Improved Sobolev's inequalities, relative entropy and fast diffusion equations

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Surface of the (d-1)-dimensional unit sphere

$$\texttt{S[d\_] :=} \frac{2\,\pi^{\frac{d}{2}}}{\texttt{Gamma}\left[\frac{d}{2}\right]}$$

Constants

$$\texttt{mc} = \frac{d-2}{d};\ \texttt{m1} = \frac{d-1}{d};\ \texttt{tm1} = \frac{d}{d+2};\ \gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)};$$

Mass and second moment of the Barenblatt functions

$$\texttt{S[d] Integrate}\left[r^{d-1}\left(1+r^2\right)^{-a},\ \{r,\ 0,\ \infty\},\ \texttt{Assumptions} \to d>1\ \&\&\ d<2\,a\right]$$

$$\frac{\pi^{d/2}\,\texttt{Gamma}\left[a-\frac{d}{2}\right]}{\texttt{Gamma}[a]}$$

- **Computation of M***

$$\texttt{Mstar = S[d] Integrate}\left[r^{d-1}\left(1+r^2\right)^{\frac{1}{m-1}},\ \{r,\ 0,\ \infty\},\ \texttt{Assumptions} \to d>1\ \&\&\ m>\frac{d-2}{d}\ \&\&\ m>0\ \&\&\ m<1\right]$$

$$\texttt{Mstar1 = FullSimplify}\left[\texttt{PowerExpand}\left[\%\ /.\ m \to \frac{p+1}{2\,p}\right]\right]$$

$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\%\ /.\ p \to \frac{d}{d-2}\right]\right]$$

$$\frac{\pi^{d/2}\,\texttt{Gamma}\left[\frac{2+d\,(-1+m)}{2-2\,m}\right]}{\texttt{Gamma}\left[\frac{1}{1-m}\right]}$$

$$\frac{\pi^{d/2}\,\texttt{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]}{\texttt{Gamma}\left[\frac{2\,p}{-1+p}\right]}$$

$$\frac{\pi^{d/2}\,\texttt{Gamma}\left[\frac{d}{2}\right]}{\texttt{Gamma}[d]}$$

■ Computation of C_M

$$\text{CM} = \text{FullSimplify}\left[\left(\frac{\text{Mstar}}{\text{M}}\right)^{\frac{2\,(1-m)}{d\,(m-mc)}}\right]$$

$$\left(\frac{\pi^{d/2}\,\text{Gamma}\left[-\frac{d}{2}+\frac{1}{1-m}\right]}{M\,\text{Gamma}\left[\frac{1}{1-m}\right]}\right)^{\frac{2-2m}{2+d\,(-1+m)}}$$

$$\text{FullSimplify}\left[\text{CM} \;/.\; m \to \frac{d-1}{d}\right]$$

$$\text{FullSimplify}\left[\text{CM} \;/.\; m \to \frac{p+1}{2\,p}\right]$$

$$\left(\frac{\pi^{d/2}\,\text{Gamma}\left[\frac{d}{2}\right]}{M\,\text{Gamma}[d]}\right)^{2/d}$$

$$\left(\frac{\pi^{d/2}\,\text{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]}{M\,\text{Gamma}\left[\frac{2\,p}{-1+p}\right]}\right)^{\frac{2-2p}{d\,(-1+p)-4\,p}}$$

```
CM /. M → 1;
```

$$\text{CC1} = \text{FullSimplify}\left[\text{PowerExpand}\left[\% \;/.\; m \to \frac{p+1}{2\,p}\right]\right]$$

$$\pi^{\frac{d-d\,p}{d\,(-1+p)-4\,p}}\,\text{Gamma}\left[\frac{2\,p}{-1+p}\right]^{\frac{2-2\,p}{d+4\,p-d\,p}}\,\text{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]^{\frac{2-2\,p}{d\,(-1+p)-4\,p}}$$

■ Computation of $\int B_1{}^m \, dx$

```
Mm =
 S[d] Integrate[r^(d-1) (C1 + r^2)^(m/(m-1)), {r, 0, ∞}, Assumptions → d > 1 && m > d/(d+2) && m > 0 && m < 1 && C1 > 0]
FullSimplify[PowerExpand[% /. C1 → CM]];
% /. Gamma[-m/(-1 + m)] → (1 - m)/m Gamma[1/(1 - m)];
B1m = FullSimplify[% /. Gamma[(d - (2 + d) m)/(2 (-1 + m))] → (2 (1 - m))/((d + 2) m - d) Gamma[-d/2 + 1/(1 - m)]]
FullSimplify[PowerExpand[B1m/CM]]
```

$$\frac{C1^{\frac{d}{2}+\frac{m}{-1+m}}\,\pi^{d/2}\,\mathrm{Gamma}\left[\frac{d-2\,m-d\,m}{2\,(-1+m)}\right]}{\mathrm{Gamma}\left[-\frac{m}{-1+m}\right]}$$

$$\frac{2\,m\,M^{\frac{d\,(-1+m)+2\,m}{2+d\,(-1+m)}}\,\pi^{-1+\frac{2}{2+d\,(-1+m)}}\,\mathrm{Gamma}\left[-\frac{d}{2}+\frac{1}{1-m}\right]^{\frac{2-2\,m}{2+d\,(-1+m)}}\,\mathrm{Gamma}\left[\frac{1}{1-m}\right]^{\frac{2\,(-1+m)}{2+d\,(-1+m)}}}{d\,(-1+m)+2\,m}$$

$$\frac{2\,m\,M}{d\,(-1+m)+2\,m}$$

■ Computation of K_M

```
KM = (d (1 - m))/(2 m) B1m
KM /. M → 1;
K1 = FullSimplify[PowerExpand[% /. m → (p + 1)/(2 p)]];
FullSimplify[PowerExpand[1/K1 (d (p - 1))/(d + 2 - p (d - 2)) ((π^(d/2) Gamma[(2 p)/(-1 + p) - d/2])/Gamma[(2 p)/(-1 + p)])^((2 (p - 1))/(d - p (d - 4)))]]
```

$$\frac{d\,(1-m)\,M^{\frac{d\,(-1+m)+2\,m}{2+d\,(-1+m)}}\,\pi^{-1+\frac{2}{2+d\,(-1+m)}}\,\mathrm{Gamma}\left[-\frac{d}{2}+\frac{1}{1-m}\right]^{\frac{2-2\,m}{2+d\,(-1+m)}}\,\mathrm{Gamma}\left[\frac{1}{1-m}\right]^{\frac{2\,(-1+m)}{2+d\,(-1+m)}}}{d\,(-1+m)+2\,m}$$

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■ A verification

```
FullSimplify[PowerExpand[1/B1m (2 m)/((d + 2) m - d) M^(((d + 2) m - d)/(d (m - mc))) (π^(d/2) Gamma[(d (m - mc))/(2 (1 - m))]/Gamma[1/(1 - m)])^((2 (1 - m))/(d (m - mc)))]]
```

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Sobolev's inequality

```
FullSimplify[PowerExpand[S[d]^(-2/d) (Integrate[r^(d-1) (1 + r^2)^-d, {r, 0, ∞}, Assumptions → d > 2])^(1-2/d) /
    ((d - 2)^2 Integrate[r^(d+1) (1 + r^2)^-d, {r, 0, ∞}, Assumptions → d > 2])]];
% /. Gamma[1 + d/2] → d/2 Gamma[d/2];
% /. Gamma[-1 + d/2] → 2/(d - 2) Gamma[d/2];
Sobolev = FullSimplify[PowerExpand[% /. Gamma[(1 + d)/2] → 2^(1-d) Sqrt[π] Gamma[d]/Gamma[d/2]]]
```

$$\frac{\text{Gamma}\left[\frac{d}{2}\right]^{-2/d}\,\text{Gamma}[d]^{2/d}}{(-2+d)\,d\,\pi}$$

Gagliardo-Nirenberg inequalities

```
B[a_, b_, d_] := S[d] Integrate[r^b (1 + r^2)^-a, {r, 0, ∞}, Assumptions → d > 0 && a > (b + 1)/2]
θ[p_, d_] := (p - 1)/p d/(d + 2 - p (d - 2))
FullSimplify[
  PowerExpand[B[2 p/(p-1), d - 1, d]^(1/(2 p)) /
    (((2/(p-1))^2 B[2 p/(p-1), d + 1, d])^(θ[p,d]/2) (B[(p+1)/(p-1), d - 1, d])^((1-θ[p,d])/(p+1)))] /. Gamma[1 + d/2] → d/2 Gamma[d/2]];
FullSimplify[PowerExpand[% /. Gamma[2 p/(-1 + p)] → (p + 1)/(p - 1) Gamma[(p + 1)/(p - 1)]]];
FullSimplify[PowerExpand[% /. Gamma[1 - d/2 + 2/(-1 + p)] → Gamma[-d/2 + (p + 1)/(-1 + p)]]];
FullSimplify[PowerExpand[% /. Gamma[-d/2 + 2 p/(-1 + p)] → (-d/2 + (p + 1)/(-1 + p)) Gamma[-d/2 + (p + 1)/(-1 + p)]]];
CGN = FullSimplify[PowerExpand[% /. Gamma[1 - d/2 + 2/(-1 + p)] → Gamma[-d/2 + (p + 1)/(-1 + p)]]];
CGN1 = (Gamma[(1+p)/(-1+p)] / ((2 π d)^(d/2) Gamma[1 - d/2 + 2/(-1+p)]))^((p-1)/(p (d+2-p (d-2)))) ((p - 1)^(p+1)/(p + 1)^(d+1-p (d-1)))^(1/(p (d+2-p (d-2)))) ((d + 2 - p (d - 2))/(2 (p - 1)))^(1/(2 p));
CGN1/CGN;
FullSimplify[PowerExpand[%^(2 p)]]
```

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- **The Sobolev case**

$$\texttt{Factor}\left[\theta\left[\frac{d}{d-2},\ d\right]\right]$$

$$\texttt{CGN}^2\ /.\ p \to \frac{d}{d-2};$$

$$\texttt{FullSimplify[PowerExpand[\%]];}$$

$$\%\ /.\ \texttt{Gamma}[-1+d] \to \frac{\texttt{Gamma}[d]}{d-1};$$

$$\%\ /.\ \texttt{Gamma}\left[-1+\frac{d}{2}\right] \to \frac{2\ \texttt{Gamma}\left[\frac{d}{2}\right]}{d-2};$$

$$\texttt{Sobolev = FullSimplify[PowerExpand[\%]]}$$

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$$\frac{\texttt{Gamma}\left[\frac{d}{2}\right]^{-2/d}\ \texttt{Gamma}[d]^{2/d}}{(-2+d)\ d\ \pi}$$

$$\texttt{Limit}[\texttt{CGN},\ p \to 1]$$

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The Csiszár-Kullback inequality

$$\texttt{CCK = FullSimplify}\left[\frac{p-1}{p+1}\ \frac{m}{8\ \texttt{B1m}}\ \texttt{CM}^2\ \left(\frac{\texttt{M2}}{\texttt{KM}}\right)^{\frac{d}{2}(1-m)}\ /.\ m \to \frac{p+1}{2\,p}\right];$$

$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\frac{1}{\texttt{CCK}}\ \frac{p-1}{p+1}\ \frac{d+2-p\ (d-2)}{32\ p}\ \left(\frac{d+2-p\ (d-2)}{d\ (p-1)}\ \texttt{M2}\right)^{d\,\frac{p-1}{4\,p}}\ \texttt{Mstar1}^{\frac{p-1}{2\,p}}\ M^{\frac{2+d-(6+d)\,p}{4\,p}}\right]\right];$$

$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\%^{\frac{4\,p}{d\,(p-1)}}\right]\right]$$

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$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\texttt{CCK}\ /.\ p \to \frac{d}{d-2}\right]\right]$$

$$-\frac{M^{-\frac{3}{2}-\frac{1}{d}}\ \sqrt{\texttt{M2}}\ \sqrt{\pi}\ \texttt{Gamma}\left[\frac{d}{2}\right]^{\frac{1}{d}}\ \texttt{Gamma}[d]^{-1/d}}{16\ \left(\frac{1}{-2+d}\right)^{3/2}\ (-1+d)\ d^{3/2}}$$

Homogeneity and scalings

- **Exponents in Theorem 3**

$$\texttt{Solve}\left[\left\{\gamma == 2\,m - \frac{\alpha+\beta\,\gamma}{2\,p},\ 2\,d\ (m-1) + \frac{\alpha}{p} == 0\right\},\ \{\alpha,\ \beta\}\right][[1]];$$

$$\texttt{Simplify}\left[\texttt{Expand}\left[\%\ /.\ m \to \frac{p+1}{2\,p}\right]\right]$$

$$\{\beta \to d + 2\,p - d\,p,\ \alpha \to d\,(-1+p)\}$$

Exponents in the Csiszár-Kullback inequality

$$\texttt{Solve}\left[\left\{2\,p\,m == \frac{\alpha+\delta}{2} + 4\,p,\ d\,(m-1) + \frac{\alpha}{2\,p} == 0\right\},\ \{\alpha,\ \delta\}\right][[1]];$$

$$\texttt{Simplify}\left[\texttt{Expand}\left[\%\ /.\ m \to \frac{p+1}{2\,p}\right]\right]$$

$$\{\delta \to 2 + d - 6\,p - d\,p,\ \alpha \to d\,(-1+p)\}$$

- **Exponents in Corollary 4**

$$(\delta - \beta\,\gamma) - 2\,p\,(\gamma - 4)\ /.\ \delta \to 2 + d - 6\,p - d\,p;$$
$$\texttt{Simplify}[\%\ /.\ \beta \to d + 2\,p - d\,p]$$

$$0$$

$$\texttt{Simplify}\left[2\,p\,(\gamma-4)\ /.\ p \to \frac{d}{d-2}\right]$$

$$\frac{4+6\,d}{2-d}$$

Improved Gagliardo-Nirenberg and Sobolev inequalities

$$\texttt{Cmd} = d^2\,\frac{(1-m)^2\,(m-\texttt{mc})}{8\,m};$$

$$\texttt{mc} = \frac{d-2}{d};$$

$$\texttt{m1} = \frac{d-1}{d};$$

$$\texttt{Res} = \texttt{Simplify}\left[\texttt{PowerExpand}\left[\frac{m}{1-m}\,\frac{(d+2)\,m-d}{d\,(m-\texttt{mc})}\,\frac{\texttt{Cmd}}{\texttt{C1}}\,\texttt{k1}^{-d\,(m-\texttt{m1})}\ /.\ \texttt{k1} \to \frac{d\,(1-m)}{(d+2)\,m-d}\,\texttt{C1}\right]\right];$$

$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\frac{1}{\texttt{Res}}\,d\,\frac{p-1}{32\,p^2}\,(d\,(p-1))^{-\frac{d-p\,(d-2)}{2\,p}}\left(\frac{\texttt{C1}}{d+2-p\,(d-2)}\right)^{-\frac{d-p\,(d-4)}{2\,p}}\ /.\ m \to \frac{1+p}{2\,p}\right]\right]$$

$$\texttt{Simplify}[\texttt{Res}\ /.\ \texttt{C1} \to \texttt{CC1}];$$

$$\texttt{Cpd1} = \texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\%\ /.\ m \to \frac{1+p}{2\,p}\right]\right]$$

$$\left(2 - d + \frac{2+d}{p}\right)^{\frac{d\,(-1+p)}{2\,p}}\,p^{\frac{d\,(-1+p)}{2\,p}}\,(2 + d + 2\,p - d\,p)^{-\frac{d\,(-1+p)}{2\,p}}$$

$$\frac{d^{\frac{d\,(-1+p)}{2\,p}}\left(2 - d + \frac{2+d}{p}\right)^{-\frac{d\,(-1+p)}{2\,p}}\left(\frac{-1+p}{p}\right)^{\frac{d\,(-1+p)}{2\,p}}(2 + d + 2\,p - d\,p)^2\,\pi^{-\frac{d\,(-1+p)}{2\,p}}\,\texttt{Gamma}\left[\frac{2\,p}{-1+p}\right]^{\frac{-1+p}{p}}\,\texttt{Gamma}\left[-\frac{d}{2} + \frac{2\,p}{-1+p}\right]^{-1+\frac{1}{p}}}{32\,p^2}$$

$$\texttt{FullSimplify}\left[\texttt{PowerExpand}\left[\frac{\texttt{Cpd1}}{\texttt{Sobolev}}\ /.\ p \to \frac{d}{d-2}\right]\right]$$

$$\frac{1}{8}\,(-2+d)^2$$

$$\text{Simplify}\left[\frac{(2\,m-1)^2}{m\,(1-m)}\;\frac{m}{1-m}\;\frac{d\,(m-mc)}{(d+2)\,m-d}\,\text{CM}\;/.\;m\to 1-\frac{1}{d}\right];$$

$$\text{FullSimplify}[\%\;/.\;M\to 1]$$

$$\text{Simplify}\left[\frac{(2\,m-1)^2}{m\,(1-m)}\,\text{Cmd}\,(d-1)^2\;/.\;m\to 1-\frac{1}{d}\right]$$

$$(-2+d)\;d\left(\frac{\pi^{d/2}\,\text{Gamma}\left[\frac{d}{2}\right]}{\text{Gamma}[d]}\right)^{2/d}$$

$$\frac{1}{8}\,(-2+d)^2$$

Non-homogeneous Gagliardo-Nirenberg inequalities

▪ The inhomogenous version of Gagliardo-Nirenberg inequalities

$$f[A\_,\,B\_,\,M\_,\,F\_,\,\sigma 0\_]:=\frac{m\,(1-m)}{(2\,m-1)^2}\,\sigma^{\frac{d}{2}\,(m-mc)}\,\lambda^{2-d}\,A+\frac{d\,(m-m1)}{1-m}\,\lambda^{-d}\,B-$$
$$\frac{m}{1-m}\;\frac{d\,(m-mc)}{(d+2)\,m-d}\,\sigma^{\frac{d}{2}\,(1-m)}\,\text{C1}\,\left(\lambda^{-d}\,M\right)^{\gamma}-\text{Cmd1}\,\sigma^{\frac{d}{2}\,(m-mc)}\,\frac{\left(\lambda^{-d}\,F\right)^2}{\left(\lambda^{-d}\right)^{\gamma}\,\text{KM}\,\sigma}\;/.\;\sigma\to\lambda^{-(d+2)}\,\sigma 0$$

$$a=f[1,\,0,\,1,\,0,\,\sigma]\;/.\;\text{C1}\to 0;$$

$$g[A\_,\,B\_,\,M\_,\,F\_,\,\sigma\_]:=\text{Simplify}\left[\frac{f[A,\,B,\,M,\,F,\,\sigma]}{a}\right]$$

```
b1 = FullSimplify[PowerExpand[g[0, 1, 1, 0, σ] /. C1 → 0]];
c1 = FullSimplify[PowerExpand[g[0, 0, 1, 0, σ]]];
FullSimplify[PowerExpand[g[0, 0, 1, 1, σ]]];
d1 = FullSimplify[PowerExpand[g[0, 0, 1, 1, σ] /. C1 → 0]];
```

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$$\text{Simplify}\left[\text{b1}\;/.\;m\to\frac{1+p}{2\,p}\right];$$

```
λ1 = FullSimplify[PowerExpand[λ /. Solve[% == d - p (d - 2), λ][[1]]]]
```

$$4^{\frac{4\,p}{d\,(d\,(-1+p)-2\,(1+p))}}\,(-1+p)^{\frac{8\,p}{d\,(2+d+2\,p-d\,p)}}\,(1+p)^{\frac{4\,p}{d\,(2+d+2\,p-d\,p)}}\,\sigma^{\frac{d+4\,p-d\,p}{d\,(2+d+2\,p-d\,p)}}$$

$$\text{Simplify}\left[\text{b1}\;/.\;m\to\frac{1+p}{2\,p}\right];$$

```
FullSimplify[PowerExpand[% /. λ → λ1]]
```

$$d+2\,p-d\,p$$

$$\text{Kpd1} = \frac{d - p\,(d-4)}{d + 2 - p\,(d-2)} \left(\frac{p-1}{2}\right)^{-\frac{2\,d\,(p-1)}{d-p\,(d-4)}} (p+1)^{2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}} \left(\pi^{\frac{d}{2}} \frac{\text{Gamma}\left[\frac{d-p\,(d-4)}{2\,(p-1)}\right]}{\text{Gamma}\left[\frac{2\,p}{-1+p}\right]}\right)^{\frac{2\,(p-1)}{d-p\,(d-4)}};$$

$$\text{CC1} = \pi^{\frac{d-d\,p}{d\,(-1+p)-4\,p}}\,\text{Gamma}\left[\frac{2\,p}{-1+p}\right]^{\frac{2-2\,p}{d+4\,p-d\,p}}\,\text{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]^{\frac{2-2\,p}{d\,(-1+p)-4\,p}};$$

$$\text{Simplify}\left[\text{c1} \;/.\; m \to \frac{1+p}{2\,p}\right];$$

$$\text{FullSimplify}[\text{PowerExpand}[\% \;/.\; \lambda \to \lambda 1]];$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{-\%}{\text{Kpd1}} \;/.\; \text{C1} \to \text{CC1}\right]\right]$$

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- **From the inhomogenous to the homogenous version of Gagliardo-Nirenberg inequalities**

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$$f[\lambda\_] := A\,\lambda^{\frac{d}{p}-d+2} + (d - p\,(d-2))\,B\,\lambda^{d\,\frac{p+1}{2\,p}-d}$$

$$\text{PowerExpand}\left[\text{FullSimplify}\left[2\,p\,\lambda^{d+1}\,f'[\lambda]\right]\right];$$

$$\text{FullSimplify}[\lambda \;/.\; \text{Solve}[\% == 0, \lambda][[1]]]$$

$$\text{Coef1} = \text{PowerExpand}[f[\lambda 1]];$$

$$\lambda 1 = 4^{\frac{p}{d\,(-1+p)-4\,p}} \left(\frac{A}{B\,d\,(p-1)}\right)^{-\frac{2\,p}{d+4\,p-d\,p}}$$

$$4^{\frac{p}{d\,(-1+p)-4\,p}} \left(-\frac{A}{B\,d - B\,d\,p}\right)^{-\frac{2\,p}{d+4\,p-d\,p}}$$

$$4^{\frac{p}{d\,(-1+p)-4\,p}} \left(\frac{A}{B\,d\,(-1+p)}\right)^{-\frac{2\,p}{d+4\,p-d\,p}}$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{1}{\text{Coef1}}\,(d + 4\,p - d\,p)\,2^{-2\,\frac{d-p\,(d-2)}{d-p\,(d-4)}}\,(d\,(p-1))^{-\frac{d\,(p-1)}{d-p\,(d-4)}} \left(A^{\frac{\theta[p,d]}{2}}\,B^{\frac{1-\theta[p,d]}{p+1}}\right)^{2\,p\,\gamma}\right]\right]$$

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A second computation of Kpd

$$g[r\_] := \left(1 + r^2\right)^{\frac{-1}{p-1}}$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[-g''[r] - \frac{d-1}{r}\,g'[r] + \frac{2\,(d - p\,(d-2))}{(p-1)^2}\,g[r]^p - \frac{4\,p}{(p-1)^2}\,g[r]^{2\,p-1}\right]\right]$$

0

$$g[r\_] := \sigma^{\frac{d}{4p}} f\left[\sqrt{\sigma}\ r\right]$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[-g''[r] - \frac{d-1}{r} g'[r] + \frac{2\,(d - p\,(d-2))}{(p-1)^2} g[r]^p - \frac{4\,p}{(p-1)^2} g[r]^{2p-1}\right]\right]$$

$$-\frac{(-1+d)\,(d\,(-1+p) - 2\,p)\, r^{-2+\frac{1}{2} d \left(-2+\frac{1}{p}\right)} \sigma^{-\frac{d(-1+p)}{2p}} \left(B\, d\,(-1+p)\, r^{d/2} \sigma^{d/4} - 2\,A\, r^{2+\frac{d}{2p}} \sigma^{1+\frac{d}{4p}}\right)}{2\,p} +$$

$$\frac{1}{4\,p^2} (d\,(-1+p) - 2\,p)\, r^{-2+\frac{1}{2} d \left(-2+\frac{1}{p}\right)} \sigma^{-\frac{d(-1+p)}{2p}} \left(B\, d\,(-1+p)\,(d\,(-1+p)+2\,p)\, r^{d/2} \sigma^{d/4} - 4\,A\,(d\,(-1+p) - p)\, r^{2+\frac{d}{2p}} \sigma^{1+\frac{d}{4p}}\right) -$$

$$\frac{2\,(d\,(-1+p) - 2\,p)\,\sigma^{d/4} \left(r^{-d} \sigma^{-d/2} \left(A\, r^{2+\frac{d}{p}} \sigma^{1+\frac{d}{2p}} + B\,(d + 2\,p - d\,p)\, r^{\frac{d(1+p)}{2p}} \sigma^{\frac{d(1+p)}{4p}}\right)\right)^p}{(-1+p)^2} -$$

$$\frac{4\,p\,\sigma^{\frac{d(-1+2p)}{4p}} \left(r^{-d} \sigma^{-d/2} \left(A\, r^{2+\frac{d}{p}} \sigma^{1+\frac{d}{2p}} + B\,(d + 2\,p - d\,p)\, r^{\frac{d(1+p)}{2p}} \sigma^{\frac{d(1+p)}{4p}}\right)\right)^{-1+2p}}{(-1+p)^2}$$

$$\text{Const} = p \left(\frac{p-1}{2}\right)^{-\frac{2d(p-1)}{d-p(d-4)}} (p+1)^{2\frac{d-p(d-2)}{d-p(d-4)}};$$

$$\text{Kpd1} = \frac{d - p\,(d-4)}{d + 2 - p\,(d-2)} \left(\frac{p-1}{2}\right)^{-\frac{2d(p-1)}{d-p(d-4)}} (p+1)^{2\frac{d-p(d-2)}{d-p(d-4)}} \left(\pi^{\frac{d}{2}} \frac{\text{Gamma}\left[\frac{d-p(d-4)}{2(p-1)}\right]}{\text{Gamma}\left[\frac{2p}{-1+p}\right]}\right)^{\frac{2(p-1)}{d-p(d-4)}};$$

$$\text{Solve}\left[\frac{2\ \sigma^{-\frac{d-p(d-4)}{4p}}}{(-1+p)^2} == \frac{p+1}{2}, \sigma\right][[1]]$$

$$\sigma 1 = \left(\left(\frac{p-1}{2}\right)^2 (p+1)\right)^{-\frac{4p}{d-p(d-4)}};$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{4\,p\ \sigma^{-\frac{d-p(d-2)}{2p}}}{(-1+p)^2\ \text{Const}} \ /.\ \sigma \to \sigma 1\right]\right]$$

$$\text{FullSimplify}\left[\text{PowerExpand}\left[\frac{\text{Const}}{\gamma\, p\ \text{Mstar1}^{\gamma-1}} \frac{1}{\text{Kpd1}}\right]\right]$$

$$\left\{\sigma \to 2^{-\frac{4p}{-d-4p+dp}} \left(-(1-p)^2 \left(-\frac{1}{2} - \frac{p}{2}\right)\right)^{\frac{4p}{-d-4p+dp}}\right\}$$

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▪ Checking the expresion of $K_{p,d}$

```
(2/(p - 1))^2 B[2 p/(p - 1), d + 1, d] /. Gamma[1 - d/2 + 2/(-1 + p)] → Gamma[-d/2 + (p + 1)/(-1 + p)];
a = % /. Gamma[1 + d/2] → d/2 Gamma[d/2];
b = B[(p + 1)/(p - 1), d - 1, d] /. Gamma[1 - d/2 + 2/(-1 + p)] → Gamma[-d/2 + (p + 1)/(-1 + p)];
FullSimplify[PowerExpand[(a σ1^(-(d-p (d-2))/(2 p)))/(b σ1^(d (p-1)/(4 p)))] /. Gamma[2 p/(-1 + p)] → (p + 1)/(p - 1) Gamma[(p + 1)/(p - 1)]];
FullSimplify[
 PowerExpand[(b σ1^(d (p-1)/(4 p)))/(Mstar1^γ Kpd1) (% + (d - p (d - 2)))] /. Gamma[(d - (-4 + d) p)/(2 (-1 + p))] → Gamma[-d/2 + 2 p/(-1 + p)]]
```

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▪ Recovering the best constant in the Gagliardo-Nirenberg inequality

$$\mathbf{f[\lambda\_] := \lambda^{\alpha} A + \lambda^{-\beta} B}$$

$$\mathbf{Expand\left[\frac{f'[\lambda]}{\lambda^{-1-\beta}}\right];}$$

```
FullSimplify[λ /. Solve[% == 0, λ][[1]]];
FullSimplify[PowerExpand[f[λ] /. λ → %]];
% /. B → (d - p (d - 2)) B;
```

$$\mathbf{\% \;/.\; \alpha \to \frac{d - p\,(d-2)}{p};}$$

$$\mathbf{OptFactor = FullSimplify\left[PowerExpand\left[\% \;/.\; \beta \to d\,\frac{p-1}{2\,p}\right]\right]}$$

$$\mathbf{\left(\frac{\%}{Kpd1}\right)^{\frac{1}{2\,p\,\gamma}} \;/.\; A \to 1;}$$

```
% /. B → 1;
CGN1 = FullSimplify[PowerExpand[%]]
```

$$\mathbf{FullSimplify\left[PowerExpand\left[CGN1 \;/.\; Gamma\left[\frac{2\,p}{-1+p}\right] \to \frac{p+1}{p-1}\,Gamma\left[\frac{p+1}{p-1}\right]\right]\right];}$$

$$\mathbf{N1 = FullSimplify\left[PowerExpand\left[\% \;/.\; Gamma\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right] \to \left(-\frac{d}{2}+\frac{p+1}{-1+p}\right) Gamma\left[-\frac{d}{2}+\frac{p+1}{-1+p}\right]\right]\right];}$$

$$\mathbf{D1 = FullSimplify\left[PowerExpand\left[CGN \;/.\; Gamma\left[1-\frac{d}{2}+\frac{2}{-1+p}\right] \to Gamma\left[-\frac{d}{2}+\frac{p+1}{-1+p}\right]\right]\right];}$$

$$\mathbf{PowerExpand\left[FullSimplify\left[PowerExpand\left[\frac{N1}{D1}\right]\right] \;/.\; \left(2-d+\frac{4}{-1+p}\right) \to \frac{x}{p-1}\right];}$$

```
FullSimplify[PowerExpand[% /. (2 + d + 2 p - d p) → x]]
```

$$2^{\frac{-2\,d\,(-1+p)+4\,p}{d\,(-1+p)-4\,p}}\,A^{\frac{d-d\,p}{d\,(-1+p)-4\,p}}\,B^{\frac{2\,d+4\,p-2\,d\,p}{d+4\,p-d\,p}}\,d^{\frac{d\,(-1+p)}{d\,(-1+p)-4\,p}}\,(-1+p)^{\frac{d\,(-1+p)}{d\,(-1+p)-4\,p}}\,(d+4\,p-d\,p)$$

$$4^{\frac{1}{d\,(-1+p)-2\,(1+p)}}\,d^{\frac{d\,(-1+p)}{2\,p\,(d\,(-1+p)-2\,(1+p))}}\,(-1+p)^{-\frac{d\,(-1+p)}{2\,p\,(d\,(-1+p)-2\,(1+p))}}\,(1+p)^{\frac{d+2\,p-d\,p}{p\,(d\,(-1+p)-2\,(1+p))}}$$
$$(2+d+2\,p-d\,p)^{\frac{d+4\,p-d\,p}{2\,p\,(2+d+2\,p-d\,p)}}\,\pi^{-\frac{d\,(-1+p)}{2\,p\,(2+d+2\,p-d\,p)}}\,\mathrm{Gamma}\left[\frac{2\,p}{-1+p}\right]^{\frac{1-p}{p\,(d\,(-1+p)-2\,(1+p))}}\,\mathrm{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]^{\frac{-1+p}{p\,(d\,(-1+p)-2\,(1+p))}}$$

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$$\mathbf{FactorGN = \left(\frac{d - p\,(d-4)}{\left(4^{d-p\,(d-2)}\,d^{d\,(p-1)}\,(p-1)^{d\,(p-1)}\right)^{\frac{1}{d-p\,(d-4)}}}\right)^{\frac{1}{2\,p\,\gamma}};}$$

$$\mathbf{OptFactor^{\frac{1}{2\,p\,\gamma}} \;/.\; A \to 1;}$$

$$\mathbf{FullSimplify\left[PowerExpand\left[\frac{\%}{FactorGN} \;/.\; B \to 1\right]\right]}$$

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▪ Recovering the best constant in the Gagliardo-Nirenberg inequality (2)

$$\mathbf{PowerExpand}\left[\mathbf{Kpd1\ CGN1}^{2\,p\,\gamma}\ \frac{4^{\frac{d-p\,(d-2)}{d-p\,(d-4)}}\ (d\ (p-1))^{\frac{d\,(p-1)}{d-p\,(d-4)}}}{d-p\ (d-4)}\right]$$

$$\mathbf{CGN = FullSimplify}\left[\mathbf{PowerExpand}\left[\%\ /.\ \mathrm{Gamma}\left[1-\frac{d}{2}+\frac{2}{-1+p}\right]\to \mathrm{Gamma}\left[-\frac{d}{2}+\frac{p+1}{-1+p}\right]\right]\right];$$

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}[\%]\ /.\ \mathrm{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]\to\left(-\frac{d}{2}+\frac{p+1}{-1+p}\right)\mathrm{Gamma}\left[-\frac{d}{2}+\frac{p+1}{-1+p}\right]\right];$$

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\%\ /.\ \mathrm{Gamma}\left[\frac{2\,p}{-1+p}\right]\to\frac{p+1}{p-1}\ \mathrm{Gamma}\left[\frac{p+1}{p-1}\right]\right]\right];$$

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\%\ /.\ \mathrm{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]\to\left(-\frac{d}{2}+\frac{p+1}{-1+p}\right)\mathrm{Gamma}\left[-\frac{d}{2}+\frac{p+1}{-1+p}\right]\right]\right];$$

$$\mathbf{FullSimplify}\left[\mathbf{PowerExpand}\left[\%\ /.\ \left(-\frac{d}{2}+\frac{1+p}{-1+p}\right)\to\frac{x}{2\ (p-1)}\right]\right];$$

$$\mathbf{FullSimplify}[\mathbf{PowerExpand}[\%\ /.\ x\to d+2-p\ (d-2)]]$$

$$\frac{1}{2+d-(-2+d)\ p}\ 2^{\frac{2\,d\,(-1+p)}{d-(-4+d)\,p}+\frac{2\,(d-(-2+d)\,p)}{d-(-4+d)\,p}+\frac{4\,p\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}\ d^{\frac{d\,(-1+p)}{d-(-4+d)\,p}+\frac{d\,(-1+p)\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}\ (-1+p)^{-\frac{d\,(-1+p)}{d-(-4+d)\,p}-\frac{d\,(-1+p)\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}$$
$$(1+p)^{\frac{2\,(d-(-2+d)\,p)}{d-(-4+d)\,p}+\frac{2\,(2+d-(-2+d)\,p)\,(d+2\,p-d\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}\ (2+d+2\,p-d\,p)^{\frac{(2+d-(-2+d)\,p)\,(d+4\,p-d\,p)}{(d-(-4+d)\,p)\,(2+d+2\,p-d\,p)}}\ \pi^{\frac{d\,(-1+p)}{d-(-4+d)\,p}-\frac{d\,(-1+p)\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(2+d+2\,p-d\,p)}}$$
$$\mathrm{Gamma}\left[\frac{2\,p}{-1+p}\right]^{-\frac{2\,(-1+p)}{d-(-4+d)\,p}+\frac{2\,(1-p)\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}\ \mathrm{Gamma}\left[\frac{d-(-4+d)\,p}{2\,(-1+p)}\right]^{\frac{2\,(-1+p)}{d-(-4+d)\,p}}\ \mathrm{Gamma}\left[-\frac{d}{2}+\frac{2\,p}{-1+p}\right]^{\frac{2\,(-1+p)\,(2+d-(-2+d)\,p)}{(d-(-4+d)\,p)\,(d\,(-1+p)-2\,(1+p))}}$$

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