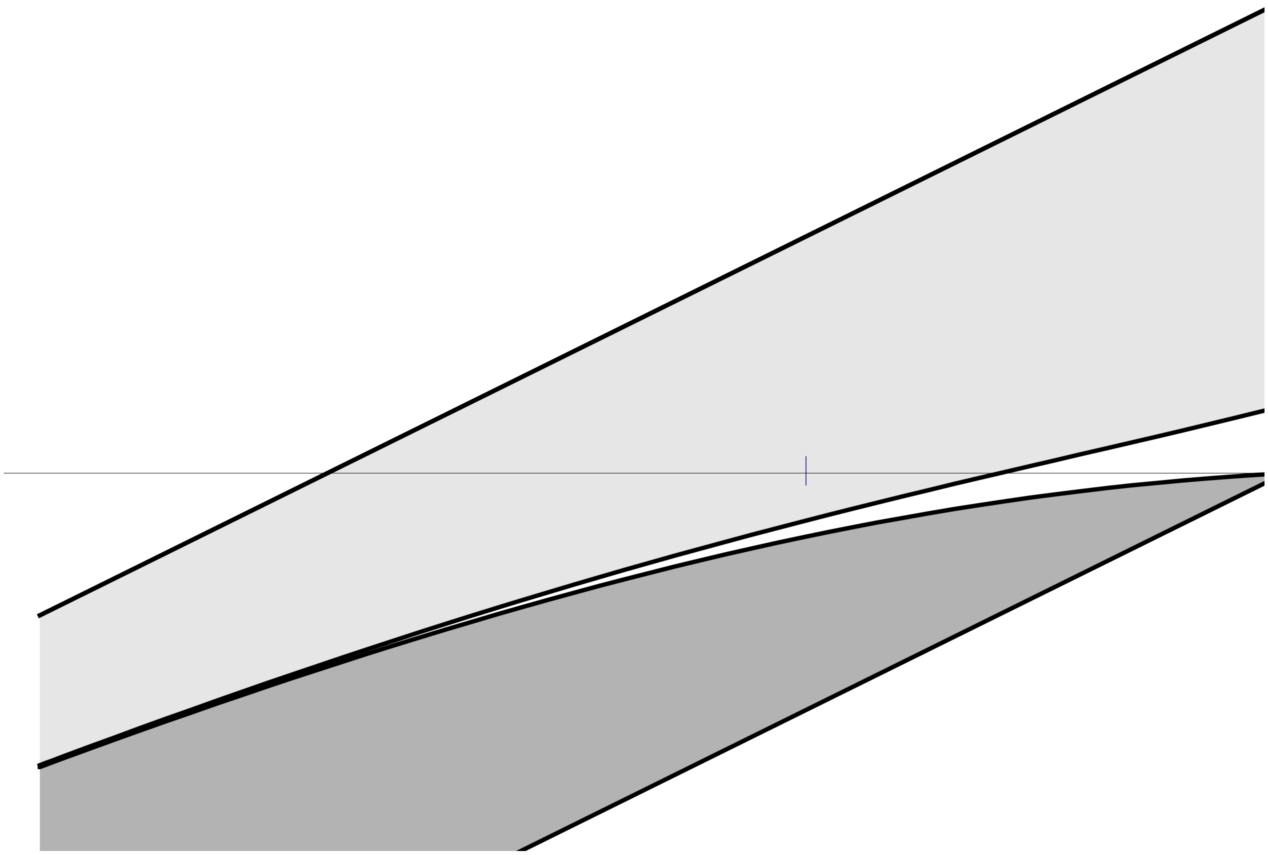

0.006
0.004
0.002

}

```
amin = -1.3;
brange = {-0.75, 1.5};
eps = 0.005;
P0[d_] := Plot[{a, a + 1}, {a, 0, (d - 2)/2}, Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9]]
P1[d_] := {Plot[FS[a, d] + a, {a, amin, 0}, PlotRange → All, PlotStyle → {Thick, Black}],
  Plot[LT[a, d] + a, {a, amin, 0}, PlotRange → All, PlotStyle → {Thick, Black}]}
P2[d_] := Plot[{LT[a, d] + a, a + 1}, {a, amin, 0, 0},
  Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9]]
P3[d_] := Plot[{FS[a, d] + a, a}, {a, amin, 0, 0},
  Filling → {1 → {2}}, FillingStyle → GrayLevel[0.7]]
Show[P0[3], P3[3], P2[3], P1[3], ListPlot[{{-0.5, -0.025}, {-0.5, 0.035}}, Joined → True],
 ListPlot[{{0.5 - eps, 0.5 - eps}, {amin, amin}}, Joined → True, PlotStyle → {Thick, Black}],
 ListPlot[{{0.5 - eps, 1.5 - eps}, {amin, 1 + amin}}, Joined → True, PlotStyle → {Thick, Black}],
 ListPlot[{{0.5, -0.025}, {0.5, 1.525}}, Joined → True, PlotStyle → Black],
 ListPlot[{{0, 0}, {0, 1/7}}, Joined → True, PlotStyle → Black],
 ListPlot[{{0, 1}, {amin, 1 + amin}}, Joined → True, PlotStyle → {Thick, Black}],
 PlotRange → {All, brange}, Ticks →
  None]
```



A verification

$$\textbf{Solve}\left[\lambda == 4\,\frac{\mathbf{d-1}}{\mathbf{p^2-4}},\ \mathbf{p}\right]\mathbf{[[2]]}$$

$$\mathbf{a - ac + \frac{d}{p}\ /.\ \%;}$$

```
BFS = FullSimplify[% /. λ → (ac - a)^2]
FullSimplify[a + FS[a, d] - BFS]
```

$$\left\{p \to \frac{2\sqrt{-1+d+\lambda}}{\sqrt{\lambda}}\right\}$$

$$a + \frac{1}{2}\left(2 + d\left(-1 + \frac{\sqrt{(-2-2\,a+d)^2}}{\sqrt{4\,a\,(2+a) - 4\,a\,d + d^2}}\right)\right)$$

0

```
Solve[λ == (d - 1) (6 - p)/(4 (p - 2)), p]
a - ac + d/p /. %;
BLT = FullSimplify[% /. λ → (ac - a)^2][[1]]
FullSimplify[a + LT[a, d] - BLT]
```

$$\left\{\left\{p \to \frac{2\,(-3 + 3\,d + 4\,\lambda)}{-1 + d + 4\,\lambda}\right\}\right\}$$

$$a + \frac{1 + 4\,a\,(2 + a - d)}{1 + 4\,a\,(2 + a) - d - 4\,a\,d + d^2}$$

0

Monotonicity in tems of a

```
FullSimplify[FDelta[a, d]]
```

$$\frac{1}{2}\,d\left(1 + \frac{2 + 2\,a - d}{\sqrt{4\,a\,(2 + a) - 4\,a\,d + d^2}} + \frac{2 - 2\,d}{1 + 4\,a\,(2 + a) - d - 4\,a\,d + d^2}\right)$$

```
f[x_] := -Sqrt[x/(x + ϵ)] - 2 ϵ/(4 x + 3 ϵ)
FullSimplify[Sqrt[x/(x + ϵ)] D[f[x], x]]
```

$$\epsilon\left(-\frac{1}{2\,(x + \epsilon)^2} + \frac{8\,\sqrt{\frac{x}{x+\epsilon}}}{(4\,x + 3\,\epsilon)^2}\right)$$

```
Expand[16^2 x (x + ϵ)^3 - (4 x + 3 ϵ)^4]
```

$$-96\,x^2\,\epsilon^2 - 176\,x\,\epsilon^3 - 81\,\epsilon^4$$

▪ The optimal function for the Lieb - Thirring (1) inequality

```
u[α_, s_] := Cosh[s]^-α
```

$$A1 = \left(\frac{p\,\lambda}{2}\right)^{\frac{1}{p-2}};$$

$$B1 = \frac{p-2}{2}\sqrt{\lambda};$$

$$u1[s\_] := A1\, u\left[\frac{2}{p-2},\, B1\, s\right]$$

$$-u1''[s] + \lambda\, u1[s] - u1[s]^{p-1};$$

$$\%\ /.\ \mathrm{Cosh}\left[\frac{1}{2}(-2+p)\, s\sqrt{\lambda}\right] \to x;$$

$$\mathrm{FullSimplify}\left[\mathrm{PowerExpand}\left[A\,(-2+p)^2\, x^{2+\frac{2}{-2+p}}\ \%\ /.\ \mathrm{Sinh}\left[\frac{1}{2}(-2+p)\, s\sqrt{\lambda}\right]^2 \to x^2-1\right]\right]$$

0

▪ Computation of the best constant in the Lieb - Thirring (1) inequality

```
Integrate[u[α, s], {s, -∞, ∞}, Assumptions → α > 0]
```

$$I1[\alpha\_] := \frac{2^{-1+\alpha}\,\mathrm{Gamma}\left[\frac{\alpha}{2}\right]^2}{\mathrm{Gamma}[\alpha]}$$

$$\frac{2^{-1+\alpha}\,\mathrm{Gamma}\left[\frac{\alpha}{2}\right]^2}{\mathrm{Gamma}[\alpha]}$$

$$\mathrm{Solve}\left[\gamma == \frac{p+2}{2\,(p-2)},\, p\right][[1]];$$

$$\mathrm{FullSimplify}\left[\mathrm{PowerExpand}\left[\frac{\lambda^{\gamma}}{\frac{A1^p}{B1}\, I1\left[\frac{2p}{p-2}\right]}\ /.\ \%\right]\right]$$

$$CLT[\gamma\_] := \frac{\left(1-\frac{2}{1+2\gamma}\right)^{-\frac{1}{2}+\gamma}\,\mathrm{Gamma}[1+\gamma]}{\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}+\gamma\right]}$$

$$\mathrm{FullSimplify}\left[\frac{1}{CLT[\gamma]}\left(\frac{2\gamma-1}{2\gamma+1}\right)^{\gamma-\frac{1}{2}}\frac{2\gamma}{2\gamma+1}\,\frac{\mathrm{Gamma}[\gamma]}{\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{1}{2}+\gamma\right]}\right]$$

$$\frac{\left(1-\frac{2}{1+2\gamma}\right)^{-\frac{1}{2}+\gamma}\,\mathrm{Gamma}[1+\gamma]}{\sqrt{\pi}\,\mathrm{Gamma}\left[\frac{3}{2}+\gamma\right]}$$

1

```
FullSimplify[1/CLT[γ] (p^2 - 4)/(8 Sqrt[π]) Gamma[(p+2)/(2 (p-2))]/Gamma[2/(p-2)] (2/p)^(p/(p-2)) /. p -> 2 (2 γ + 1)/(2 γ - 1)]
```

$$\text{FullSimplify}\left[\frac{1}{\text{CLT}[\gamma]} \frac{p^2-4}{8\sqrt{\pi}} \frac{\text{Gamma}\left[\frac{p+2}{2(p-2)}\right]}{\text{Gamma}\left[\frac{2}{p-2}\right]} \left(\frac{2}{p}\right)^{\frac{p}{p-2}} \text{/. } p \to 2\,\frac{2\gamma+1}{2\gamma-1}\right]$$

1

■ The ODE corresponding to $\theta < 1$

$$u[s\_] := A\,\text{Cosh}[B\,s]^{-\frac{2}{p-2}}$$

$$-\theta\,u''[s] + \eta\,u[s] - u[s]^{p-1};$$

$$\%\ \text{/. Cosh}[B\,s] \to x;$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[A\,(-2+p)^2\,x^{2+\frac{2}{-2+p}}\ \%\ \text{/. Sinh}[B\,s]^2 \to x^2-1\right]\right]$$

$$-A^p\,(-2+p)^2 + A^2\left((-2+p)^2\,x^2\,\eta + 2\,B^2\left(p-2\,x^2\right)\theta\right)$$

$$f[x\_] := -A^p\,(-2+p)^2 + A^2\left((-2+p)^2\,x^2\,\eta + 2\,B^2\left(p-2\,x^2\right)\theta\right)$$

```
Off[Solve::"incnst"]
Off[Solve::"ifun"]
Solve[{f[0] == 0, f''[0] == 0}, {A, B}][[4]]
```

$$\left\{B \to \frac{\sqrt{4-4\,p+p^2}\,\sqrt{\eta}}{2\sqrt{\theta}},\ A \to 2^{\frac{1}{2-p}}\left(\frac{1}{p\,\eta}\right)^{\frac{1}{2-p}}\right\}$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[-\theta\,u''[s] + \eta\,u[s] - u[s]^{p-1}\ \text{/. }\left\{B \to \frac{\sqrt{4-4\,p+p^2}\,\sqrt{\eta}}{2\sqrt{\theta}},\ A \to 2^{\frac{1}{2-p}}\left(\frac{1}{p\,\eta}\right)^{\frac{1}{2-p}}\right\}\right]\right]$$

0

■ Various constants

$$\gamma\theta = \gamma\ \text{/. Solve}\left[\left(\gamma+\frac{1}{2}\right)(p-2) == p\,\theta,\ \gamma\right][[1]]$$

$$\frac{2-p+2\,p\,\theta}{2\,(-2+p)}$$

$$\eta\theta = \eta\ \text{/. Solve}\left[\eta == (1-\theta)\,\frac{p-2}{p+2}\,\frac{\eta}{\theta} + \lambda,\ \eta\right][[1]]$$

$$\frac{(2\,\theta + p\,\theta)\,\lambda}{2-p+2\,p\,\theta}$$

$$A\theta = \left(\frac{p\,\eta\theta}{2}\right)^{\frac{1}{p-2}}$$

$$B\theta = \text{PowerExpand}\left[\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{(p-2)\sqrt{\eta\theta}}{2\sqrt{\theta}}\right]\right]\right]$$

$$2^{-\frac{1}{-2+p}}\left(\frac{p\,(2\,\theta+p\,\theta)\,\lambda}{2-p+2\,p\,\theta}\right)^{\frac{1}{-2+p}}$$

$$\frac{(-2+p)\sqrt{2+p}\,\sqrt{\lambda}}{2\sqrt{2+p\,(-1+2\,\theta)}}$$

$$q\theta = q\ /.\ \text{Solve}\left[q+1 == \frac{2\,\gamma}{\gamma-1}\ /.\ \gamma\to\gamma\theta,\ q\right][[1]]$$

$$\frac{-2+p+2\,p\,\theta}{6-3\,p+2\,p\,\theta}$$

$$\text{Simplify}\left[\frac{N-1}{q\theta-1}\right]$$

$$\frac{(-1+N)\,(6+p\,(-3+2\,\theta))}{4\,(-2+p)}$$

$$\text{Simplify}\left[\frac{(1-\theta)\,p}{\gamma\theta\,(p-2)}\right]$$

$$\text{Simplify}\left[\frac{\theta\,p-2}{\gamma\theta\,(p-2)}\right]$$

$$-\frac{2\,p\,(-1+\theta)}{2+p\,(-1+2\,\theta)}$$

$$\frac{2\,(-2+p\,\theta)}{2+p\,(-1+2\,\theta)}$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\left(\frac{A\theta^{p}}{B\theta}\right)^{\frac{\theta\,p-2}{\gamma\theta\,(p-2)}}\left(\frac{A\theta^{p}}{B\theta}\right)^{\frac{(1-\theta)\,p}{\gamma\theta\,(p-2)}}\right]\right]$$

$$16^{\frac{1}{-2+p-2\,p\,\theta}}\,(-2+p)^{\frac{4-2\,p}{2-p+2\,p\,\theta}}\,p^{\frac{2\,p}{2-p+2\,p\,\theta}}\,(2+p)^{\frac{2-p}{2-p+2\,p\,\theta}}\,((2+p)\,\theta)^{\frac{2\,p}{2-p+2\,p\,\theta}}\,(2+p\,(-1+2\,\theta))^{-\frac{2+p}{2-p+2\,p\,\theta}}\,\lambda^{\frac{2+p}{2-p+2\,p\,\theta}}$$

▪ The computation of the constant for $\theta = 1$

$$\text{LegendeDupl}[z\_] := 2^{1-2\,z}\sqrt{\pi}\,\frac{\text{Gamma}[2\,z]}{\text{Gamma}[z]}$$

$$\text{FullSimplify}\left[\text{FullSimplify}\left[\text{PowerExpand}\left[\text{CLT}[\gamma]^{\frac{1}{\gamma}}\left(\frac{\text{A}\theta^{\theta\,p}}{\text{B}\theta}\,\text{I1}\left[\frac{2\,p}{p-2}\right]\right)^{\frac{1}{\gamma}}\right]\right]\;/.\;\gamma\to\gamma\theta\right];$$

$$\text{FullSimplify}[\text{PowerExpand}[\,\%\;/.\;\theta\to 1]]$$
$$\text{FullSimplify}[\text{PowerExpand}[\,\%\;/.\;\lambda\to 1]];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[\frac{1}{2}+\frac{p}{-2+p}\right]\to\text{LegendeDupl}\left[\frac{p}{-2+p}\right]\right]\right]$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[1+\frac{p}{-2+p}\right]\to\frac{p}{-2+p}\,\text{Gamma}\left[\frac{p}{-2+p}\right]\right]\right]$$

$$4\left(\frac{-2+p}{p}\right)^{-2+\frac{8}{2+p}}\pi^{-1+\frac{4}{2+p}}\,\lambda\,\text{Gamma}\left[\frac{p}{-2+p}\right]^{\frac{4\,(-2+p)}{2+p}}$$

$$\text{Gamma}\left[\frac{2\,p}{-2+p}\right]^{-2+\frac{8}{2+p}}\text{Gamma}\left[\frac{1}{2}+\frac{p}{-2+p}\right]^{2-\frac{8}{2+p}}\text{Gamma}\left[1+\frac{p}{-2+p}\right]^{-2+\frac{8}{2+p}}$$

$$\left(\frac{-2+p}{p}\right)^{-2+\frac{8}{2+p}}\text{Gamma}\left[\frac{p}{-2+p}\right]^{2-\frac{8}{2+p}}\text{Gamma}\left[1+\frac{p}{-2+p}\right]^{-2+\frac{8}{2+p}}$$

1

■ The computation of the constant for $\theta < 1$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\text{CLT}[\gamma]^{\frac{1}{\gamma}}\left(\left(\frac{\text{A}\theta^{p}}{\text{B}\theta}\,\text{I1}\left[\frac{2\,p}{p-2}\right]\right)^{\theta-\frac{2}{p}}\left(\frac{\text{A}\theta^{2}}{\text{B}\theta}\,\text{I1}\left[\frac{4}{p-2}\right]\right)^{1-\theta}\right)^{\frac{p}{p-2}\,\frac{1}{\gamma}}\right]\right];$$

$$\text{FullSimplify}[\text{PowerExpand}[\%\;/.\;\gamma\to\gamma\theta]];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[\frac{1}{2}+\frac{p\,\theta}{-2+p}\right]\to\text{LegendeDupl}\left[\frac{p\,\theta}{-2+p}\right]\right]\right];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[1+\frac{p\,\theta}{-2+p}\right]\to\frac{p\,\theta}{-2+p}\,\text{Gamma}\left[\frac{p\,\theta}{-2+p}\right]\right]\right];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[\frac{4}{-2+p}\right]\to\frac{(-2+p)^2}{4\,(p+2)}\,\text{Gamma}\left[\frac{2\,p}{-2+p}\right]\right]\right];$$

$$\text{Cp}\theta=\text{FullSimplify}\left[\text{PowerExpand}\left[\%\;/.\;\text{Gamma}\left[\frac{2}{-2+p}\right]\to\frac{p-2}{2}\,\text{Gamma}\left[\frac{p}{-2+p}\right]\right]\right]$$

$$\text{FullSimplify}[\text{PowerExpand}[\,\%\;/.\;\theta\to 1]]$$

$$\frac{1}{2+p\,(-1+2\,\theta)}\,2^{-1+\frac{-2+p}{2+p\,(-1+2\,\theta)}}\,(2+p)^{\frac{2+p}{2-p+2\,p\,\theta}}\,(2+p\,(-1+\theta))^{1+\frac{2-p}{2-p+2\,p\,\theta}}\,\lambda$$

$$\text{Gamma}\left[\frac{p}{-2+p}\right]^{\frac{4\,(-2+p)}{2+p\,(-1+2\,\theta)}}\text{Gamma}\left[\frac{2\,p}{-2+p}\right]^{\frac{4-2\,p}{2-p+2\,p\,\theta}}\text{Gamma}\left[\frac{p\,\theta}{-2+p}\right]^{\frac{8-4\,p}{2-p+2\,p\,\theta}}\text{Gamma}\left[\frac{2\,p\,\theta}{-2+p}\right]^{\frac{2\,(-2+p)}{2+p\,(-1+2\,\theta)}}$$

λ

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\frac{1}{\text{Cp}\theta}\,\lambda\,(p+2)^{\frac{2+p}{2-p+2p\theta}}\,\frac{1}{2+p\,(-1+2\theta)}\left(\frac{2}{2+p\,(-1+\theta)}\right)^{-\frac{2\,(2-p+p\theta)}{2-p+2p\theta}}\left(\frac{\text{Gamma}\left[\frac{p\theta}{-2+p}\right]}{\text{Gamma}\left[\frac{p}{-2+p}\right]}\right)^{\frac{4\,(2-p)}{2-p+2p\theta}}\left(\frac{\text{Gamma}\left[\frac{2p\theta}{-2+p}\right]}{\text{Gamma}\left[\frac{2p}{-2+p}\right]}\right)^{\frac{2\,(-2+p)}{2+p\,(-1+2\theta)}}\Big]\Big]$$

1

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\text{CLT}[\gamma\theta]^{\frac{1}{\gamma\theta}}\left(\frac{A\theta^p}{B\theta}\,\frac{4}{p+2}\,\text{I1}\left[\frac{4}{p-2}\right]\right)^{\frac{2\,(\theta p-2)}{2+p\,(-1+2\theta)}}\left(\frac{A\theta^2}{B\theta}\,\text{I1}\left[\frac{4}{p-2}\right]\right)^{\frac{2p\,(1-\theta)}{2+p\,(-1+2\theta)}}\Big]\Big];$$

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\%\;/.\;\text{Gamma}\left[\frac{1}{2}+\frac{p\theta}{-2+p}\right]\to\text{LegendeDupl}\left[\frac{p\theta}{-2+p}\right]\Big]\Big];$$

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\%\;/.\;\text{Gamma}\left[1+\frac{p\theta}{-2+p}\right]\to\frac{p\theta}{-2+p}\,\text{Gamma}\left[\frac{p\theta}{-2+p}\right]\Big]\Big];$$

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\%\;/.\;\text{Gamma}\left[\frac{4}{-2+p}\right]\to\frac{(-2+p)^2}{4\,(p+2)}\,\text{Gamma}\left[\frac{2p}{-2+p}\right]\Big]\Big];$$

$$\texttt{FullSimplify}\Big[\texttt{PowerExpand}\Big[\frac{\%}{\text{Cp}\theta}\;/.\;\text{Gamma}\left[\frac{2}{-2+p}\right]\to\frac{p-2}{2}\,\text{Gamma}\left[\frac{p}{-2+p}\right]\Big]\Big]$$

1

■ More results in the case $\theta < 1$

$$p\;/.\;\text{Solve}\left[\frac{(2\theta+1)\,p-2}{(2\theta-3)\,p+6}+1==\frac{2\,(d-1)}{d-3},\,p\right][[1]]$$

$$\theta\;/.\;\text{Solve}\left[p==\frac{2d}{d-2\theta},\,\theta\right][[1]]$$

$$\text{Factor}\left[\frac{d-1}{q-1}\;/.\;q\to\frac{(2\theta+1)\,p-2}{(2\theta-3)\,p+6}\right]$$

$$\frac{2d}{d-2\theta}$$

$$\frac{-2d+dp}{2p}$$

$$\frac{(-1+d)\,(6-3p+2p\theta)}{4\,(-2+p)}$$