# Multiplicity results for Fermi-Dirac statistics
# Jean Dolbeault and Robert Stanczy

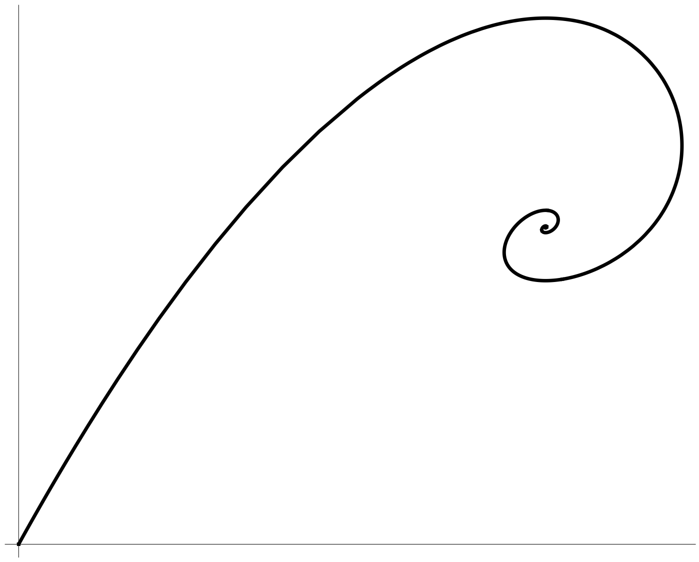
- Preliminary definitions

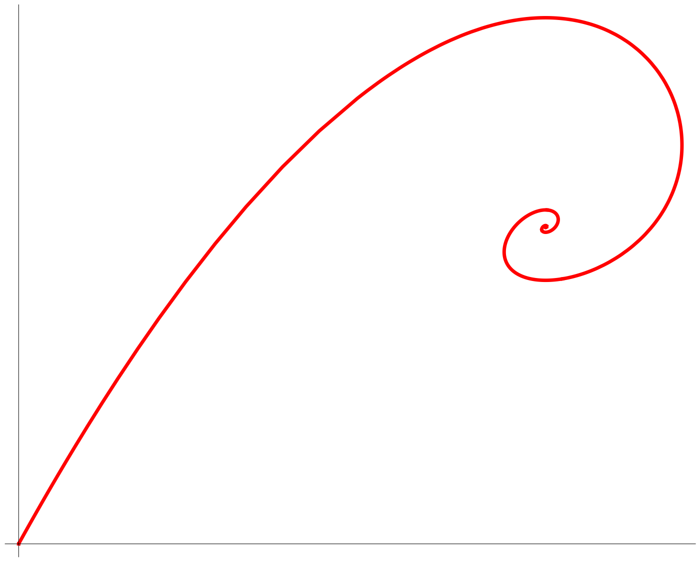
```
ϵ = 10^-6;
β[X_, η_] := X Log[X] - X + 9/10 η X^(5/3)
R[x_, η_] := 1/(1/x + η x^(-1/3))
```

## Figure 1: Maxwell-Boltzmann distribution, computation of the heteroclinic orbit

```
F[T_, Tmax_, d_, Col_] := ParametricPlot[Evaluate[
   {w[s], v[s]} /. NDSolve[{v'[s] == (2 - w[s]) v[s], w'[s] == (2 - d) w[s] + v[s],
      v[-T] == Exp[-T], w[-T] == Exp[-T] / d}, {v, w}, {s, -T, Tmax}]],
  {s, -T, Tmax}, PlotRange → All, Ticks → None, AspectRatio → 0.8,
  PlotPoints → 200, PlotStyle → {Thickness[0.005], Col}]
```

```
F[10, 10, 3, Black]
F[10, 10, 3, Red]
```

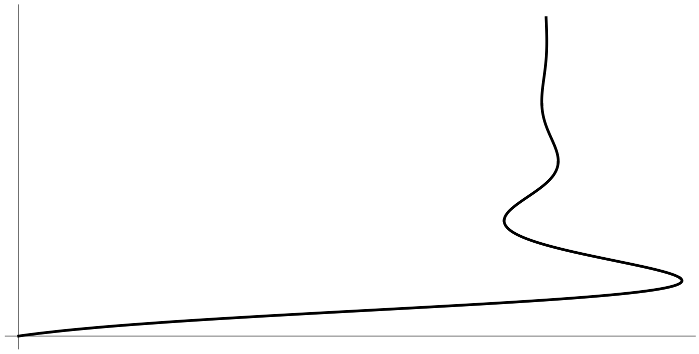




```
BifurcationMaxwDiag[pmin_, pmax_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[
   {q[0], Log[1 + p0]} /.NDSolve[{p′[s] == - p[s] ⅇ^(2 s) q[s], q′[s] == p[s] - 3 q[s],
      p[t0] == p0, q[t0] == p0/3}, {p, q}, {s, t0, 0}],
   {p0, pmin, pmax}, PlotPoints → 100, PlotStyle → PS, PlotRange → All]]
```

```
Show[BifurcationMaxwDiag[0.0001, 100, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxwDiag[100, 10 000, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxwDiag[10 000, 1 000 000, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxwDiag[1 000 000, 100 000 000,
  {Thickness[0.005], Red, Opacity[0.75]}], BifurcationMaxwDiag[
  100 000 000, 100 000 000 000, {Thickness[0.005], Red, Opacity[0.75]}],
 PlotRange → All, AxesOrigin → {0, 0}, AspectRatio → 0.5, Ticks → None]
```

```
Show[BifurcationMaxwDiag[0.0001, 100, {Thickness[0.004], Black}],
 BifurcationMaxwDiag[100, 10 000, {Thickness[0.004], Black}],
 BifurcationMaxwDiag[10 000, 1 000 000, {Thickness[0.004], Black}],
 BifurcationMaxwDiag[1 000 000, 100 000 000, {Thickness[0.004], Black}],
 BifurcationMaxwDiag[100 000 000, 100 000 000 000, {Thickness[0.004], Black}],
 PlotRange → All, AxesOrigin → {0, 0}, AspectRatio → 0.5, Ticks → None]
```

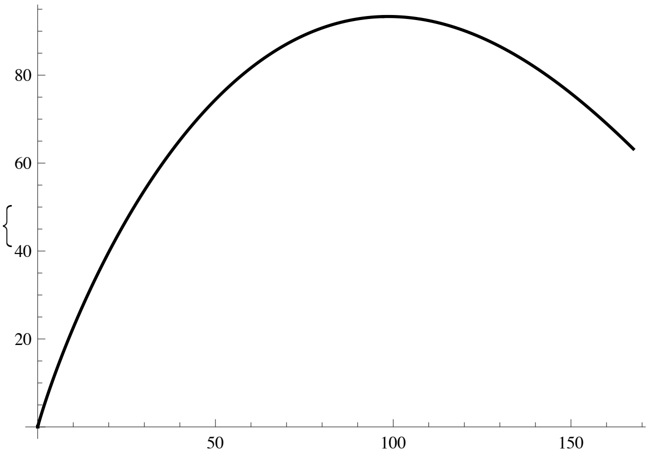


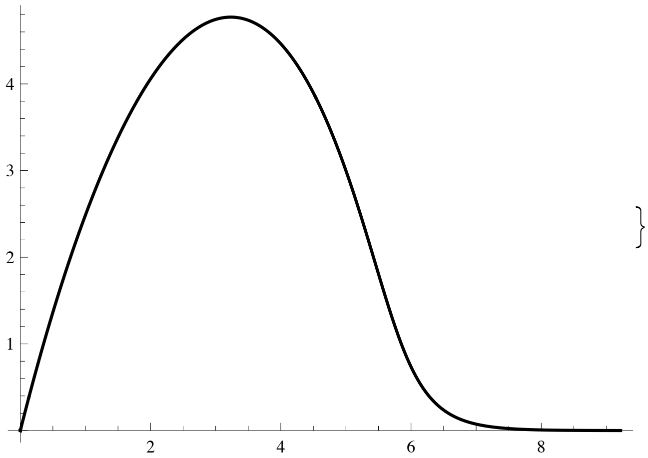
# Figure 2: simplified Fermi-Dirac distribution, computation of the orbits

- The non-autonomous dynamical system: phase diagrams

```
fPlotxy[η_, t_, eps_, d_] := {x[t], y[t]} /. NDSolve[
   {x'[s] == -(d-2) x[s] + y[s], y'[s] == 2 y[s] - R[ⅇ^(-2 (s-t)) y[s], η] ⅇ^(2 (s-t)) x[s],
    x[0] == eps, y[0] == d eps}, {x, y}, {s, 0, t}]

FPhase[η_, tmax_, eps_, d_, PR_, PP_, PS_] :=
 ParametricPlot[fPlotxy[η, t, eps, d], {t, 0, tmax},
  PlotRange → PR, PlotPoints → PP, AspectRatio → 0.7, PlotStyle → PS]
```

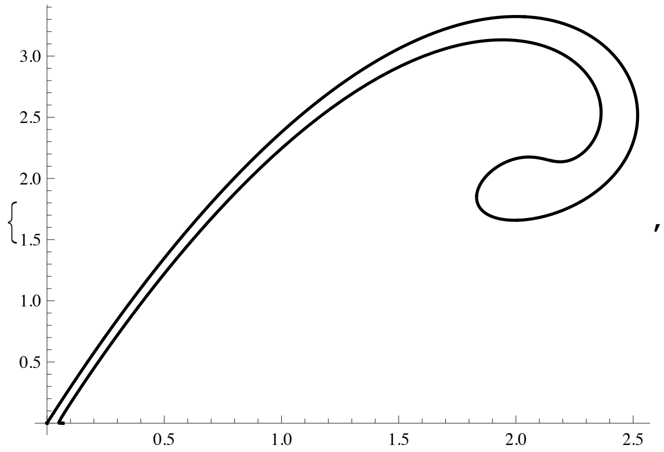
```
{FP0 = FPhase[0.8, 17, 10^-12, 3, All, $DisplayFunction,
   2000, {Thickness[0.005], Black}], FP1 = FPhase[0.1, 17, 10^-12,
   3, All, $DisplayFunction, 2000, {Thickness[0.005], Black}]}
```

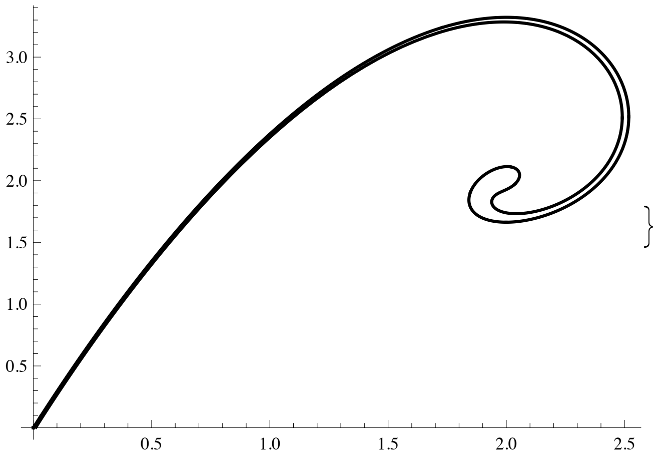
{  ,

 }

```
FP2 = FPhase[0.05, 17, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];

FP3 = FPhase[0.03, 19, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];

FP4 = FPhase[0.01, 19, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];

FP5 = FPhase[0.002, 20.5, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];

FP6 = FPhase[0.001, 20.5, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];

FP7 = FPhase[0.0005, 22, 10^-12, 3, All, 3000, {Thickness[0.005], Black}];
```

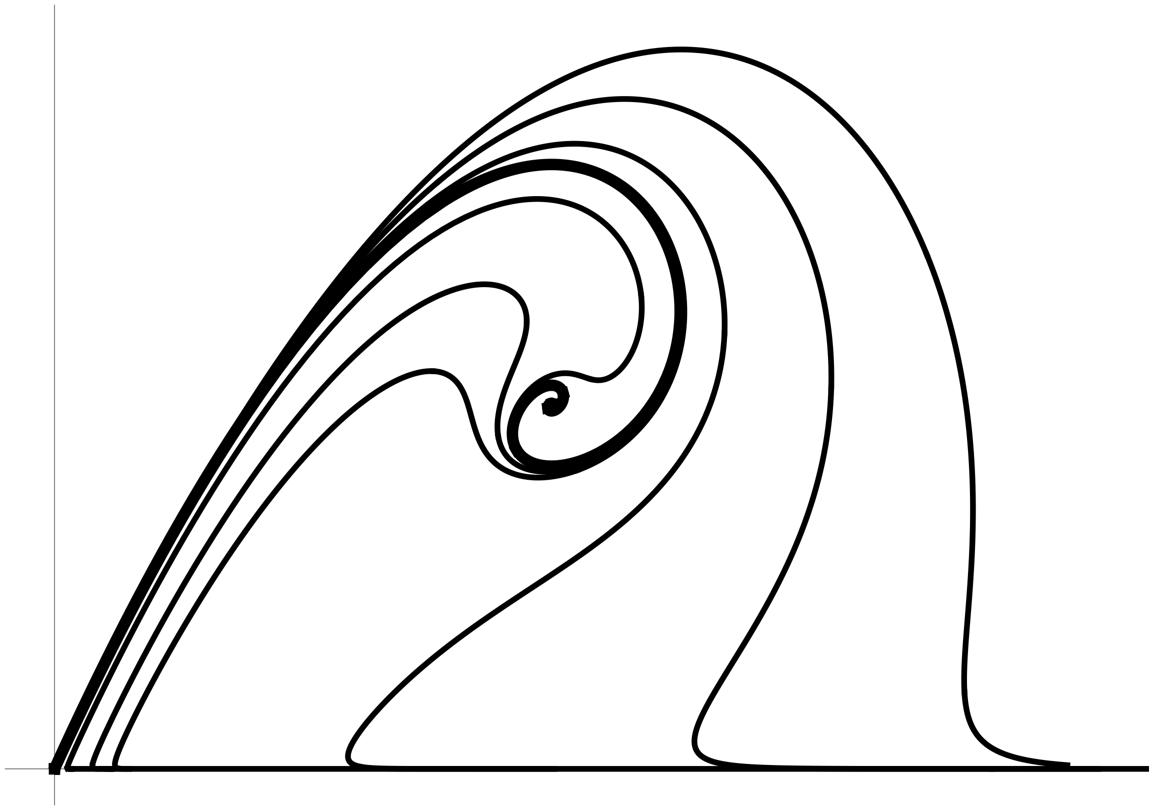
```
{FP8 = FPhase[0.0001, 24, 10^-12, 3, All, 3000, {Thickness[0.005], Black}],
 FP9 = FPhase[0.00001, 24, 10^-12, 3, All, 5000, {Thickness[0.005], Black}]}
```



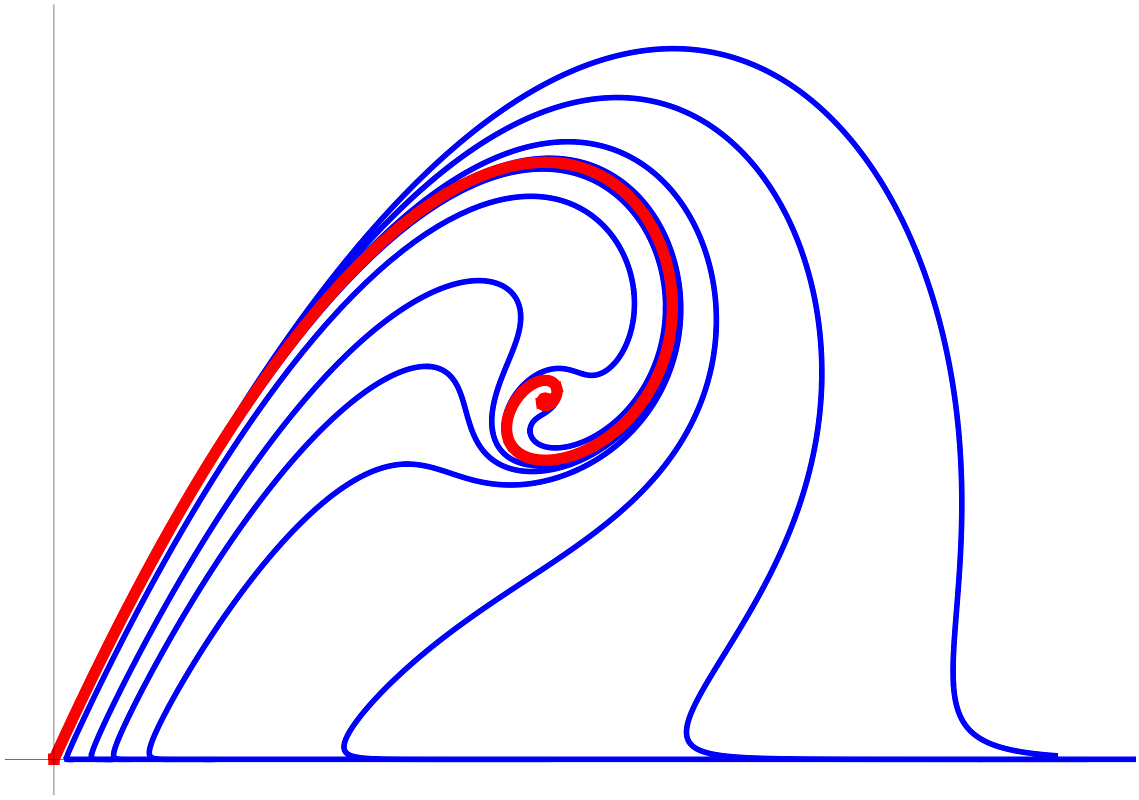

```
FPlim = FPhase[0, 50, 10^-12, 3, All, 1000, {Thickness[0.01], Black}];
```

```
Show[Show[FP2, FP3, FP4, FP5, FP6, FP7, FPlim,
  ListLinePlot[{{0.09, 0}, {4.2, 0}}, PlotStyle → {Thickness[0.005], Black}],
  PlotRange → {{0, 4.2}, All}, Ticks → None],
 PlotRange → {{-0.2, 4.4}, {-0.2, 4.2}}]
```



```
FP2col = FPhase[0.05, 17, 10^-12, 3, All, 2000, {Thickness[0.005], Blue}];

FP3col = FPhase[0.03, 19, 10^-12, 3, All, 2000, {Thickness[0.005], Blue}];

FP4col = FPhase[0.01, 19, 10^-12, 3, All, 2000, {Thickness[0.005], Blue}];

FP5col = FPhase[0.002, 20.5, 10^-12, 3, All, 2000, {Thickness[0.005], Blue}];

FP6col = FPhase[0.001, 20.5, 10^-12, 3, All, 2000, {Thickness[0.005], Blue}];

FP7col = FPhase[0.0005, 22, 10^-12, 3, All, 3000, {Thickness[0.005], Blue}];

FP8col = FPhase[0.0001, 24, 10^-12, 3, All, 3000, {Thickness[0.005], Blue}];

FP9col = FPhase[0.00001, 24, 10^-12, 3, All, 5000, {Thickness[0.005], Blue}];

FPlimcol = FPhase[0, 50, 10^-12, 3, All, 1000, {Thickness[0.01], Red}];
```

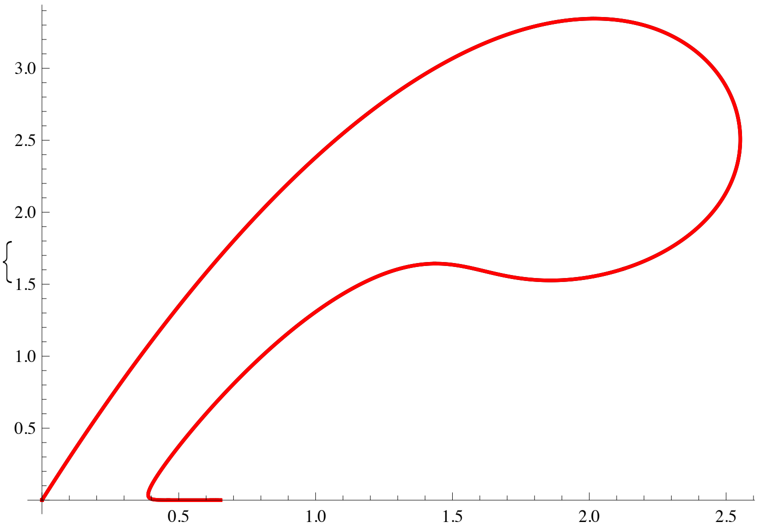
```
Show[Show[FP2col, FP3col, FP4col,
  FP5col, FP6col, FP7col, FP8col, FP9col, FPlimcol,
  ListLinePlot[{{0.09, 0}, {4.2, 0}}, PlotStyle → {Thickness[0.005], Blue}],
  PlotRange → {{0, 4.2}, All}, Ticks → None],
 PlotRange → {{-0.2, 4.4}, {-0.2, 4.2}}]
```

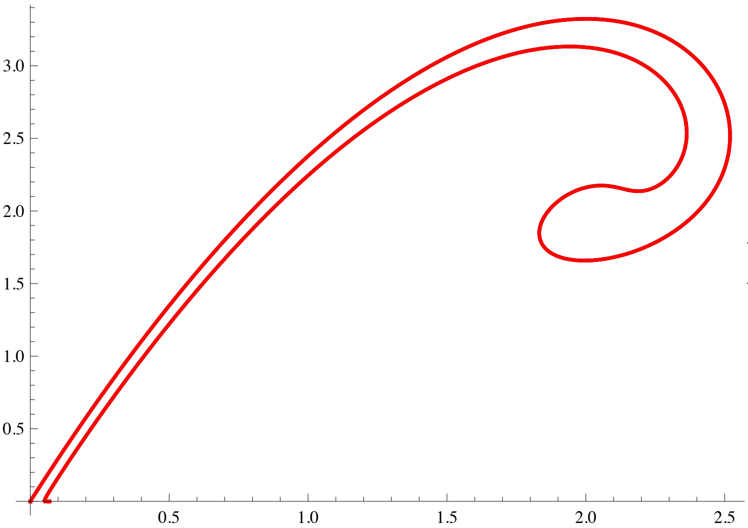


- A test for numerical stability

```
δ = Log[10] / 2;

FP5 = FPhase[0.002, 20.5, 10^-12, 3, All, 2000, {Thickness[0.005], Black}];
FP5bis = FPhase[0.002, 20.5 - δ, 10^-11, 3, All, 2000, {Thickness[0.005], Red}];

FP8 = FPhase[0.0001, 24, 10^-12, 3, All, 3000, {Thickness[0.005], Black}];
FP8bis = FPhase[0.0001, 24 - δ, 10^-11, 3, All, 3000, {Thickness[0.005], Red}];
```

```
{Show[FP5, FP5bis], Show[FP8, FP8bis]}
```

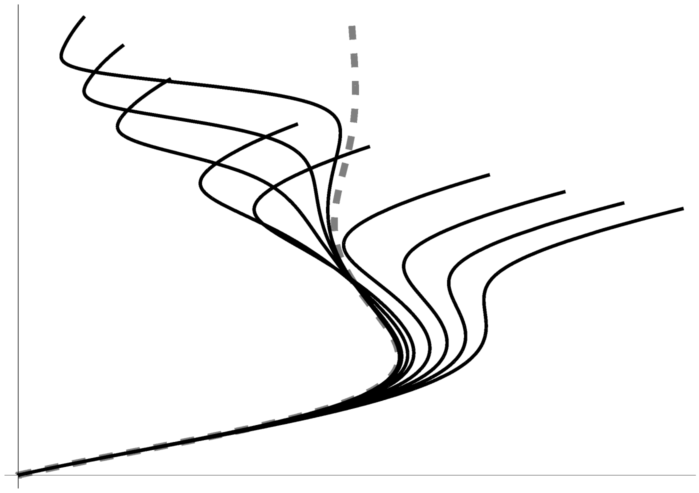




# Figure 3:

```
F[η_, tmax_, eps_, d_, PR_, PS_] :=
 ParametricPlot[{Log[1 + x[t]], Log[1 + d eps ⅇ^(2 t)]} /. NDSolve[
    {x'[s] == -(d - 2) x[s] + y[s], y'[s] == 2 y[s] - R[ⅇ^(-2 (s-t)) y[s], η] ⅇ^(2 (s-t)) x[s],
     x[0] == eps, y[0] == d eps}, {x, y}, {s, 0, t}],
  {t, 0, tmax}, PlotRange → PR, AspectRatio → 0.7, PlotStyle → PS]

DP1 = F[0.05, 18, 10^-12, 3, All, {Thickness[0.005], Black}];

DP0 = F[0.04, 18.1, 10^-12, 3, All, {Thickness[0.005], Black}];

DP00 = F[0.03, 18.3, 10^-12, 3, All, {Thickness[0.005], Black}];

DP10 = F[0.02, 18.6, 10^-12, 3, All, {Thickness[0.005], Black}];

DP2 = F[0.01, 19.1, 10^-12, 3, All, {Thickness[0.005], Black}];
```

```
DP20 = F[0.006, 19.5, 10^-12, 3, All, {Thickness[0.005], Black}];

DP3 = F[0.002, 20.3, 10^-12, 3, All, {Thickness[0.005], Black}];

DP4 = F[0.001, 20.9, 10^-12, 3, All, {Thickness[0.005], Black}];

DP5 = F[0.0005, 21.4, 10^-12, 3, All, {Thickness[0.005], Black}];

DPlim =
  F[0, 21.4, 10^-12, 3, All, {Thickness[0.01], Opacity[0.5], Black, Dashing[0.02]}];

Show[DPlim, DP00, DP0, DP1, DP10, DP2, DP20, DP3, DP4, DP5, Ticks → None]
```



```
DP1col = F[0.05, 18, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP0col = F[0.04, 18.1, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP00col = F[0.03, 18.3, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP10col = F[0.02, 18.6, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP2col = F[0.01, 19.1, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP20col = F[0.006, 19.5, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP3col = F[0.002, 20.3, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP4col = F[0.001, 20.9, 10^-12, 3, All, {Thickness[0.005], Blue}];

DP5col = F[0.0005, 21.4, 10^-12, 3, All, {Thickness[0.005], Blue}];

DPlimcol = F[0, 21.4, 10^-12, 3, All, {Thickness[0.01], Opacity[0.8], Red}];
```

```
Show[DPlimcol, DP00col, DP0col, DP1col, DP10col,
 DP2col, DP20col, DP3col, DP4col, DP5col, Ticks → None]
```

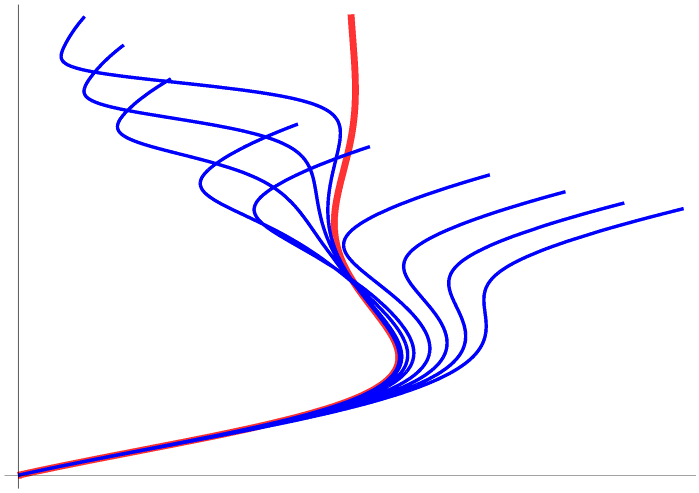
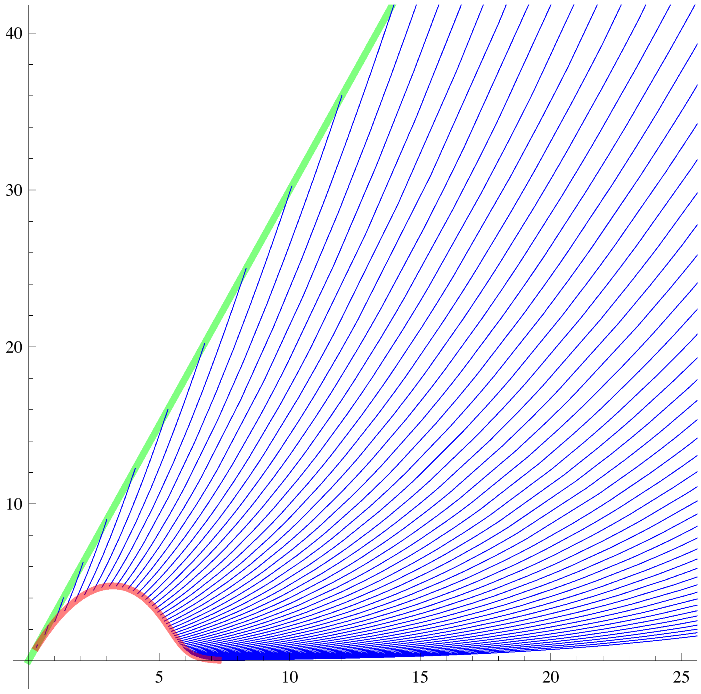


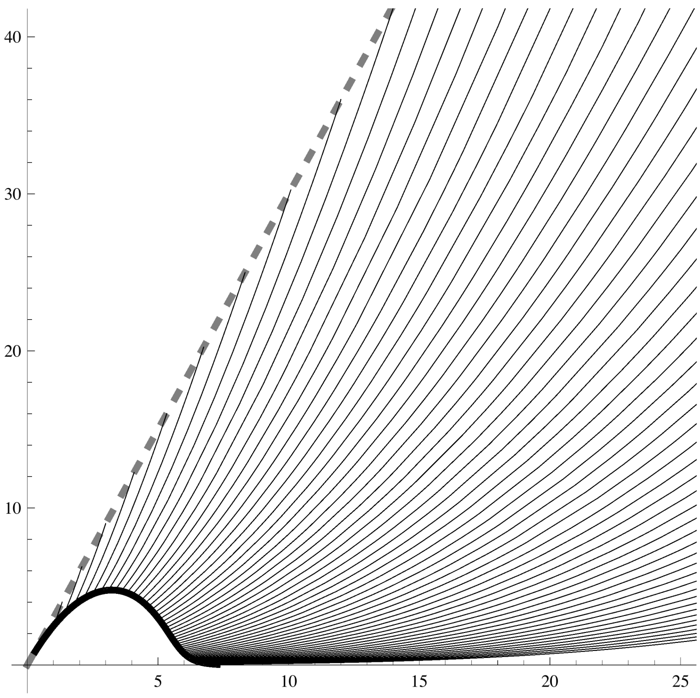
Figure 4:

```
Fphase[p0_, η_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{q[t], p[t]} /.NDSolve[
     {p'[s] == -R[p[s], η] e^(2 s) q[s], q'[s] == p[s] - 3 q[s], p[t0] == p0, q[t0] == p0/3},
     {p, q}, {s, t0, 0}], {t, t0, 0}, PlotStyle → PS]]
FphaseFinal[η_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{q[0], p[0]} /.NDSolve[
     {p'[s] == -R[p[s], η] e^(2 s) q[s], q'[s] == p[s] - 3 q[s], p[t0] == p0, q[t0] == p0/3},
     {p, q}, {s, t0, 0}], {p0, 1, 1000}, PlotStyle → PS, PlotRange → All]]
```

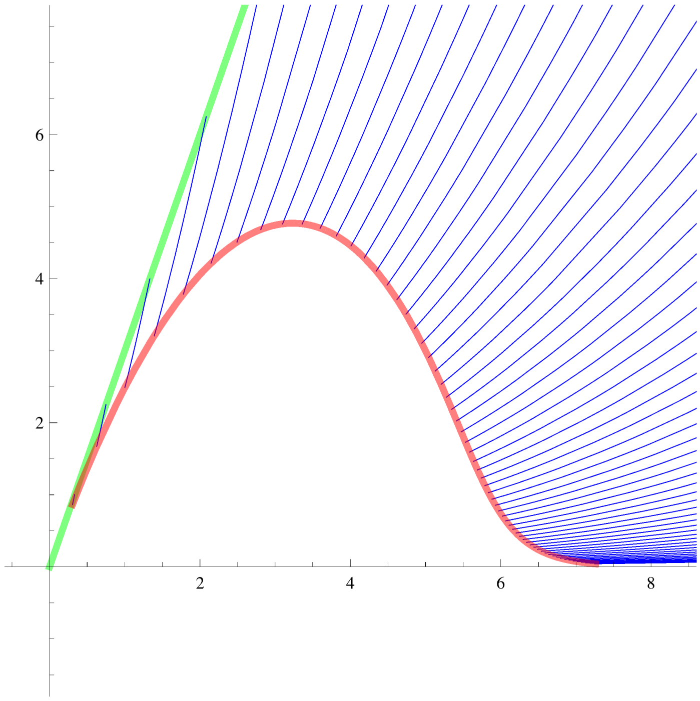
```
Show[ListLinePlot[{{0, 0}, {30, 90}},
  PlotStyle → {Thickness[0.01], Green, Opacity[0.5]}],
 Table[Fphase[p1², 0.1, Blue], {p1, 1, √1000 , 0.5}],
 FphaseFinal[0.1, {Thickness[0.01], Red, Opacity[0.5]}],
 PlotRange → {{0, 25}, {0, 40}}, AspectRatio → 1]
```



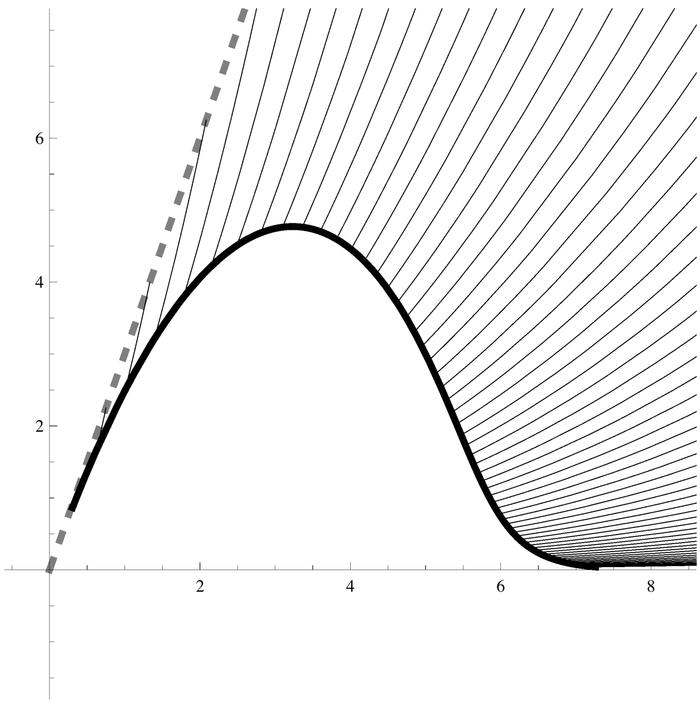
```
Show[ListLinePlot[{{0, 0}, {30, 90}},
  PlotStyle → {Thickness[0.01], Opacity[0.5], Black, Dashing[0.02]}],
 Table[Fphase[p1^2, 0.1, Black], {p1, 1, √1000 , 0.5}],
 FphaseFinal[0.1, {Thickness[0.01], Black}],
 PlotRange → {{0, 25}, {0, 40}}, AspectRatio → 1]
```



```
Show[ListLinePlot[{{0, 0}, {30, 90}},
   PlotStyle → {Thickness[0.01], Green, Opacity[0.5]}],
  Table[Fphase[p1², 0.1, Blue], {p1, 1, √1000 , 0.5}],
  FphaseFinal[0.1, {Thickness[0.01], Red, Opacity[0.5]}],
  PlotRange → {{0, 8}, {0, 6}}, AspectRatio → 1]
```



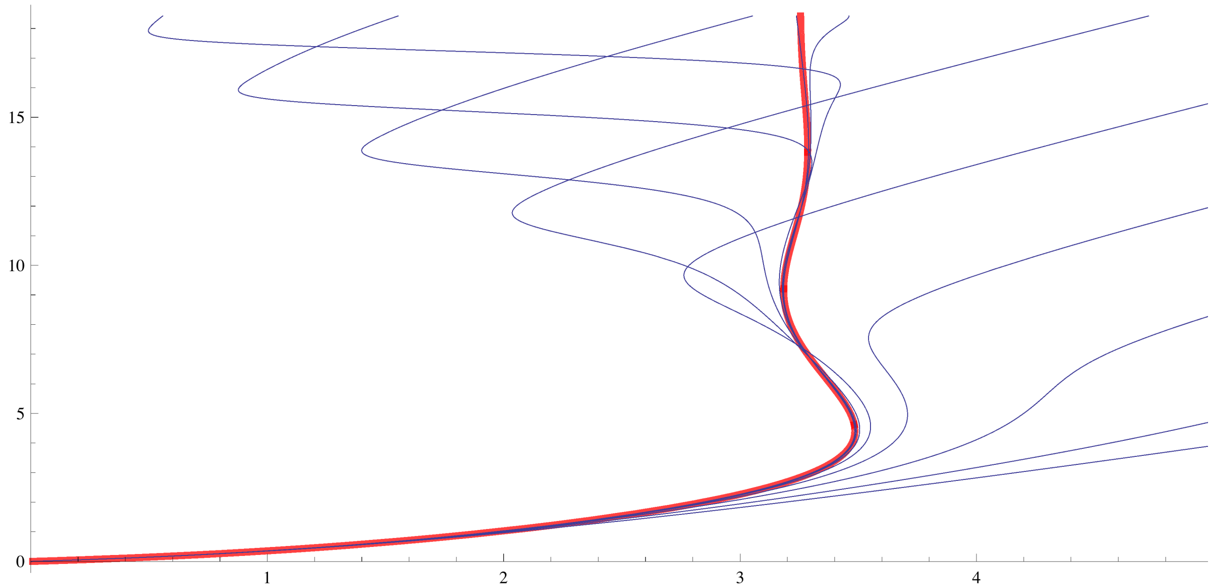
```
Show[ListLinePlot[{{0, 0}, {30, 90}},
   PlotStyle → {Thickness[0.01], Opacity[0.5], Black, Dashing[0.02]}],
  Table[Fphase[p1^2, 0.1, Black], {p1, 1, √1000 , 0.5}],
  FphaseFinal[0.1, {Thickness[0.01], Black}],
  PlotRange → {{0, 8}, {0, 6}}, AspectRatio → 1]
```



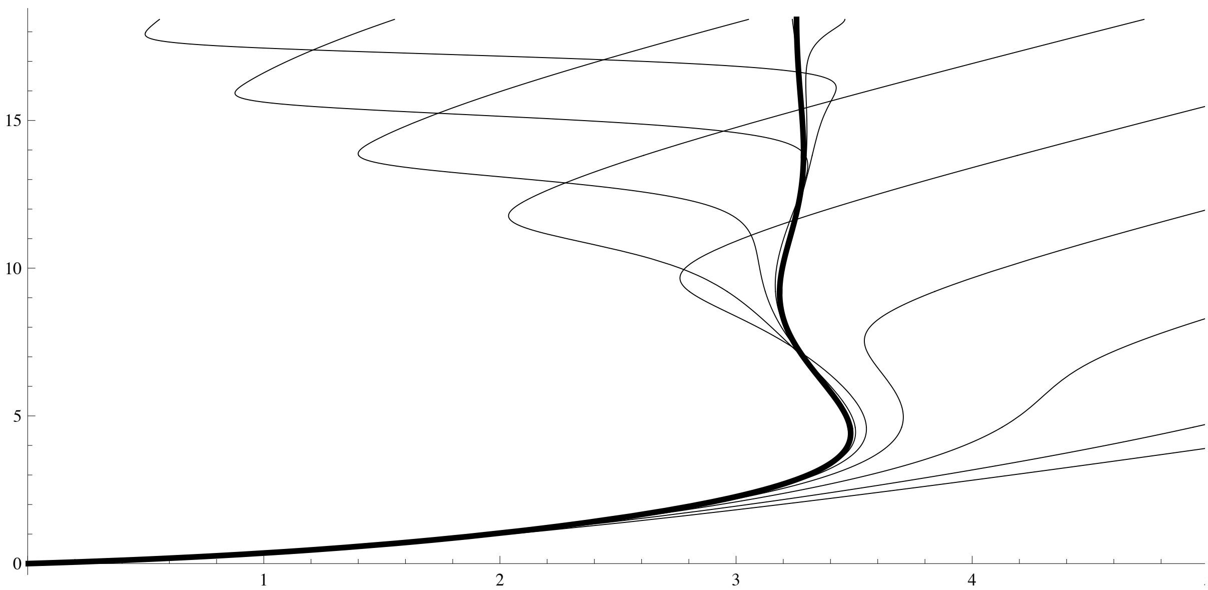

# Figure 5:

```
Bifurcation[Col_, η_, pmin_, pmax_] :=
  Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{Log[1 + 4 π q[0]], Log[1 + p0]} /.
     NDSolve[{p′[s] == -R[ p[s], η] ⅇ^(2 s) q[s], q′[s] == p[s] - 3 q[s], p[t0] == p0,
        q[t0] == p0/3}, {p, q}, {s, t0, 0}, Method → "ExplicitRungeKutta"],
    {p0, pmin, pmax}, PlotPoints → 100, PlotStyle → Col]]
BifurcationMaxw[pmin_, pmax_, PS_] := Module[{t0 = 1/2 Log[ϵ/p0]},
   ParametricPlot[{Log[1 + 4 π q[0]], Log[1 + p0]} /.NDSolve[{p′[s] == - p[s] ⅇ^(2 s) q[s],
        q′[s] == p[s] - 3 q[s], p[t0] == p0, q[t0] == p0/3}, {p, q}, {s, t0, 0}],
    {p0, pmin, pmax}, PlotPoints → 100, PlotStyle → PS, PlotRange → All]]
```

```
Show[BifurcationMaxw[0.0001, 100, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxw[100, 10000, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxw[10000, 1000000, {Thickness[0.005], Red, Opacity[0.75]}],
 BifurcationMaxw[1000000, 100000000, {Thickness[0.005], Red, Opacity[0.75]}],
 Table[Bifurcation[Blue, 10^k, 0.0001, 100], {k, -5, 0, 0.5}],
 Table[Bifurcation[Blue, 10^k, 100, 10000], {k, -5, 0, 0.5}],
 Table[Bifurcation[Blue, 10^k, 10000, 1000000], {k, -5, 0, 0.5}],
 Table[Bifurcation[Blue, 10^k, 1000000, 100000000], {k, -5, 0, 0.5}],
 PlotRange → {{0, 6}, All}, AxesOrigin → {0, 0}, AspectRatio → 0.4]
```



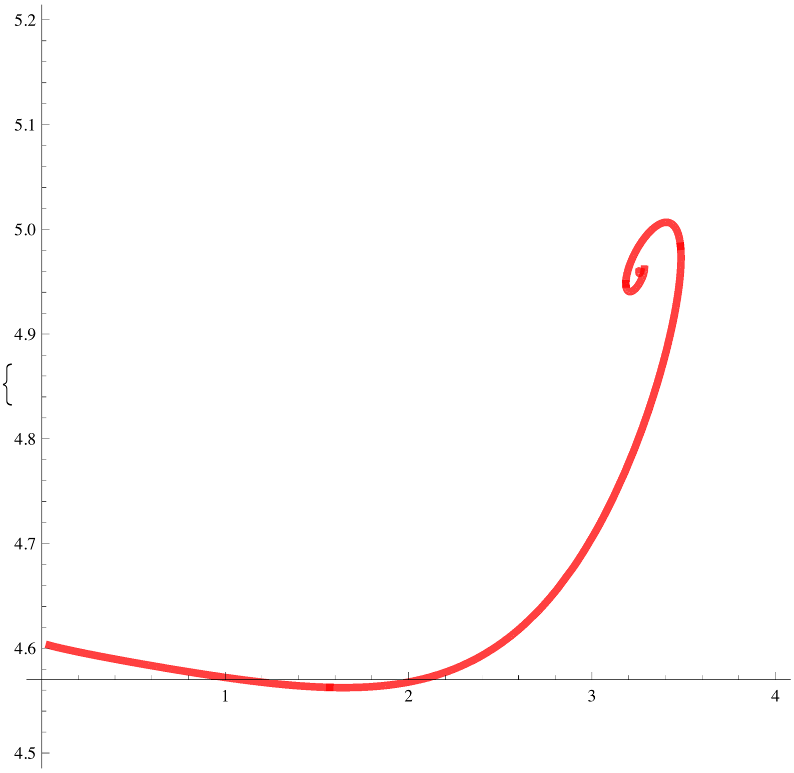

```
Show[BifurcationMaxw[0.0001, 100, {Thickness[0.004], Black}],
 BifurcationMaxw[100, 10000, {Thickness[0.004], Black}],
 BifurcationMaxw[10000, 1000000, {Thickness[0.004], Black}],
 BifurcationMaxw[1000000, 100000000, {Thickness[0.004], Black}],
 Table[Bifurcation[Black, 10^k, 0.0001, 100], {k, -5, 0, 0.5}],
 Table[Bifurcation[Black, 10^k, 100, 10000], {k, -5, 0, 0.5}],
 Table[Bifurcation[Black, 10^k, 10000, 1000000], {k, -5, 0, 0.5}],
 Table[Bifurcation[Black, 10^k, 1000000, 100000000], {k, -5, 0, 0.5}],
 PlotRange → {{0, 6}, All}, AxesOrigin → {0, 0}, AspectRatio → 0.4]
```



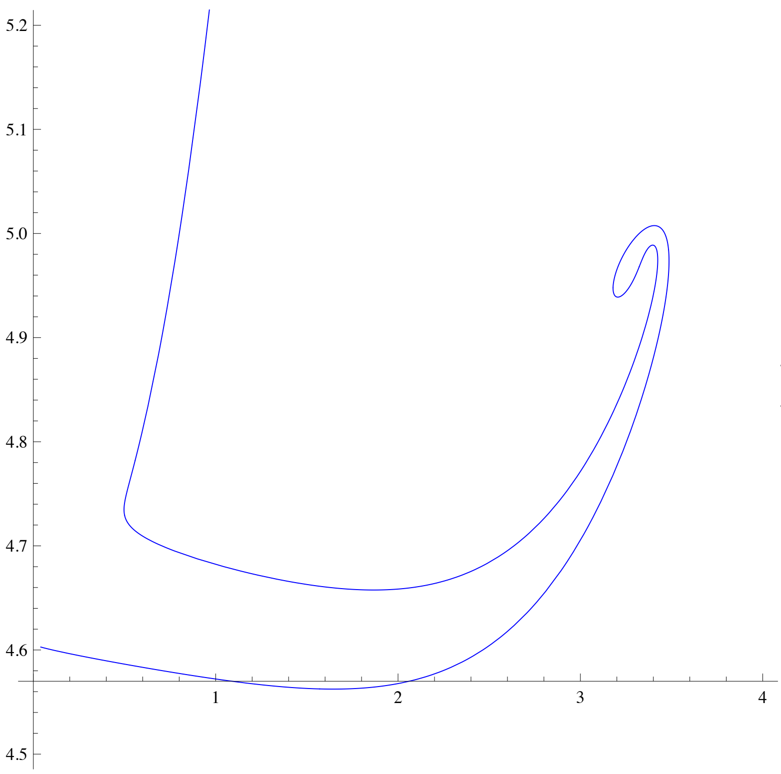

## Entropy

```
Fentropy[η_, pmin_, pmax_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{Log[1 + 4 π q[0]], Log[100 + 4 π z[0]]} /.
    NDSolve[{p′[s] == -R[p[s], η] ⅇ^(2 s) q[s], q′[s] == p[s] - 3 q[s],
       z′[s] == ⅇ^(3 s) β[p[s], η], p[t0] == p0, q[t0] == p0/3, z[t0] == 0},
      {p, q, z}, {s, t0, 0}, Method → "ExplicitRungeKutta"],
    {p0, pmin, pmax}, AspectRatio → 1, PlotStyle → PS]]

{Show[Fentropy[0, 0.01, 1, {Thickness[0.01], Red, Opacity[0.75]}],
  Fentropy[0, 1, 100, {Thickness[0.01], Red, Opacity[0.75]}],
  Fentropy[0, 100, 10 000, {Thickness[0.01], Red, Opacity[0.75]}],
  Fentropy[0, 10 000, 1 000 000, {Thickness[0.01], Red, Opacity[0.75]}],
  Fentropy[0, 100 000 000, 10 000 000 000, {Thickness[0.01], Red, Opacity[0.75]}],
  PlotRange → {{0, 4}, {4.5, 5.2}}],
 Show[Fentropy[0.0001, 0.01, 1, Blue], Fentropy[0.0001, 1, 100, Blue],
  Fentropy[0.0001, 100, 10 000, Blue], Fentropy[0.0001, 10 000, 1 000 000, Blue],
  Fentropy[0.0001, 1 000 000, 100 000 000, Blue], Fentropy[0.0001,
   100 000 000, 10 000 000 000, Blue], PlotRange → {{0, 4}, {4.5, 5.2}}]}
```

{

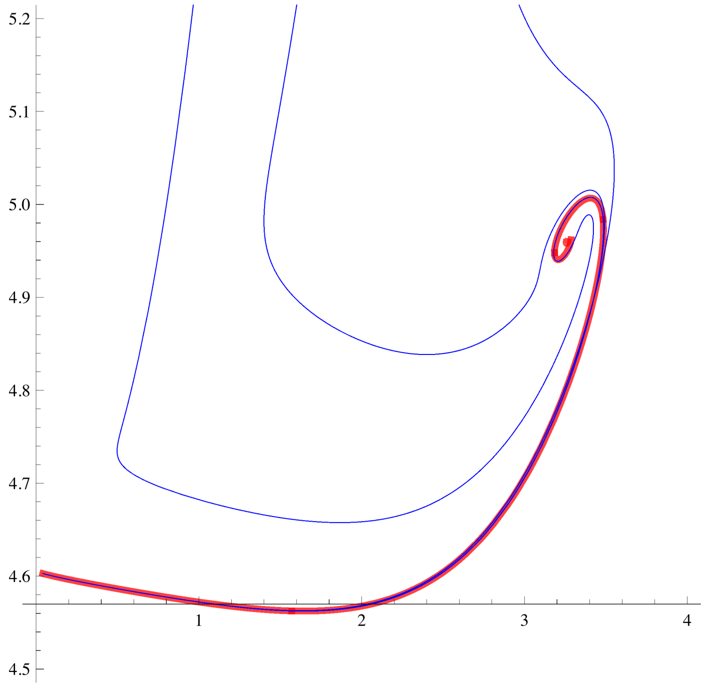

5.2
5.1
5.0
4.9
4.8
4.7
4.6
4.5
1
2
3
4

,

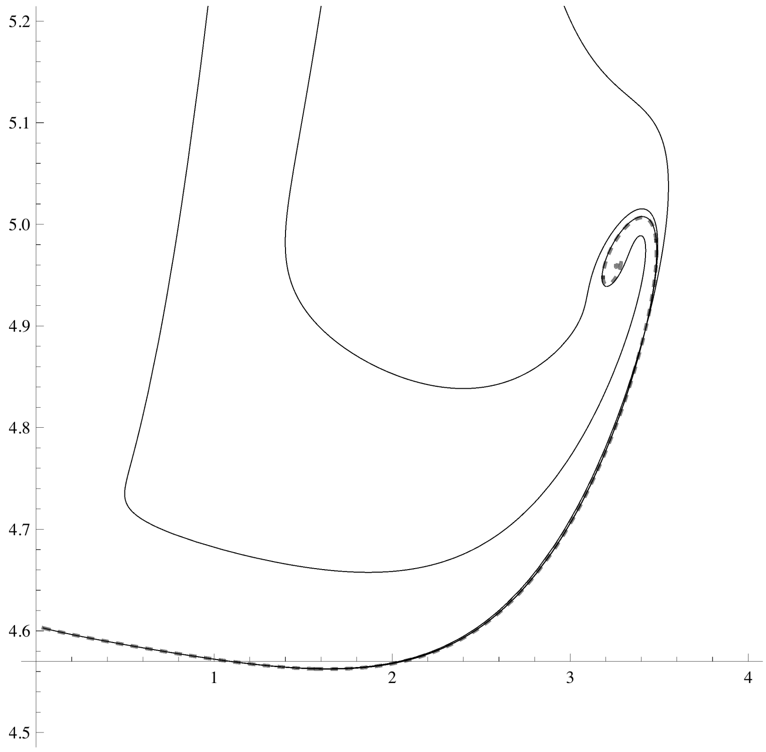

5.2
5.1
5.0
4.9
4.8
4.7
4.6
4.5
1
2
3
4

}

```
Show[Fentropy[0, 0.01, 1, {Thickness[0.01], Red, Opacity[0.75]}],
 Fentropy[0, 1, 100, {Thickness[0.01], Red, Opacity[0.75]}],
 Fentropy[0, 100, 10 000, {Thickness[0.01], Red, Opacity[0.75]}],
 Fentropy[0, 10 000, 1 000 000, {Thickness[0.01], Red, Opacity[0.75]}],
 Fentropy[0, 100 000 000, 10 000 000 000, {Thickness[0.01], Red, Opacity[0.75]}],
 Fentropy[0.0001, 0.01, 1, Blue], Fentropy[0.0001, 1, 100, Blue],
 Fentropy[0.0001, 100, 10 000, Blue], Fentropy[0.0001, 10 000, 1 000 000, Blue],
 Fentropy[0.0001, 1 000 000, 100 000 000, Blue],
 Fentropy[0.0001, 100 000 000, 10 000 000 000, Blue],
 Fentropy[0.001, 0.01, 1, Blue], Fentropy[0.001, 1, 100, Blue],
 Fentropy[0.001, 100, 10 000, Blue], Fentropy[0.001, 10 000, 1 000 000, Blue],
 Fentropy[0.001, 1 000 000, 100 000 000, Blue],
 Fentropy[0.001, 100 000 000, 10 000 000 000, Blue], Fentropy[0.01, 0.01, 1, Blue],
 Fentropy[0.01, 1, 100, Blue], Fentropy[0.01, 100, 10 000, Blue],
 Fentropy[0.01, 10 000, 1 000 000, Blue], Fentropy[0.01, 1 000 000, 100 000 000, Blue],
 Fentropy[0.01, 100 000 000, 10 000 000 000, Blue], PlotRange → {{0, 4}, {4.5, 5.2}}]
```



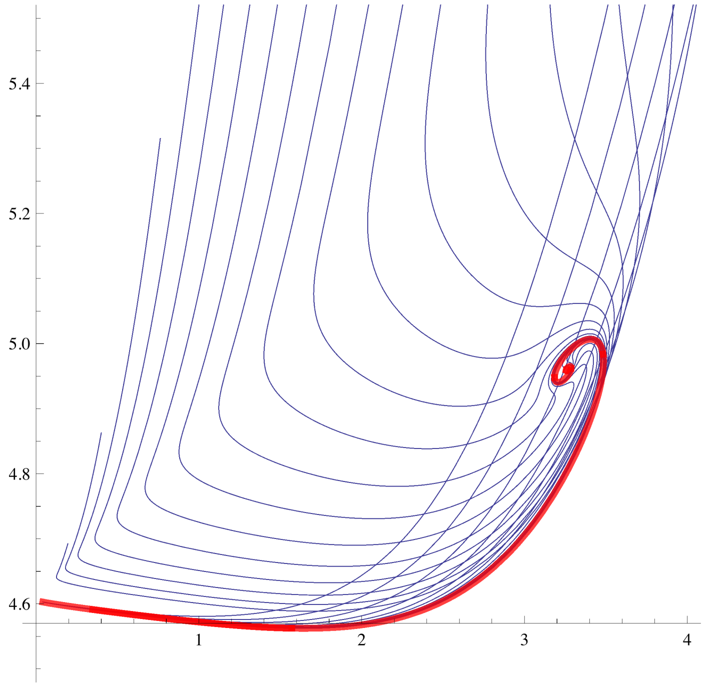
```
Show[Fentropy[0, 0.01, 1,
   {Thickness[0.005], Opacity[0.5], Black, Dashing[{0.01, 0.01}]}], Fentropy[0, 1,
   100, {Thickness[0.005], Opacity[0.5], Black, Dashing[{0.01, 0.01}]}], Fentropy[
   0, 100, 10 000, {Thickness[0.005], Opacity[0.5], Black, Dashing[{0.01, 0.01}]}],
 Fentropy[0, 10 000, 1 000 000, {Thickness[0.005], Opacity[0.5], Black,
    Dashing[{0.01, 0.01}]}], Fentropy[0, 100 000 000, 10 000 000 000,
   {Thickness[0.005], Opacity[0.5], Black, Dashing[{0.01, 0.01}]}],
 Fentropy[0.0001, 0.01, 1, Black], Fentropy[0.0001, 1, 100, Black],
 Fentropy[0.0001, 100, 10 000, Black], Fentropy[0.0001, 10 000, 1 000 000, Black],
 Fentropy[0.0001, 1 000 000, 100 000 000, Black],
 Fentropy[0.0001, 100 000 000, 10 000 000 000, Black],
 Fentropy[0.001, 0.01, 1, Black], Fentropy[0.001, 1, 100, Black],
 Fentropy[0.001, 100, 10 000, Black], Fentropy[0.001, 10 000, 1 000 000, Black],
 Fentropy[0.001, 1 000 000, 100 000 000, Black],
 Fentropy[0.001, 100 000 000, 10 000 000 000, Black],
 Fentropy[0.01, 0.01, 1, Black], Fentropy[0.01, 1, 100, Black],
 Fentropy[0.01, 100, 10 000, Black], Fentropy[0.01, 10 000, 1 000 000, Black],
 Fentropy[0.01, 1 000 000, 100 000 000, Black],
 Fentropy[0.01, 100 000 000, 10 000 000 000, Black], PlotRange → {{0, 4}, {4.5, 5.2}}]
```



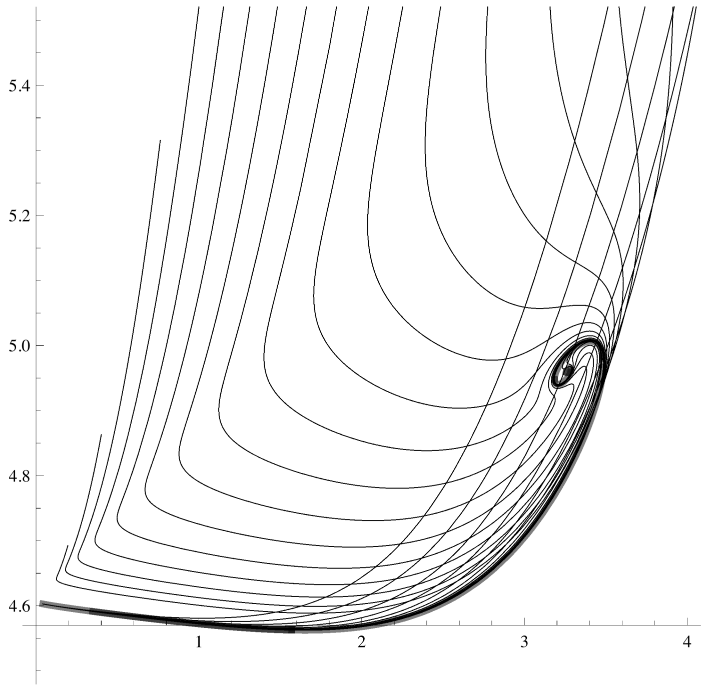

```
Fentropy[η_, pmin_, pmax_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{Log[1 + 4 π q[0]], Log[100 + 4 π z[0]]} /.
     NDSolve[{p′[s] == -R[p[s], η] e^(2 s) q[s], q′[s] == p[s] - 3 q[s],
        z′[s] == e^(3 s) β[p[s], η], p[t0] == p0, q[t0] == p0/3, z[t0] == 0},
       {p, q, z}, {s, t0, 0}, Method → "ExplicitRungeKutta"],
    {p0, pmin, pmax}, AspectRatio → 1, PlotStyle → PS]]
```

```
FentropyAll[η_, PS_] := Show[Fentropy[η, 0.01, 1, PS],
  Fentropy[η, 0.1, 1, PS], Fentropy[η, 1, 100, PS], Fentropy[η, 100, 10 000, PS],
  Fentropy[η, 10 000, 1 000 000, PS], Fentropy[η, 1 000 000, 100 000 000, PS],
  Fentropy[η, 100 000 000, 10 000 000 000, PS], PlotRange → All]
```

```
Show[Table[FentropyAll[10^k, {}], {k, -5, 0, 0.25}],
 FentropyAll[0, {Thickness[0.01], Red, Opacity[0.75]}],
 PlotRange → {{0, 4}, {4.5, 5.5}}, AspectRatio → 1]
```



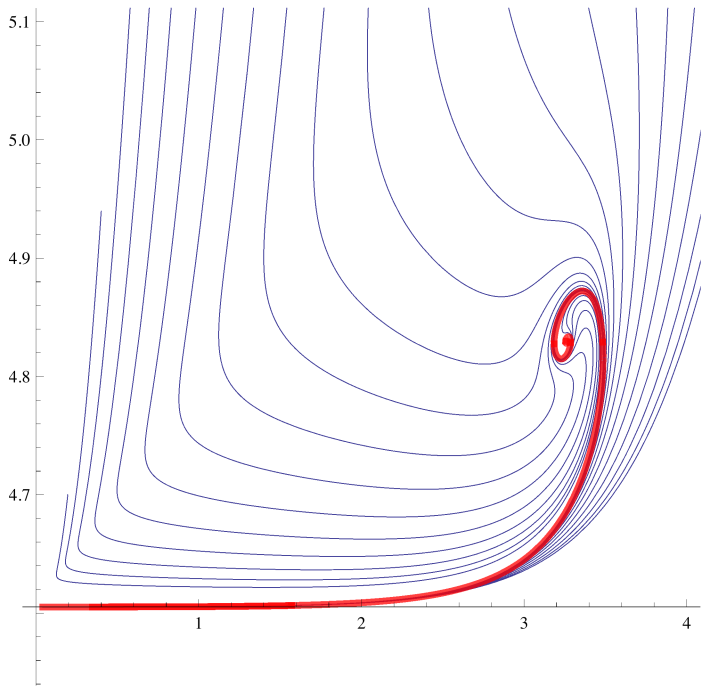

```
Show[Table[FentropyAll[10^k, Black], {k, -5, 0, 0.25}],
 FentropyAll[0, {Thickness[0.01], Opacity[0.5], Black}],
 PlotRange → {{0, 4}, {4.5, 5.5}}, AspectRatio → 1]
```



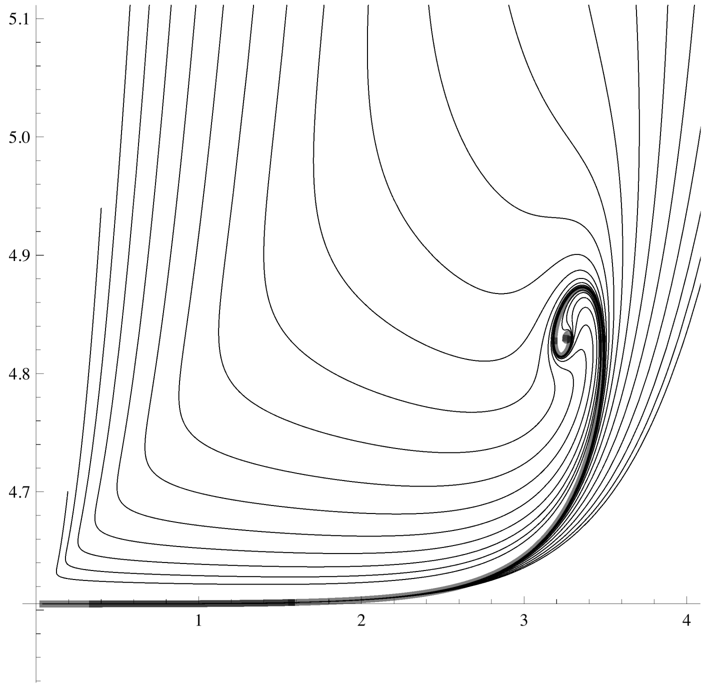
# Potential energy

## Potential energy

```
Fpotential[η_, pmin_, pmax_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{Log[1 + 4 π q[0]], Log[100 + 2 π w[0]]} /.
    NDSolve[{p′[s] == -R[p[s], η] ⅇ^(2 s) q[s], q′[s] == p[s] - 3 q[s],
      z′[s] == ⅇ^(3 s) β[p[s], η], w'[s] == ⅇ^(5 s) q[s]^2, p[t0] == p0, q[t0] == p0/3, z[t0] == 0,
      w[t0] == 0}, {p, q, z, w}, {s, t0, 0}, Method → "ExplicitRungeKutta"],
   {p0, pmin, pmax}, AspectRatio → 1, PlotStyle → PS]]

FpotentialAll[η_, PS_] :=
 Show[Fpotential[η, 0.01, 1, PS], Fpotential[η, 0.1, 1, PS],
  Fpotential[η, 1, 100, PS], Fpotential[η, 100, 10 000, PS],
  Fpotential[η, 10 000, 1 000 000, PS], Fpotential[η, 1 000 000, 100 000 000, PS],
  Fpotential[η, 100 000 000, 10 000 000 000, PS], PlotRange → All]

Show[Table[FpotentialAll[10^k, {}], {k, -5, 0, 0.25}],
 FpotentialAll[0, {Thickness[0.01], Red, Opacity[0.75]}],
 PlotRange → {{0, 4}, {4.55, 5.1}}, AspectRatio → 1]
```



```
Show[Table[FpotentialAll[10^k, Black], {k, -5, 0, 0.25}],
 FpotentialAll[0, {Thickness[0.01], Opacity[0.5], Black}],
 PlotRange → {{0, 4}, {4.55, 5.1}}, AspectRatio → 1]
```

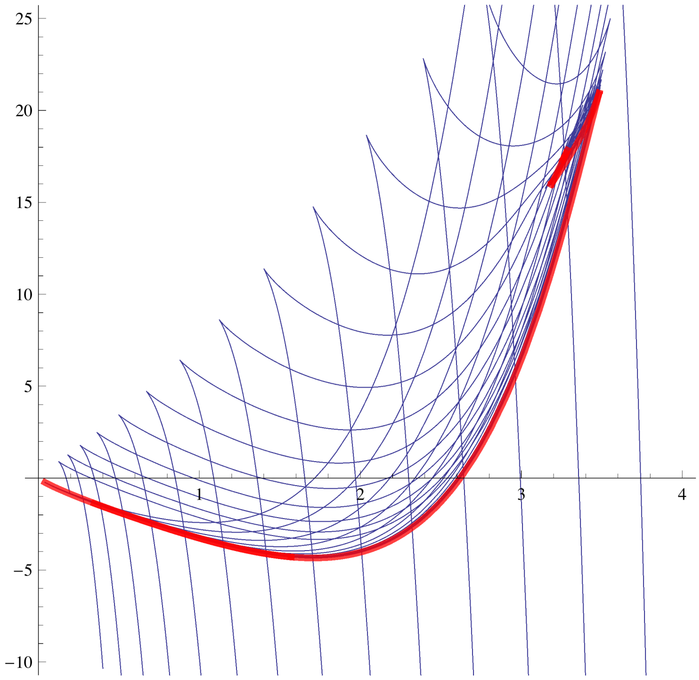



# Free energy

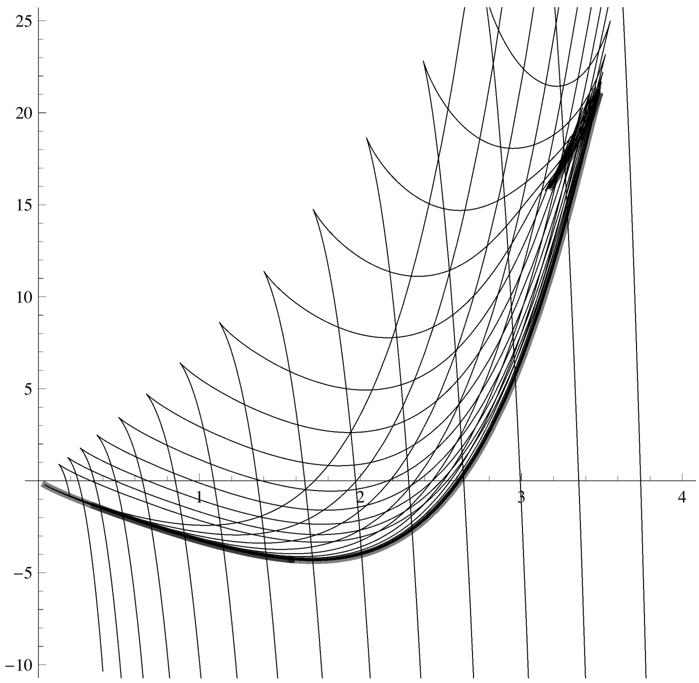
Free energy

```
Ffreeenergy[η_, pmin_, pmax_, PS_] :=
 Module[{t0 = 1/2 Log[ϵ/p0]}, ParametricPlot[{Log[1 + 4 π q[0]], 2 π (2 z[0] - w[0])} /.
    NDSolve[{p′[s] == -R[ p[s], η] ⅇ^(2 s) q[s], q′[s] == p[s] - 3 q[s],
      z′[s] == ⅇ^(3 s) β[p[s], η], w '[s] == ⅇ^(5 s) q[s]^2, p[t0] == p0, q[t0] == p0/3, z[t0] == 0,
      w[t0] == 0}, {p, q, z, w}, {s, t0, 0}, Method → "ExplicitRungeKutta"],
   {p0, pmin, pmax}, AspectRatio → 1, PlotStyle → PS]]

FfreeenergyAll[η_, PS_] :=
 Show[Ffreeenergy[η, 0.01, 1, PS], Ffreeenergy[η, 0.1, 1, PS],
  Ffreeenergy[η, 1, 100, PS], Ffreeenergy[η, 100, 10 000, PS],
  Ffreeenergy[η, 10 000, 1 000 000, PS], Ffreeenergy[η, 1 000 000, 100 000 000, PS],
  Ffreeenergy[η, 100 000 000, 10 000 000 000, PS], PlotRange → All]
```

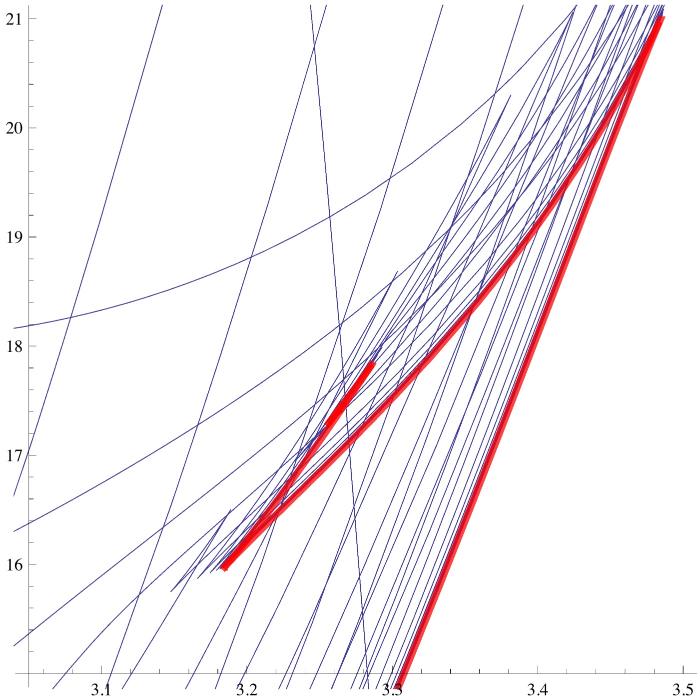
```
Show[Table[FfreeenergyAll[10^k, {}], {k, -5, 0, 0.25}],
 FfreeenergyAll[0, {Thickness[0.01], Red, Opacity[0.75]}],
 PlotRange → {{0, 4}, {-10, 25}}, AspectRatio → 1]
```




```
Show[Table[FfreeenergyAll[10^k, Black], {k, -5, 0, 0.25}],
 FfreeenergyAll[0, {Thickness[0.01], Opacity[0.5], Black}],
 PlotRange → {{0, 4}, {-10, 25}}, AspectRatio → 1]
```



```
Show[Table[FfreeenergyAll[10^k, {}], {k, -5, 0, 0.25}],
 FfreeenergyAll[0, {Thickness[0.01], Red, Opacity[0.75]}],
 PlotRange → {{3.05, 3.5}, {15, 21}}, AspectRatio → 1, AxesOrigin → {3.05, 15}]
```




```
Show[Table[FfreeenergyAll[10^k, Black], {k, -5, 0, 0.25}],
 FfreeenergyAll[0, {Thickness[0.01], Opacity[0.5], Black}],
 PlotRange → {{3.05, 3.5}, {15, 21}}, AspectRatio → 1, AxesOrigin → {3.05, 15}]
```