Improved interpolation inequalities on the sphere
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The function μ is a second order polynomial in β

```
f[β_] := ((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1)

Simplify[f[β] /. p → 1]
Simplify[f[b] /. p → 2 d/(d - 2)] /. b → β
```

$(-1+\beta)^2$

$\frac{(2-d+(-3+d)\ \beta)^2}{(-2+d)^2}$

```
Simplify[f[β] /. d → 1]
Simplify[f[β] /. d → 2]
```

$\frac{1}{3}\left(3+2\ (-4+p)\ \beta-3\ (-2+p)\ \beta^2\right)$

$\frac{1}{16}\left(16+8\ (-5+p)\ \beta+\left(33-18\ p+p^2\right)\ \beta^2\right)$

```
f[0]
Factor[f'[0] / 2]
f''[0] / 2
```

1

$-\frac{3+d-p}{2+d}$

$\frac{1}{2}\left(4+\frac{2\ (-1+d)^2\ (-1+p)^2}{(2+d)^2}-2\ p\right)$

```
a[p_, d_] := 1/2 (4 + (2 (-1 + d)^2 (-1 + p)^2)/(2 + d)^2 - 2 p)
b[p_, d_] := (3 + d - p)/(2 + d)
```

```
Simplify[a[1, d]]
Simplify[a[2 d/(d - 2), d]]
```

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

$$\frac{(-3+d)^2}{(-2+d)^2}$$

1

The discriminant

```
Simplify[b[p, d]^2 - a[p, d]]
```

$$-\frac{d\ (d\ (-2+p)-2\ p)\ (-1+p)}{(2+d)^2}$$

```
FullSimplify[(2 + d)^2 a[aa, d]] /. aa → p
```

$$3\ (3+d\ (2+d))-3\ \left(2+d^2\right)\ p+(-1+d)^2\ p^2$$

```
g[p_] := 3 (3 + d (2 + d)) - 3 (2 + d^2) p + (-1 + d)^2 p^2
```

```
MatrixForm[Simplify[Table[{d, a[p, d], b[p, d]}, {d, 1, 4}]]]
```

$$\begin{pmatrix} 1 & 2-p & \frac{4-p}{3} \\ 2 & \frac{1}{16}\left(33-18\,p+p^2\right) & \frac{5-p}{4} \\ 3 & \frac{1}{25}\left(54-33\,p+4\,p^2\right) & \frac{6-p}{5} \\ 4 & \frac{1}{4}\,(-3+p)^2 & \frac{7-p}{6} \end{pmatrix}$$

```
MatrixForm[
 Simplify[Table[{d, b[p, d] /. Simplify[Solve[g[p] == 0, p]]}, {d, 1, 4}]]]
```

$$\begin{pmatrix} 1 & \left\{\frac{2}{3}\right\} \\ 2 & \left\{-1+\sqrt{3},\ -1-\sqrt{3}\right\} \\ 3 & \left\{\frac{3}{4},\ 0\right\} \\ 4 & \left\{\frac{2}{3},\ \frac{2}{3}\right\} \end{pmatrix}$$

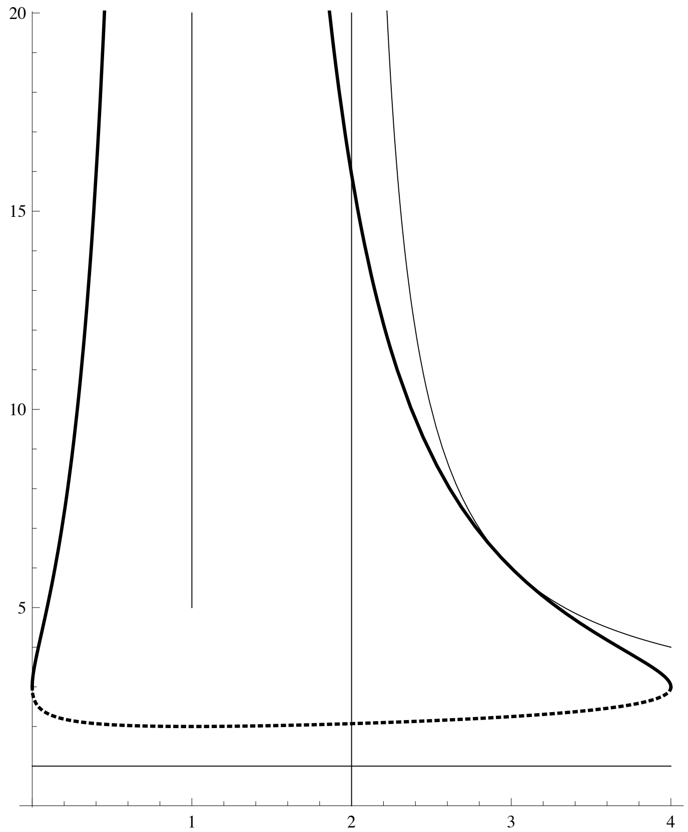
```
MatrixForm[
 Simplify[Table[{d, 1/(2 b[p, d]) /. Simplify[Solve[g[p] == 0, p]]}, {d, 1, 4}]]]
```

$$\begin{pmatrix} 1 & \left\{\frac{3}{4}\right\} \\ 2 & \left\{\frac{1}{4}\left(1+\sqrt{3}\right),\ \frac{1}{4}\left(1-\sqrt{3}\right)\right\} \\ 3 & \left\{\frac{2}{3},\ \text{ComplexInfinity}\right\} \\ 4 & \left\{\frac{3}{4},\ \frac{3}{4}\right\} \end{pmatrix}$$

The functions $p_{\pm}$

```
p /. Solve[g[p] == 0, p]
Show[ListLinePlot[{{0, 1}, {4, 1}}, PlotStyle → Black],
 ListLinePlot[{{1, 5}, {1, 20}}, PlotStyle → Black],
 ListLinePlot[{{2, 0}, {2, 20}}, PlotStyle → Black],
 Plot[2 d/(d - 2), {d, 2, 4}, PlotStyle → Black, PlotRange → {All, {0, 50}}],
 Plot[%, {d, 0, 4}, PlotStyle →
   {{Black, Dashing[Tiny], Thickness[0.005]}, {Black, Thickness[0.005]}}],
 PlotRange → {All, {0, 20}}, AxesOrigin → {0, 0}, AspectRatio → 1.2]
```

$$\left\{\frac{6+3\,d^2-\sqrt{3}\,\sqrt{16\,d+12\,d^2-d^4}}{2\,\left(1-2\,d+d^2\right)},\ \frac{6+3\,d^2+\sqrt{3}\,\sqrt{16\,d+12\,d^2-d^4}}{2\,\left(1-2\,d+d^2\right)}\right\}$$




```
psoln[d_] := {d, (6 + 3 d^2 - Sqrt[3] Sqrt[16 d + 12 d^2 - d^4])/(2 (1 - 2 d + d^2)), (6 + 3 d^2 + Sqrt[3] Sqrt[16 d + 12 d^2 - d^4])/(2 (1 - 2 d + d^2))}
MatrixForm[{psoln[0], psoln[2], psoln[3], psoln[4]}]
```

$$\begin{pmatrix} 0 & 3 & 3 \\ 2 & \frac{1}{2}\left(18-8\sqrt{3}\right) & \frac{1}{2}\left(18+8\sqrt{3}\right) \\ 3 & \frac{9}{4} & 6 \\ 4 & 3 & 3 \end{pmatrix}$$

```
Limit[(6 + 3 d^2 - Sqrt[3] Sqrt[16 d + 12 d^2 - d^4])/(2 (1 - 2 d + d^2)), d → 1]
```

2

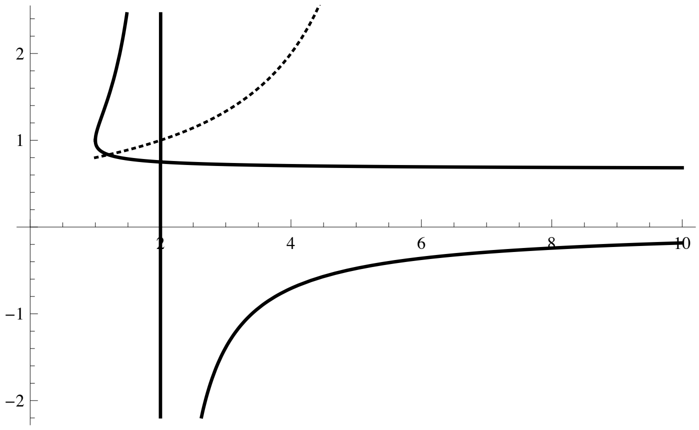
The functions $\beta_{\pm}$

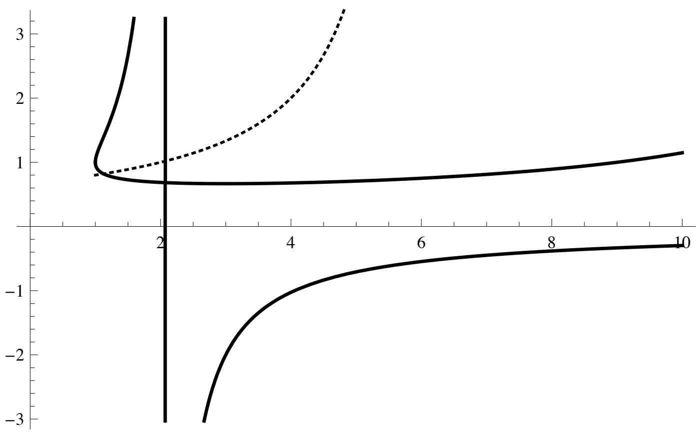
```
h[p_, d_] := FullSimplify[β /. Solve[a[p, d] β^2 - 2 b[p, d] β + 1 == 0, β]]
h[1, d]
Factor[h[2 d/(d - 2), d]]
```

{1, 1}

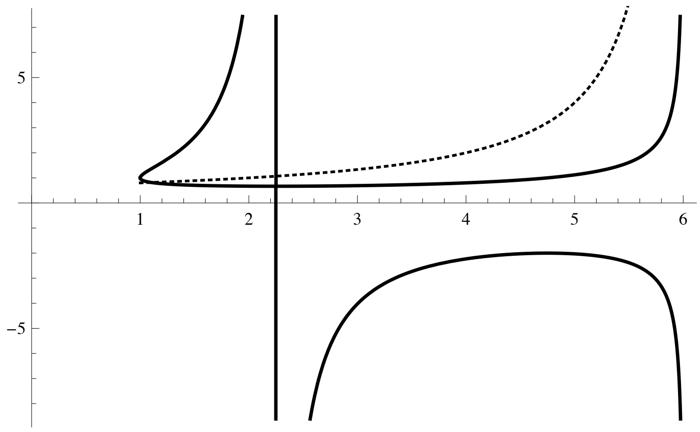
$$\left\{\frac{-2+d}{-3+d}, \frac{-2+d}{-3+d}\right\}$$

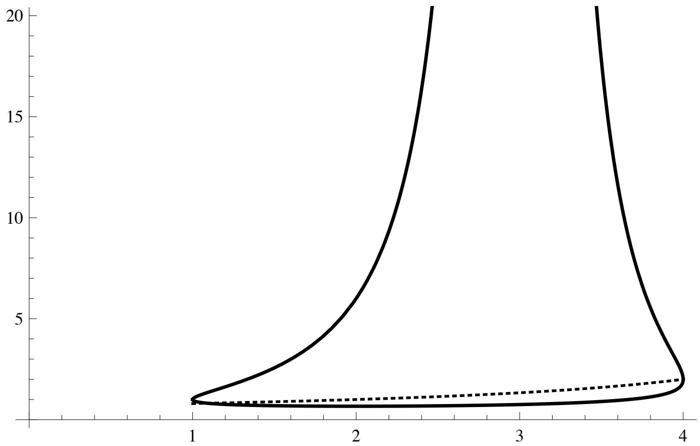
```
Plow[d_] := Show[Plot[h[p, d], {p, 1, 10}, PlotStyle →
    {{Black, Thickness[0.005]}, {Black, Thickness[0.005]}}, AxesOrigin → {0, 0}],
  Plot[4/(6 - p), {p, 1, 6}, PlotStyle → {Black, Dashing[Tiny], Thickness[0.004]}]]
Plow[1]
Plow[2]
```

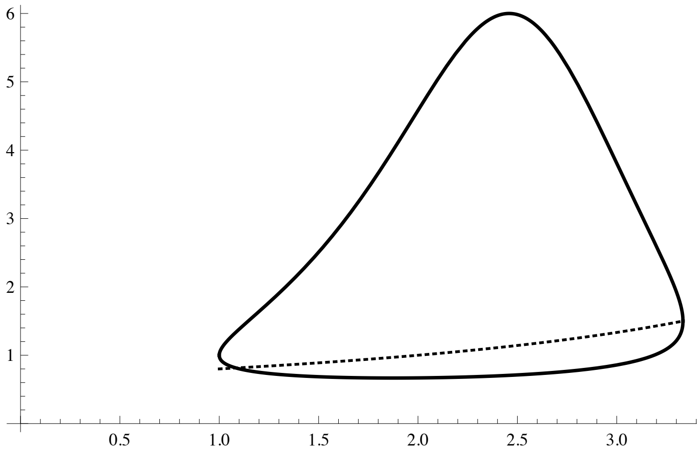


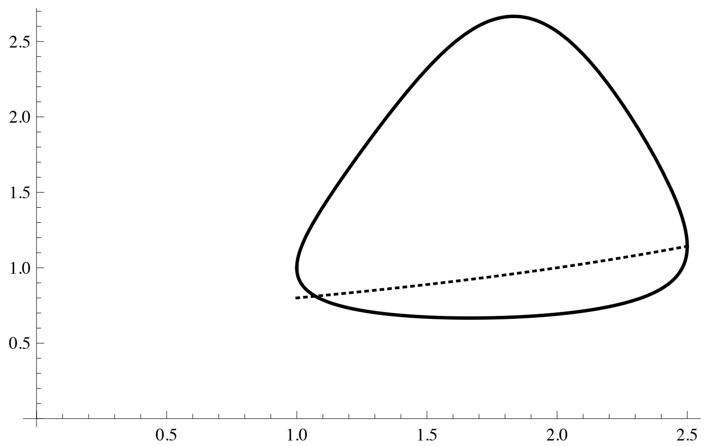


```
Pl[d_] := Show[Plot[h[p, d], {p, 1, 2 d/(d - 2)}, PlotStyle →
    {{Black, Thickness[0.005]}, {Black, Thickness[0.005]}}, AxesOrigin → {0, 0}],
  Plot[4/(6 - p), {p, 1, 2 d/(d - 2)}, PlotStyle → {Black, Dashing[Tiny], Thickness[0.004]}]]
Pl[3]
Show[Pl[4], PlotRange → {All, {0, 20}}]
Pl[5]
Pl[10]
```









Range covered by the function $\beta = \frac{4}{6-p}$

$$\mathbf{Solve}\left[\mathbf{f}\left[\frac{4}{6-p}\right] == 0,\ p\right]$$

$$\left\{\left\{p \to \frac{2\,d}{-2+d}\right\},\ \left\{p \to \frac{18\,d}{-2+17\,d}\right\}\right\}$$

Some preliminary computations for the admissible range for the improvement

$$\mathbf{Solve}\left[\left\{\frac{1}{q} + \frac{1}{q1} + \frac{1}{2} == 1,\ \varkappa == 2\,\beta - 1,\ \varkappa\,\theta\,q1 == \beta\,p,\ \varkappa\,(1-\theta)\,q == 2\,\beta\right\},\ \{q,\ q1,\ \varkappa,\ \theta\}\right]$$

$$\mathbf{Simplify}[\{\varkappa\,\theta,\ \varkappa\,(1-\theta)\}\ /.\ \%]$$

$$\left\{\left\{q \to \frac{2\,(-2\,\beta + p\,\beta)}{2 - 4\,\beta + p\,\beta},\ q1 \to \frac{(-2+p)\,\beta}{-1+\beta},\ \varkappa \to -1 + 2\,\beta,\ \theta \to \frac{p\,(-1+\beta)}{(-2+p)\,(-1+2\,\beta)}\right\}\right\}$$

$$\left\{\left\{\frac{p\,(-1+\beta)}{-2+p},\ \frac{2 + (-4+p)\,\beta}{-2+p}\right\}\right\}$$

$$\mathbf{Evaluate}\left[\frac{1}{q}\ /.\ q \to \frac{2\,(-2\,\beta + p\,\beta)}{2 - 4\,\beta + p\,\beta}\right];$$

$$\left\{\mathbf{Solve}[\% == 0,\ \beta][[1]],\ \mathbf{Solve}\left[\% == \frac{1}{2},\ \beta\right][[1]]\right\}$$

$$\left\{\left\{\beta \to -\frac{2}{-4+p}\right\},\ \{\beta \to 1\}\right\}$$

$$\mathbf{Solve}\left[\left\{\frac{1}{q} + \frac{1}{q1} == 1,\ \varkappa == 2\,(2\,\beta - 1),\ \varkappa\,\theta\,q1 == \beta\,p,\ \varkappa\,(1-\theta)\,q == 2\,\beta\right\},\ \{q,\ q1,\ \varkappa,\ \theta\}\right]$$

$$\mathbf{Simplify}[\{\varkappa\,\theta,\ \varkappa\,(1-\theta)\}\ /.\ \%]$$

$$\left\{\left\{q \to \frac{(-2+p)\,\beta}{2 - 4\,\beta + p\,\beta},\ q1 \to \frac{(-2+p)\,\beta}{2\,(-1+\beta)},\ \varkappa \to 2\,(-1 + 2\,\beta),\ \theta \to \frac{p\,(-1+\beta)}{(-2+p)\,(-1+2\,\beta)}\right\}\right\}$$

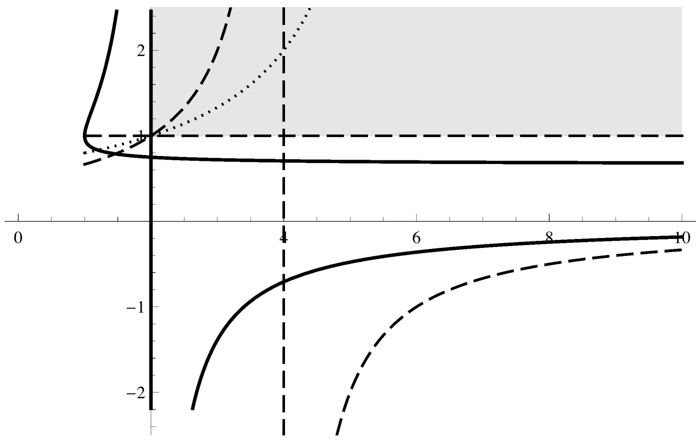
$$\left\{\left\{\frac{2\,p\,(-1+\beta)}{-2+p},\ \frac{4 + 2\,(-4+p)\,\beta}{-2+p}\right\}\right\}$$

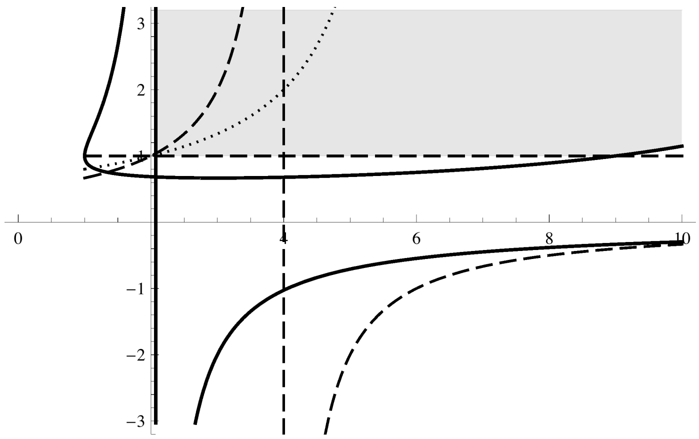
```
a[p_, d_] := 1/2 (4 + (2 (-1 + d)^2 (-1 + p)^2)/(2 + d)^2 - 2 p)
b[p_, d_] := (3 + d - p)/(2 + d)
```

The functions $\beta_{\pm}$ and the admissible range for the improvement

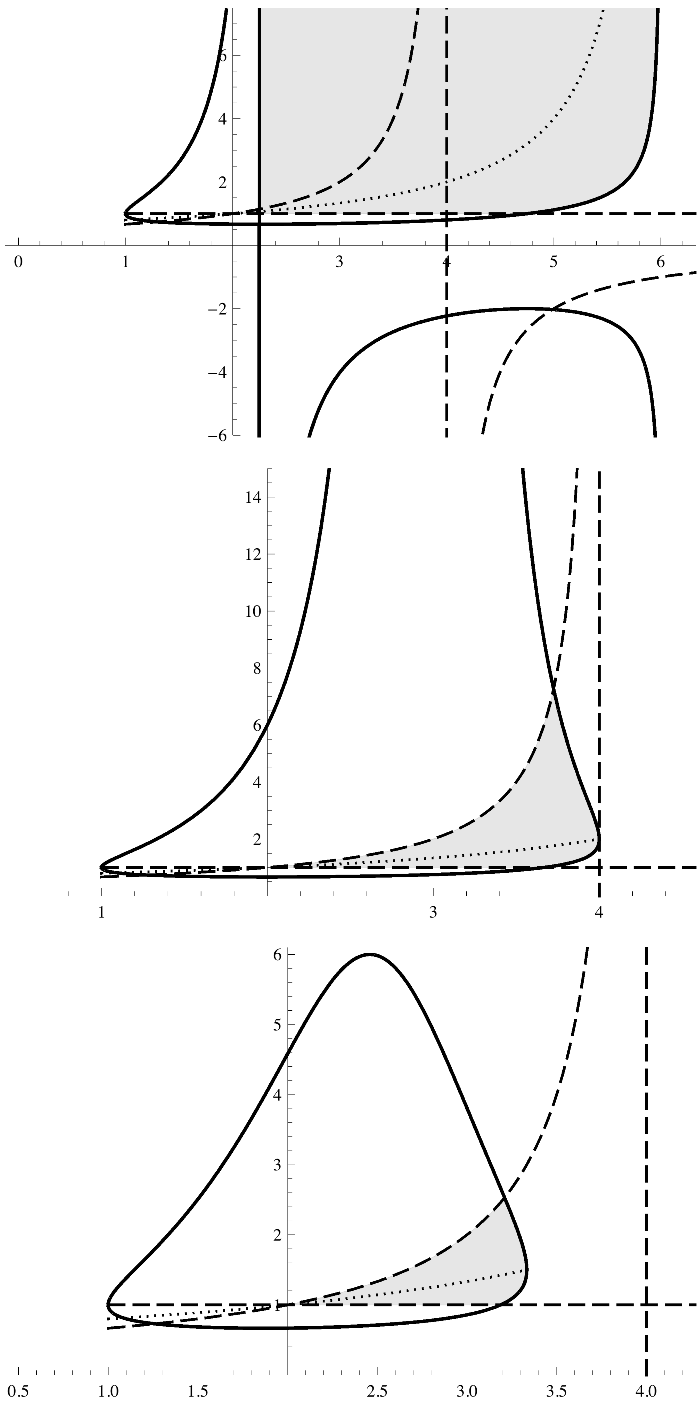
```
h[p_, d_] := FullSimplify[β /. Solve[a[p, d] β^2 - 2 b[p, d] β + 1 == 0, β]]
h[1, d]
Factor[h[(2 d)/(d - 2), d]]
{1, 1}
{(-2 + d)/(-3 + d), (-2 + d)/(-3 + d)}

Rng[p_, d_, hlim_] := {Min[h[p, d][[1]], h[p, d][[2]]], If[p < 4,
    Min[2/(4 - p), Max[h[p, d][[1]], h[p, d][[2]]]], Max[h[p, d][[1]], h[p, d][[2]]]]}
RngLim[p_, d_, hlim_] := Module[{M = Rng[p, d, hlim]}, If[M[[1]] < 0 || M[[2]] ≤ 1,
   {Max[M[[2]], 1], hlim}, {Max[1, M[[1]]], Max[1, Min[M[[2]], hlim]]}]]
RngGrey[d_, hlim_] := Module[{M = RngLim[p, d, hlim]},
  Plot[{M[[1]], M[[2]]}, {p, 2, If[d ≤ 2, 10, (2 d)/(d - 2)]},
   PlotStyle → {{Black, Thickness[0.001], Opacity[0.1]},
     {Black, Thickness[0.001], Opacity[0.1]}}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.9], PlotRange → {Automatic, {0, hlim}}]]
```

```
Plow[d_] := Show[Plot[h[p, d], {p, 1, 10}, PlotStyle →
    {{Black, Thickness[0.005]}, {Black, Thickness[0.005]}}, AxesOrigin → {0, 0}],
  Plot[4/(6 - p), {p, 1, 6}, PlotStyle → {Black, Dashing[Tiny], Thickness[0.004]}]]
PRange[d_, PR_] := Plot[{2/(4 - p), 1}, {p, 1, 10}, PlotRange →  PR,
  PlotStyle → {{Black, Thickness[0.004], Dashing[{0.025, 0.01}]},
    {Black, Thickness[0.004], Dashing[{0.025, 0.01}]}}]
PlowRange[d_, PR_] := Show[Plow[d], PRange[d, PR]]
Show[Show[RngGrey[1, 2.5], PlotRange → {{0, 10}, {-2.5, 2.5}}],
 PlowRange[1, {All, {-2.5, 2.5}}]]
Show[Show[RngGrey[2, 3.2], PlotRange → {{0, 10}, {-3.2, 3.25}}],
 PlowRange[2, {All, {-3.2, 3.25}}]]
```

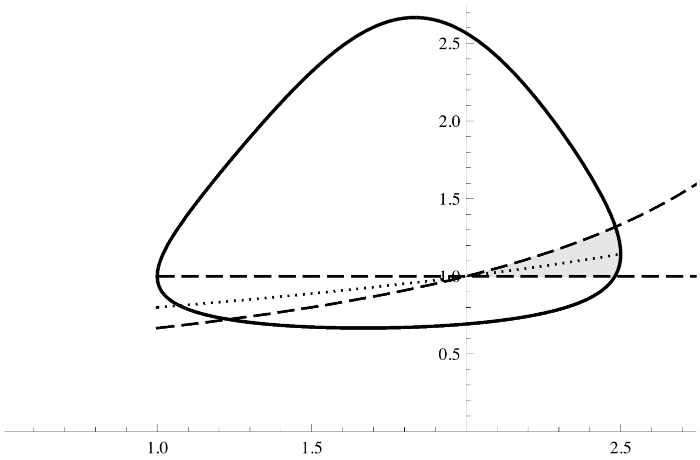





```
Pl[d_, PR_] := Show[Plot[h[p, d], {p, 1, 2 d/(d - 2)},
   PlotStyle → {{Black, Thickness[0.005]}, {Black, Thickness[0.005]}},
   AxesOrigin → {0, 0}], Plot[4/(6 - p), {p, 1, 2 d/(d - 2)},
   PlotStyle → {Black, Dashing[Tiny], Thickness[0.004]}], PRange[d, PR]]
Show[RngGrey[3, 7.5], Pl[3, {All, {-9, 8}}], PlotRange →  {{0, 6.2}, {-6, 7.5}}]
Show[RngGrey[4, 20], Pl[4, {All, {0, 20}}], PlotRange →  {{0.5, 4.5}, {0, 15}}]
Show[RngGrey[5, 6], Pl[5, {All, {0, 6.1}}], PlotRange →  {{0.5, 4.2}, {0, 6.1}}]
Show[RngGrey[10, 3], Pl[10, {All, {0, 2.75}}],
 PlotRange →  {{0.55, 2.7}, {0, 2.75}}]
```



6
4
2
0
1
3
4
5
6
−2
−4
−6
14
12
10
8
6
4
2
1
3
4
6
5
4
3
2
0.5
1.0
1.5
2.5
3.0
3.5
4.0



The two-dimensional case

```
Simplify[a[p, 2]]
```

$$\frac{1}{16}\left(33-18\,p+p^2\right)$$

```
1/16 (33 - 18 p + p^2)
Solve[% == 0, p]
N[%]
```

$$\frac{1}{16}\left(33-18\,p+p^2\right)$$

$$\{\{p\to 9-4\sqrt{3}\},\{p\to 9+4\sqrt{3}\}\}$$

{{p → 2.0718}, {p → 15.9282}}

```
h[p, 2]
```

$$\left\{\frac{4}{5+2\sqrt{2}\sqrt{-1+p}-p},-\frac{4}{-5+2\sqrt{2}\sqrt{-1+p}+p}\right\}$$

```
Solve[4/(5 + 2 Sqrt[2] Sqrt[-1 + p] - p) == 1, p]
```

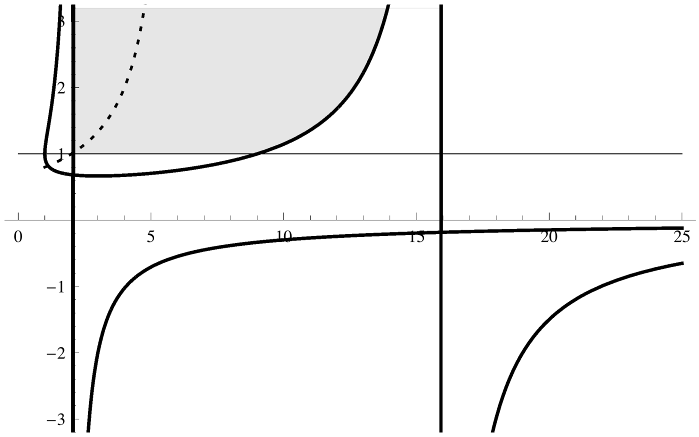
{{p → 1}, {p → 9}}

```
1 / h[p, 2]
{Solve[%[[1]] == 0, p][[1]], Solve[%[[2]] == 0, p][[1]]}
```

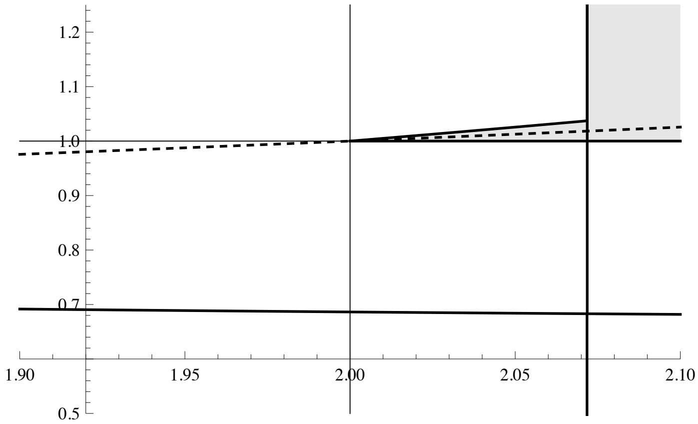
$$\left\{\frac{1}{4}\left(5+2\sqrt{2}\sqrt{-1+p}-p\right),\frac{1}{4}\left(5-2\sqrt{2}\sqrt{-1+p}-p\right)\right\}$$

$$\{\{p\to 9+4\sqrt{3}\},\{p\to 9-4\sqrt{3}\}\}$$

```
Plow2[d_] := Show[Plot[h[p, d], {p, 1, 25}, PlotStyle →
    {{Black, Thickness[0.005]}, {Black, Thickness[0.005]}}, AxesOrigin → {0, 0}],
  Plot[4/(6 - p), {p, 1, 6}, PlotStyle → {Black, Dashing[Small], Thickness[0.004]}]]
RngGreyExtended[d_, hlim_] := Module[{M = RngLim[p, d, hlim]},
  Plot[{M[[1]], M[[2]]}, {p, 2, If[d == 2, 9 + 4 √3, 2 d/(d - 2)]},
   PlotStyle → {{Black, Thickness[0.001], Opacity[0.1]},
     {Black, Thickness[0.001], Opacity[0.1]}}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.9], PlotRange → {Automatic, {0, hlim}}]]
Show[Show[RngGreyExtended[2, 3.2], PlotRange → {{0, 25}, {-3.2, 3.25}}],
 Plow2[2], ListLinePlot[{{0, 1}, {25, 1}}, PlotStyle → Black]]
```



```
Plow1[d_] := Show[Plot[h[p, d], {p, 1.9, 2.1},
   PlotStyle → {{Black, Thickness[0.004]}, {Black, Thickness[0.004]}},
   AxesOrigin → {0, 0}], Plot[4/(6 - p), {p, 1.9, 2.1},
   PlotStyle → {Black, Dashing[Small], Thickness[0.004]}]]
Show[Plot[{If[p < 9 - 4 √3, 2/(4 - p), 2], 1}, {p, 2, 2.1},
  PlotStyle → {{Black, Thickness[0.004]}, {Black, Thickness[0.004]}},
  Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9], PlotRange → {All, {0, 2}}],
 Show[ListLinePlot[{{2, 0.5}, {2, 1.25}}, PlotStyle → Black],
  ListLinePlot[{{1.9, 1}, {2.1, 1}}, PlotStyle → Black], Plow1[2],
  PlotRange →  {{1.9, 2.1}, {0.5, 1.25}}, AspectRatio →  0.5],
 PlotRange →  {{1.9, 2.1}, {0.5, 1.25}}, AxesOrigin → {1.92, 0.6}]
```



Case p>2, optimization on β>1

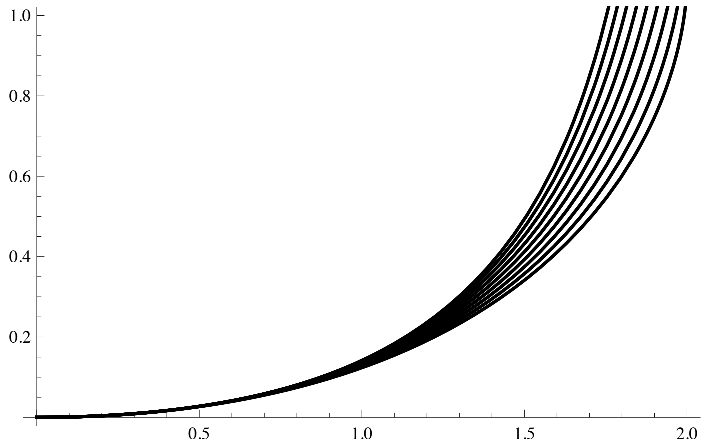
```
γ[β_, p_, d_] :=
 -Simplify[((d - 1)/(d + 2) (κ + β - 1))^2 - κ (β - 1) - d/(d + 2) (κ + β - 1) /. κ → β (p - 2) + 1]
δ[β_, p_] := Simplify[(p - (4 - p) β)/(2 β (p - 2))]
phi[p_, d_, β_, x_] := NIntegrate[Exp[
    γ[β, p, d]/(2 β^2 (1 - δ[β, p]) (p - 2)) ((1 - (p - 2) z)^(1-δ[β,p]) - (1 - (p - 2) x)^(1-δ[β,p]))], {z, 0, x}]
Rng[p_, d_] := {Max[1, Min[h[p, d][[1]], h[p, d][[2]]]], If[p < 4,
    Min[2/(4 - p), Max[h[p, d][[1]], h[p, d][[2]]]], Max[h[p, d][[1]], h[p, d][[2]]]]}
BetaTable[p_, d_] := Module[{M = Rng[p, d]}, Table[β,
    {β, M[[1]] + (M[[2]] - M[[1]])/Nbre, M[[2]] - (M[[2]] - M[[1]])/Nbre, (M[[2]] - M[[1]])/Nbre}]]
Nbre = 10;
β0[p_] := 4/(6 - p)
```

```
BetaTable[2.5, 5]
```

{2.38333, 2.26667, 2.15, 2.03333, 1.91667, 1.8, 1.68333, 1.56667, 1.45}

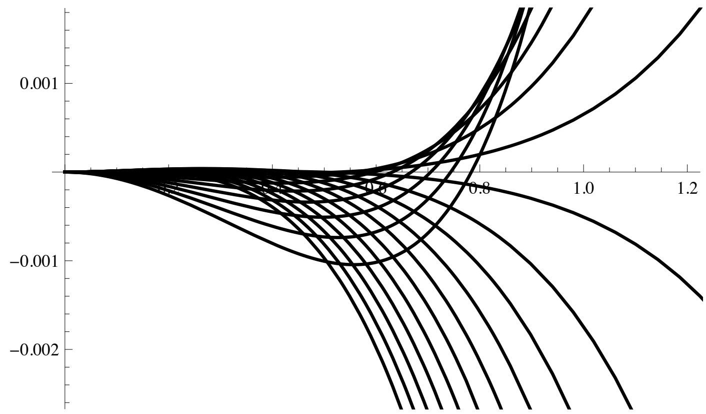
```
PMax[p_, d_, PR_] :=
 Show[Module[{M = Rng[p, d]}, Table[Plot[phi[p, d, β, x] - x, {x, 0, 1/(p - 2)},
     PlotStyle → {Black, Thickness[0.005]}, DisplayFunction → Identity],
    {β, M[[1]] + (M[[2]] - M[[1]])/Nbre, M[[2]] - (M[[2]] - M[[1]])/Nbre, (M[[2]] - M[[1]])/Nbre}]],
  DisplayFunction → $DisplayFunction, PlotRange → PR]
```

```
PMax[2.5, 5, {Automatic, {0, 1}}]
```



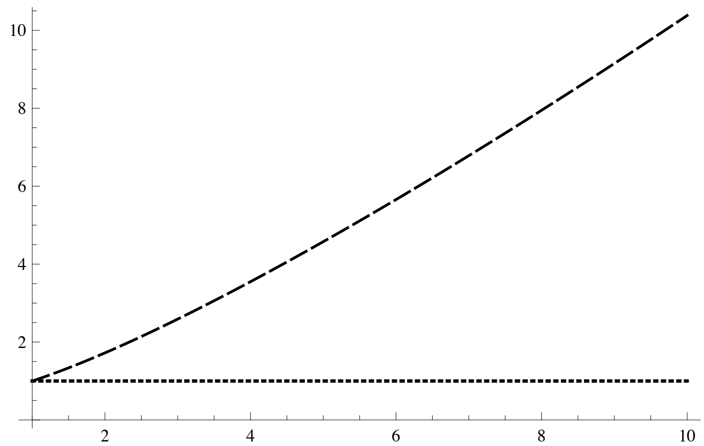

```
DeltaPMax[p_, d_, PR_] :=
 Show[Module[{M = Rng[p, d]}, Table[Plot[phi[p, d, β, x] - phi[p, d, β0[p], x], {x, 0,
      1/(p - 2)}, PlotStyle → {Black, Thickness[0.005]}, DisplayFunction → Identity],
    {β, M[[1]] + (M[[2]] - M[[1]])/Nbre, M[[2]] - (M[[2]] - M[[1]])/Nbre, (M[[2]] - M[[1]])/Nbre}]],
  DisplayFunction → $DisplayFunction, PlotRange → PR]
```

```
Nbre = 20;
DeltaPMax[2.5, 5, {{0, 1.2}, Automatic}]
```

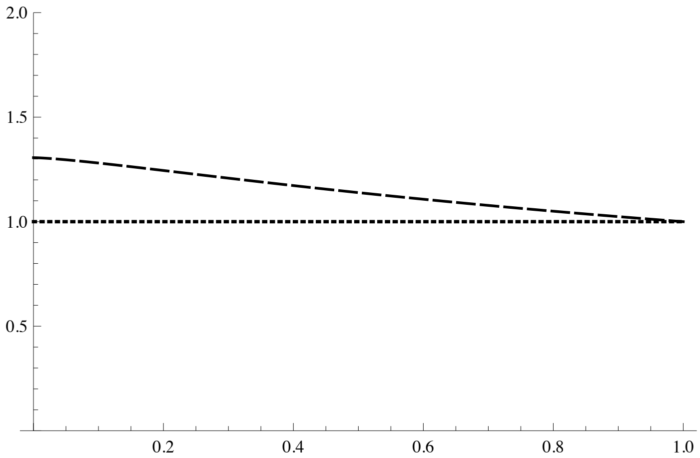


Estimates based on the case β=1

```
p1 = 5/2; d1 = 2;
TwoSharp = (2 d^2 + 1)/(d - 1)^2 /. d → d1;
((d - 1)/(d + 2))^2 (p - 1) (TwoSharp - p) /. d → d1;
γ1 = % /. p → p1;
ϕ0[x_] := 1/(γ1/2 + p1 - 2) (x^(2 + γ1/(p1 - 2)) - 1)
P0 = Plot[{1, (p1 - 2) ϕ0[x]/(x^2 - 1)}, {x, 1, 10},
  PlotStyle → {{Black, Thickness[0.005], Dashing[Tiny]},
    {Black, Thickness[0.004], Dashing[{0.025, 0.01}]}},
  PlotRange → {All, Automatic}, AxesOrigin → {1, 0}]
```

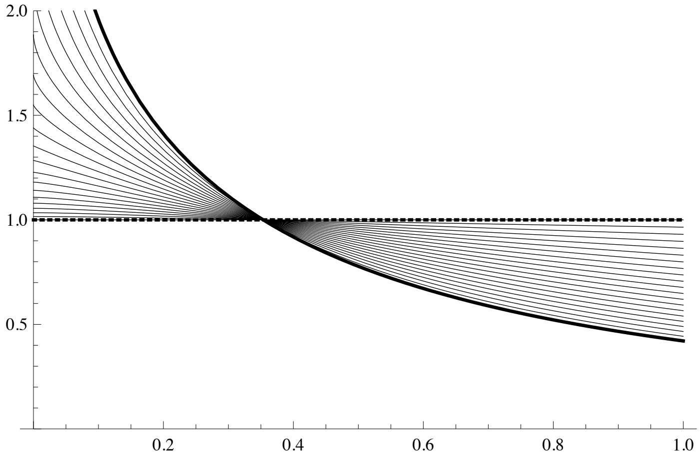


```
TwoSharp = (2 d^2 + 1)/((d - 1)^2) /. d → d1;
p1 = 3/2; d1 = 2;
((d - 1)/(d + 2))^2 (p - 1) (TwoSharp - p) /. d → d1;
γ1 = % /. p → p1;
ϕ0[x_] := 1/(γ1/2 + p1 - 2) (x^(2 + γ1/(p1-2)) - 1)
P1 = Plot[{1, (2 - p1) ϕ0[x]/(1 - x^2)}, {x, 0, 1},
  PlotStyle → {{Black, Thickness[0.005], Dashing[Tiny]},
    {Black, Thickness[0.004], Dashing[{0.025, 0.01}]}}, PlotRange → {All, {0, 2}}]
```




```
α1 = 3;
ϕ1[x_] := α1/(1 - (p1 - 1)^α1) (1 - x^2)
ϕ2[x_, γ_] := α1/(1 - (p1 - 1)^(α1 γ/2)) (1 - x^γ)
ϕ3[x_] := 2/Log[p1 - 1] Log[x]
```

```
P2 = Plot[Table[ϕ2[x, γ]/ϕ1[x], {γ, 0.1, 2, 0.1}], {x, 0, 1},
   PlotStyle → {Black, Thickness[0.001]}, PlotRange → {All, {0, 2}}];
P3 = Plot[{1, ϕ3[x]/ϕ1[x]}, {x, 0, 1}, PlotRange → {All, {0, 2}}, PlotStyle →
    {{Black, Thickness[0.005], Dashing[Tiny]}, {Black, Thickness[0.005]}}];
Show[
 P2,
 P3]
```




```
Show[P1, P2, P3]
```