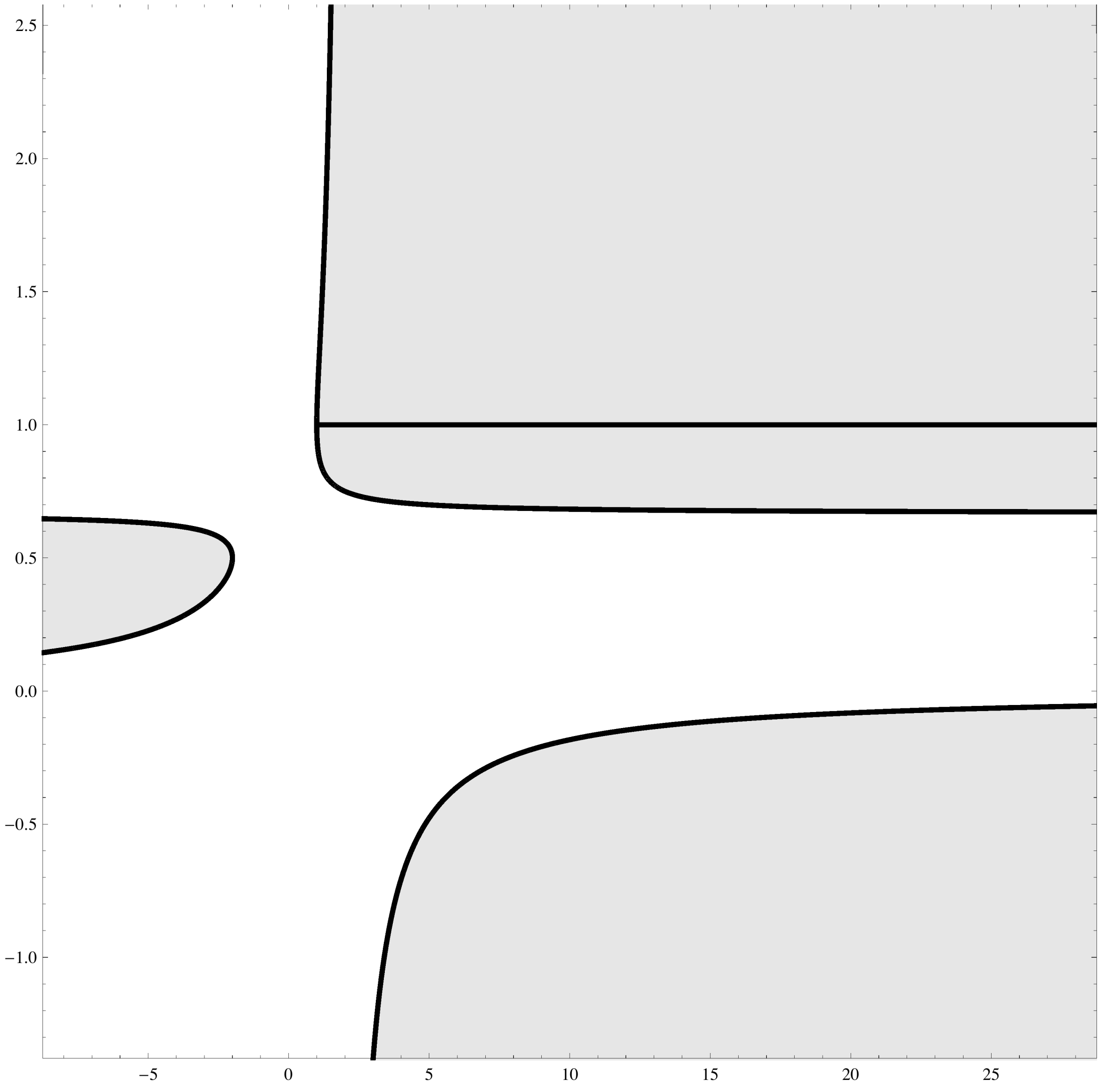
Definitions

```
f[β_, p_, d_] := ((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1)

Betapm[p_, d_] := β /. Solve[f[β, p, d] == 0, β]

BetaPlot[d_] :=
 Module[{M = Betapm[p, d]}, Show[ListLinePlot[{{1, 0}, {1, 3}}, PlotStyle → Black],
   ListLinePlot[{{2 d/(d - 2), 0}, {2 d/(d - 2), 3}}, PlotStyle → Black],
   Plot[M, {p, 1, 2 d/(d - 2)}, Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9],
    PlotStyle → {{Thickness[0.005], Black}, {Thickness[0.005], Black}}],
   PlotRange → All, AxesOrigin → {2, 1}]]

mPlot[d_] := Module[{M = 1 + 2/p (1/β - 1) /.
     Solve[((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1) == 0, β]},
  Show[ListLinePlot[{{1, 0}, {1, 2}}, PlotStyle → Black],
   ListLinePlot[{{2 d/(d - 2), 0}, {2 d/(d - 2), 2}}, PlotStyle → Black],
   Plot[M, {p, 1, 2 d/(d - 2)}, Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9],
    PlotStyle → {{Thickness[0.005], Black}, {Thickness[0.005], Black}}],
   PlotRange → All, AxesOrigin → {2, 1}]]
```

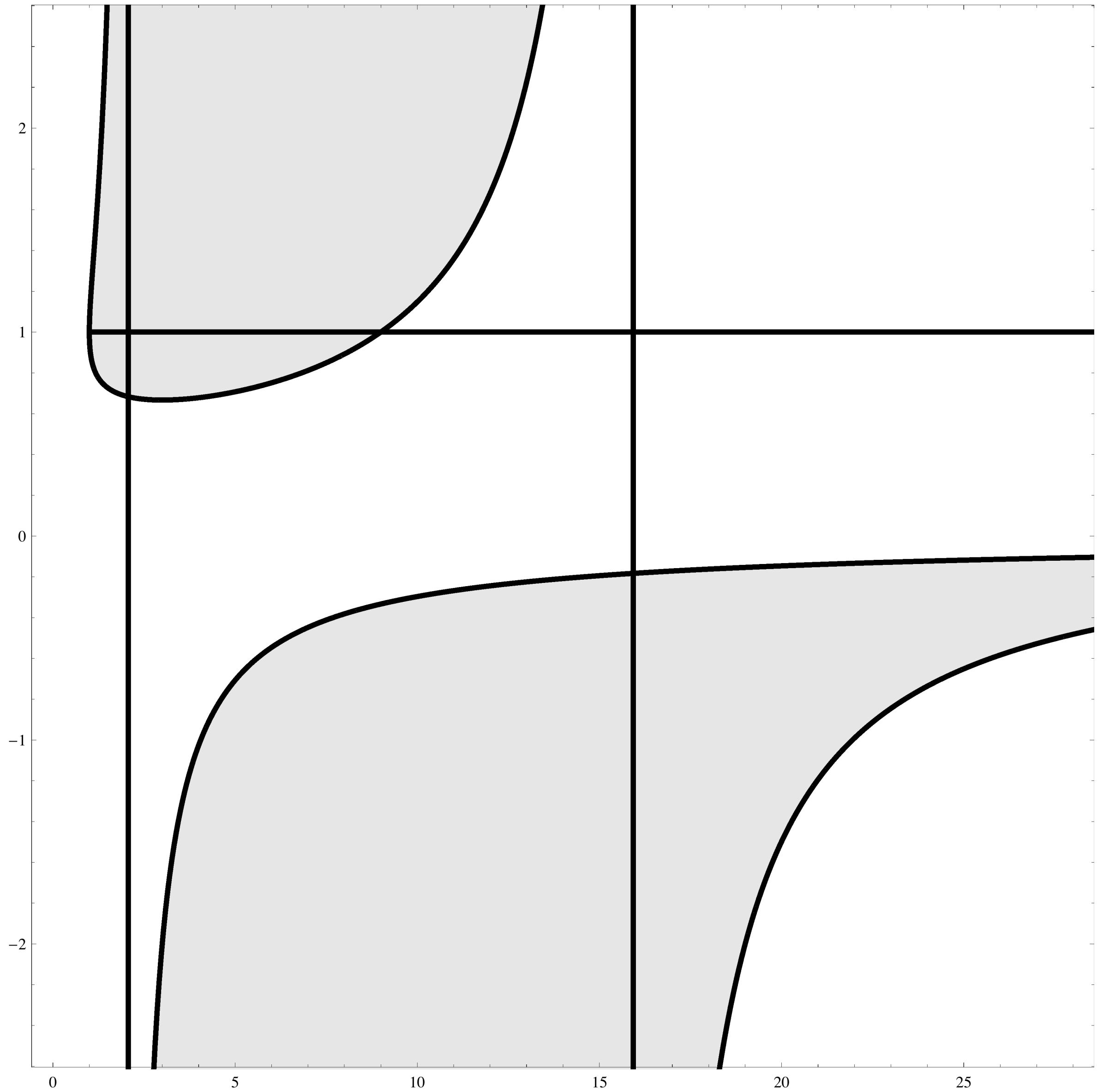
Dimension d=1

```
Show[RegionPlot[f[β, p, 1] < 0,
   {p, -10, -2}, {β, 0, 1}, PlotStyle → GrayLevel[0.9]],
 RegionPlot[f[β, p, 1] < 0, {p, 1, 30}, {β, -4, 4}, PlotStyle → GrayLevel[0.9]],
 Plot[Betapm[p, 1], {p, 1, 1.9999}, PlotPoints → 500,
  PlotStyle → {Thickness[0.005], Black}], Plot[Betapm[p, 1], {p, 2.00001, 30},
  PlotPoints → 500, PlotStyle → {Thickness[0.005], Black}], Plot[Betapm[p, 1],
  {p, -10, -2}, PlotPoints → 500, PlotStyle → {Thickness[0.005], Black}],
 ListLinePlot[{{1, 1}, {30, 1}}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {{-8, 28}, {-1.3, 2.5}}, AxesOrigin → {0, 0}]
```



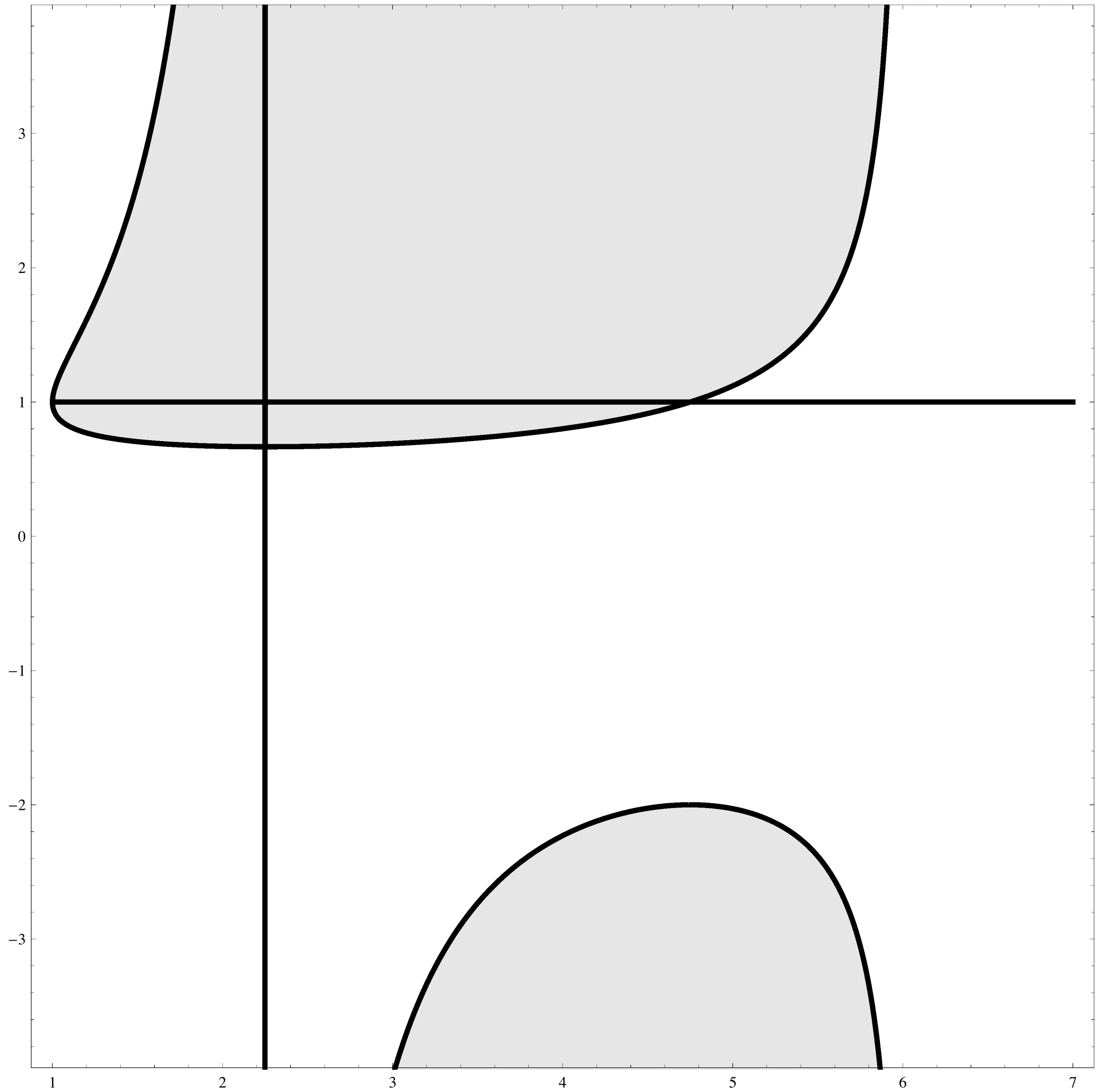
Dimension d=2

```
Show[RegionPlot[f[β, p, 2] < 0, {p, 0, 30}, {β, -4, 4}, PlotStyle → GrayLevel[0.9]],
 Plot[Betapm[p, 2], {p, 1, 30}, PlotPoints → 500,
  PlotStyle → {Thickness[0.005], Black}],
 ListLinePlot[{{1, 1}, {30, 1}}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {{0, 28}, {-2.5, 2.5}}]
```



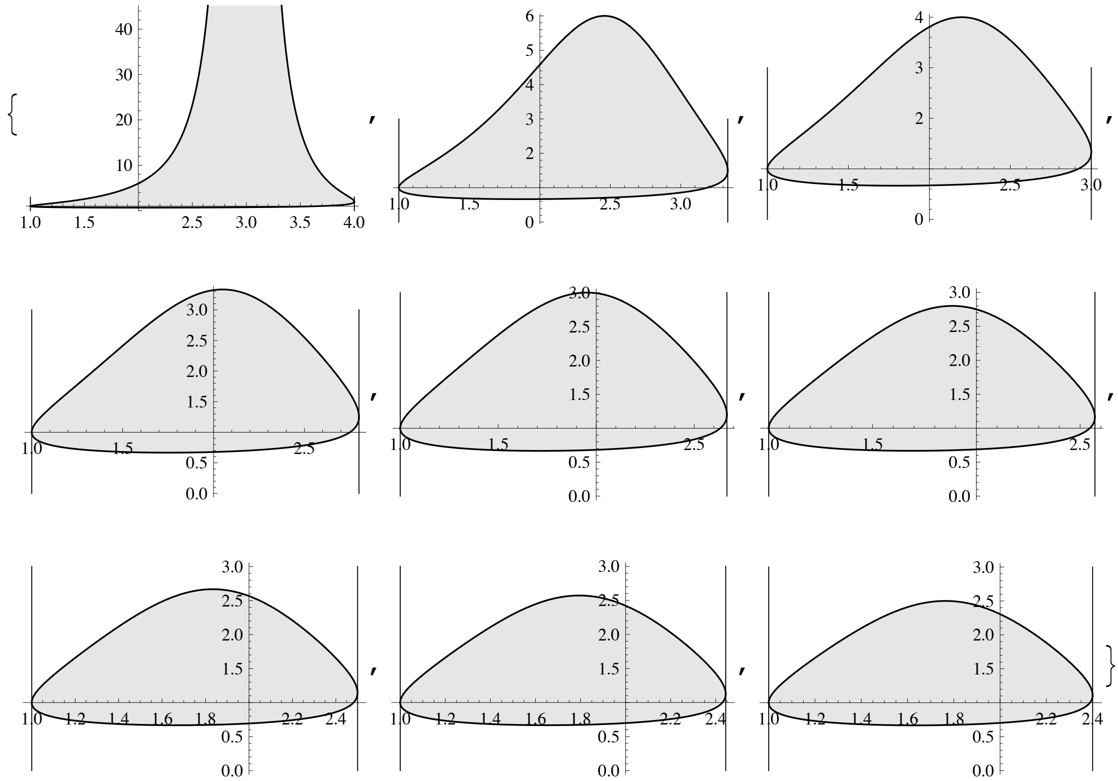
Dimension d=3

```
Show[RegionPlot[f[β, p, 3] < 0, {p, 0, 7}, {β, -4, 4}, PlotStyle → GrayLevel[0.9]],
 Plot[Betapm[p, 3], {p, 1, 7}, PlotPoints → 500,
  PlotStyle → {Thickness[0.005], Black}],
 ListLinePlot[{{1, 1}, {7, 1}}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {All, {-3.8, 3.8}}]
```



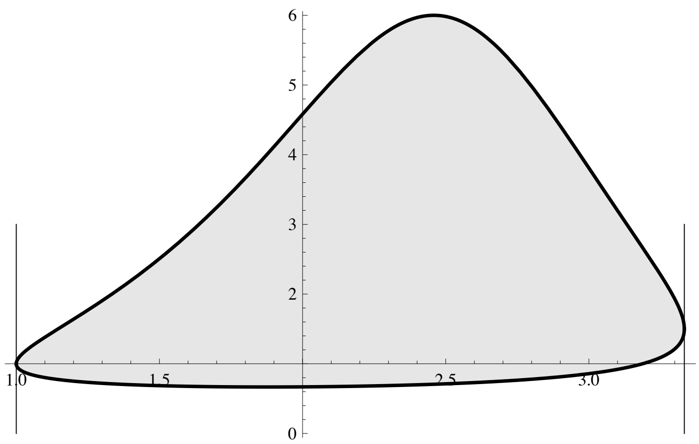
Dimension d≥4

```
Table[BetaPlot[d], {d, 4, 12}]
```

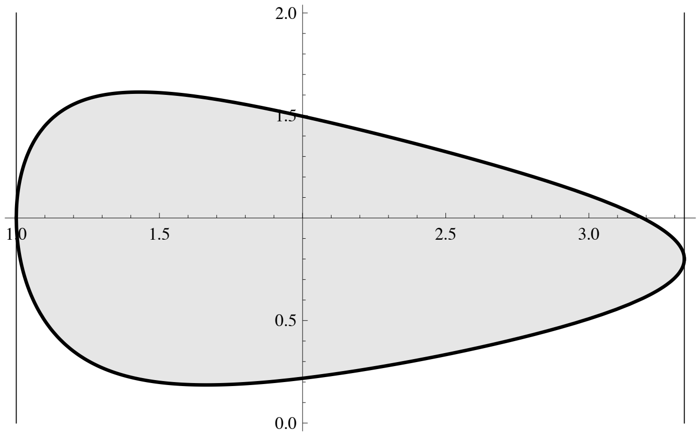


Dimension d=5

```
BetaPlot[5]
```

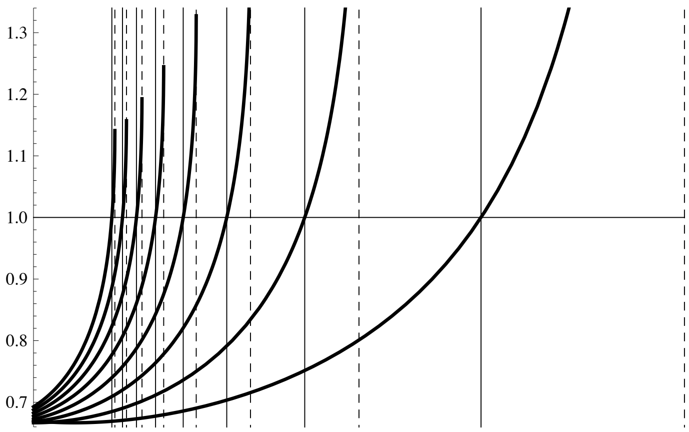


```
mPlot[5]
```



The lower branch

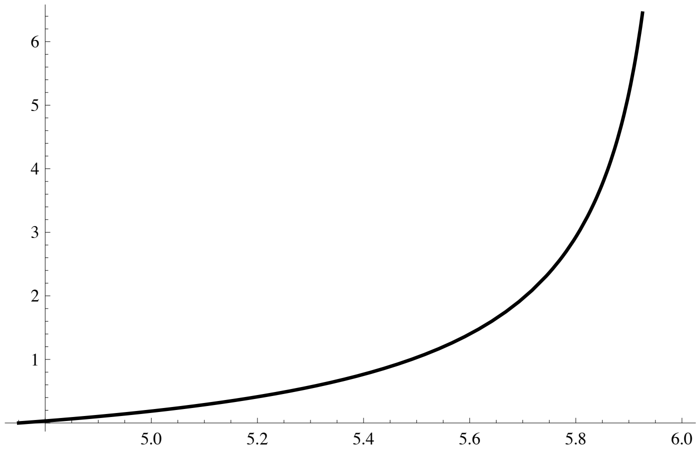
```
f[β_] := ((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1)
Res = Simplify[Solve[f[β] == 0, β]][[1]];

Show[ListLinePlot[{{2, 1}, {6, 1}}, PlotStyle → Black],
 Table[ListLinePlot[{{(2 d^2 + 1)/(d - 1)^2, 0.6}, {(2 d^2 + 1)/(d - 1)^2, 1.5}}, PlotStyle → Black],
  {d, 3, 10}], Table[ListLinePlot[{{(2 d)/(d - 2), 0.6}, {(2 d)/(d - 2), 1.5}},
   PlotStyle → {Dashed, Black}], {d, 3, 10}], Plot[
  Table[(6 + d^2 - d (-5 + p) - Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] - 2 p)/((-3 + p)^2 - 2 d (-3 + p^2) + d^2 (3 - 3 p + p^2)), {d, 3, 10}],
  {p, 2, 6}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {{2, 6}, {0.7, 1.3}}]
```

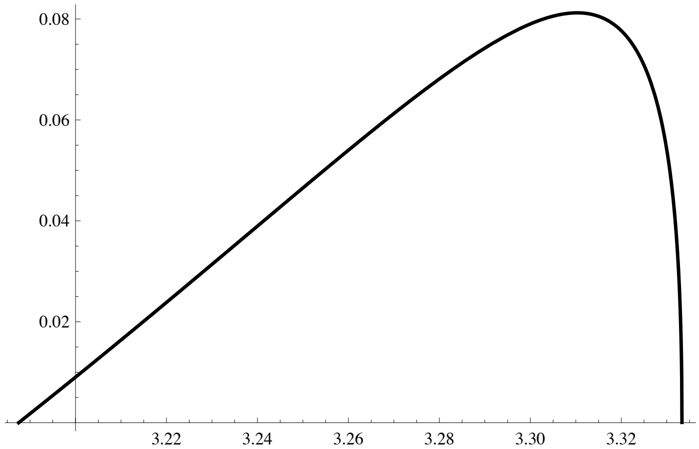


The second obstruction

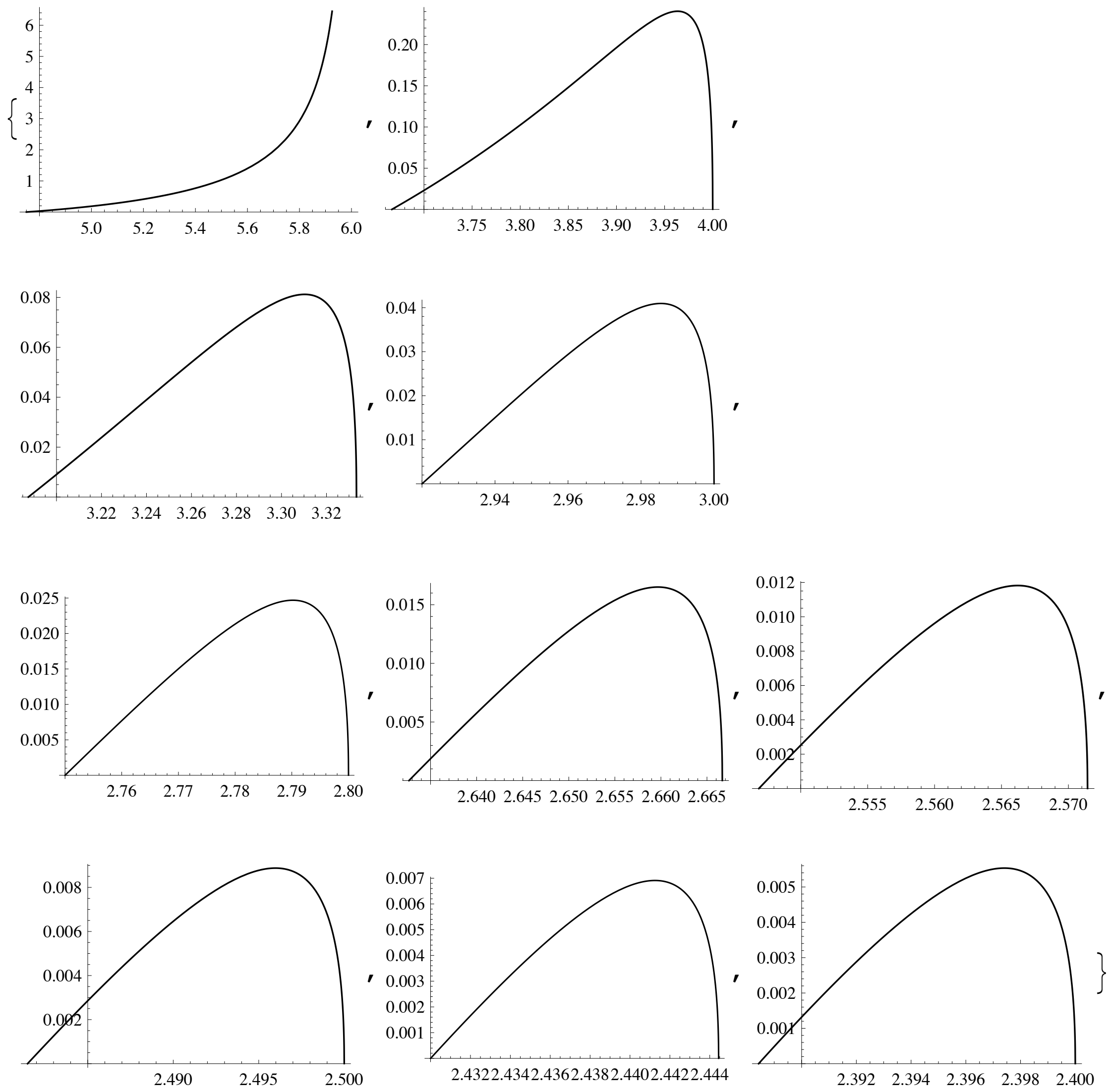
```
FullSimplify[Simplify[-(α + β - 1)^2 + 2 (d - 1)/(d + 2) (p - 1) (α + β - 1) β - d/(d + 2) (p - 1) β^2 /.
    α → (d - 1)/(d + 2) (p - 1) β] /. Res];
f[p_, d_] := -1/((2 + d) ((-3 + p)^2 + d^2 (3 + (-3 + p) p) - 2 d (-3 + p^2))^2) 2
   (-1 + p) (-(-3 + p)^2 Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] +
     d^5 (-2 + p) (1 + (-3 + p) p) + d^3 (22 - 4 Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] +
        p (-11 + 2 Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] + (16 - 3 p) p)) +
     2 d^4 (2 + p (2 + p - p^2)) + d^2 (12 + 8 p^3 - 11 Sqrt[-d (2 + d)^2 (-1 + p) (-2 d + (-2 + d) p)] +
        3 p (8 + Sqrt[-d (2 + d)^2 (-1 + p) (-2 d + (-2 + d) p)]) -
        p^2 (12 + Sqrt[-d (2 + d)^2 (-1 + p) (-2 d + (-2 + d) p)])) +
     2 d (-3 Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] +
        p (6 - 4 Sqrt[-d (2 + d)^2 (d (-2 + p) - 2 p) (-1 + p)] +
          p (4 - 2 p + Sqrt[-d (2 + d)^2 (-1 + p) (-2 d + (-2 + d) p)]))))
h[d_] := Plot[f[p, d], {p, (2 d^2 + 1)/(d - 1)^2, 2 d/(d - 2)}, PlotStyle → {Thickness[0.005], Black}]
```

```
h[3]
h[5]
```





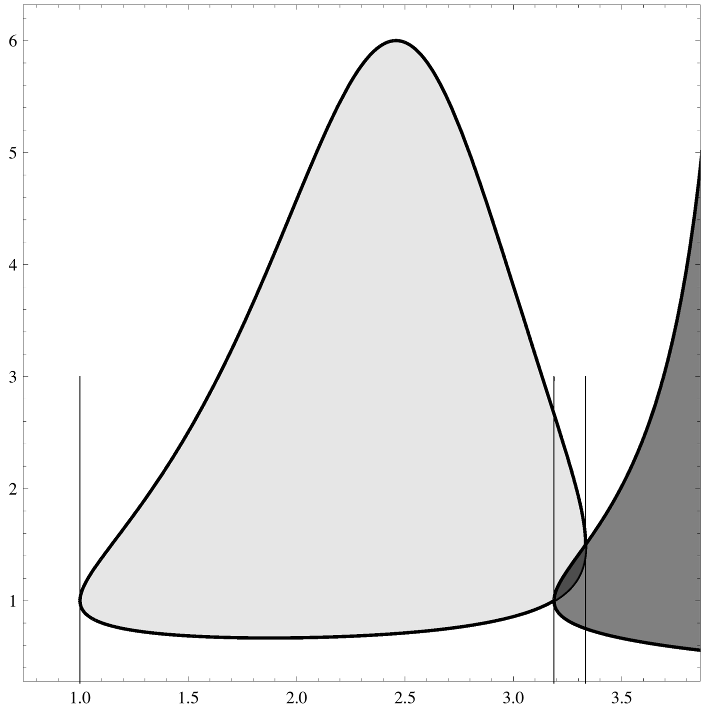
```
Table[h[d], {d, 3, 12}]
```



The second obstruction: the region in which A is positive when d=5

```
A[β_, p_, d_] :=
  (α + β - 1)^2 - 2 (d - 1)/(d + 2) (p - 1) (α + β - 1) β + d/(d + 2) (p - 1) β^2 /. α → (d - 1)/(d + 2) β (p - 1)
g[p_, d_] := β /. Solve[A[β, p, d] == 0, β]
```

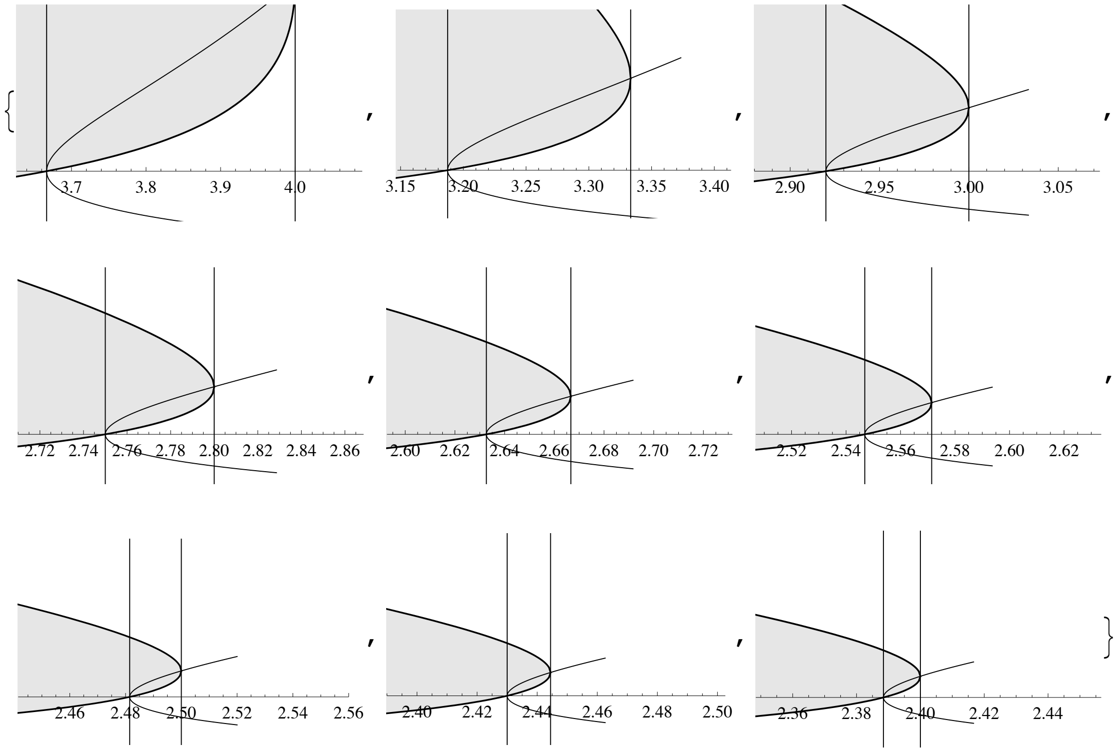
```
Show[RegionPlot[A[β, p, 5] < 0, {p, 1, 4},
  {β, -4, 8}, PlotStyle → GrayLevel[0.5]], BetaPlot[5],
 RegionPlot[A[β, p, 5] < 0 && f[β, p, 5] < 0, {p, (2 d^2 + 1)/(d - 1)^2 /. d → 5, (2 d)/(d - 2) /. d → 5},
  {β, -4, 8}, PlotStyle → GrayLevel[0.3], PlotPoints → 200],
 ListLinePlot[{{(2 d^2 + 1)/(d - 1)^2, 0}, {(2 d^2 + 1)/(d - 1)^2, 3}} /. d → 5, PlotStyle → Black],
 Plot[g[p, 5], {p, 3, 4}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {{0.8, 3.8}, {0.4, 6.2}}]
```



The second obstruction when d≥5

```
ABetaPlot[d_] :=
 Show[BetaPlot[d], ListLinePlot[{{(2 d^2 + 1)/(d - 1)^2, 0}, {(2 d^2 + 1)/(d - 1)^2, 3}}, PlotStyle → Black],
  Plot[g[p, d], {p, (2 d^2 + 1)/(d - 1)^2, (2 d)/(d - 2) + 0.2/d}, PlotStyle → Black],
  PlotRange → {{(2 d^2 + 1)/(d - 1)^2, (2 d)/(d - 2) + 0.2/d}, {0.8, 1.8}}]
```

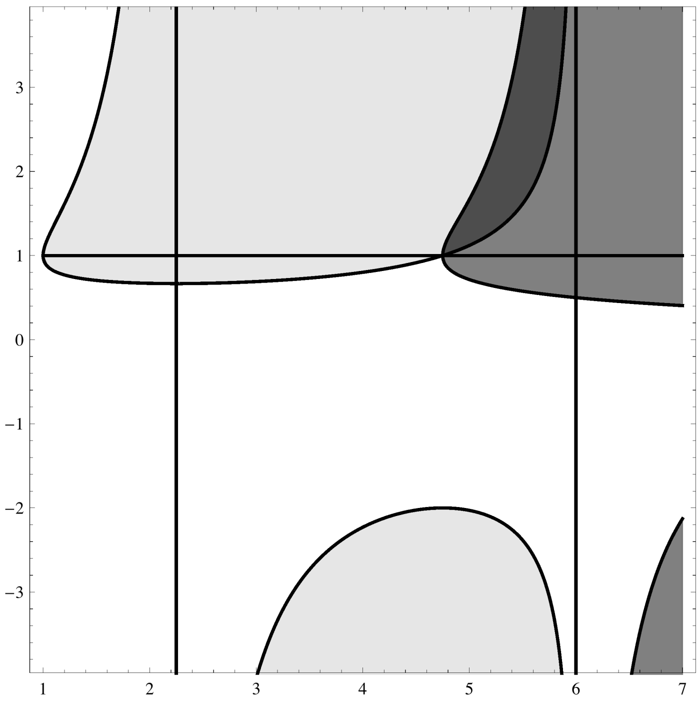
```
Table[ABetaPlot[d], {d, 4, 12}]
```



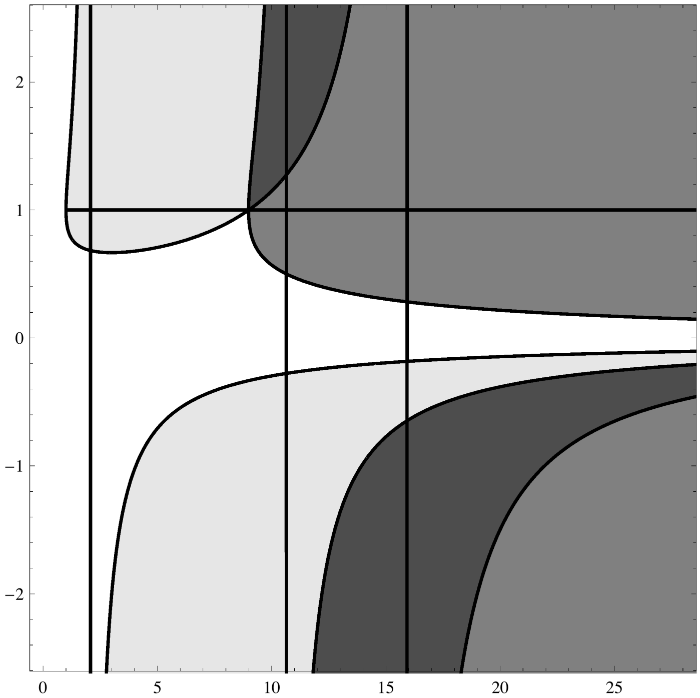
The second obstruction when d=3

```
Pg[d_, pmax_] := Plot[g[p, d], {p, (2 d^2 + 1)/(d - 1)^2, pmax},
  PlotPoints → 500, PlotStyle → {Thickness[0.005], Black}]
```

```
Show[RegionPlot[f[β, p, 3] < 0, {p, 0, 7}, {β, -4, 4}, PlotStyle → GrayLevel[0.9]],
 RegionPlot[A[β, p, 3] < 0, {p, 4.5, 7}, {β, -4, 8}, PlotStyle → GrayLevel[0.5],
  PlotPoints → 50], RegionPlot[A[β, p, 3] < 0 && f[β, p, 3] < 0, {p, 4.5, 7},
  {β, -4, 8}, PlotStyle → GrayLevel[0.3], PlotPoints → 50], Plot[Betapm[p, 3],
  {p, 1, 7}, PlotPoints → 500, PlotStyle → {Thickness[0.005], Black}],
 ListLinePlot[{{1, 1}, {7, 1}}, PlotStyle → {Thickness[0.005], Black}],
 Pg[3, 7], PlotRange → {All, {-3.8, 3.8}}]
```



```
Show[RegionPlot[A[β, p, 2] < 0, {p, 1, 30},
  {β, -4, 8}, PlotStyle → GrayLevel[0.5], PlotPoints → 50],
 RegionPlot[f[β, p, 2] < 0, {p, 0, 30}, {β, -4, 4}, PlotStyle → GrayLevel[0.9]],
 RegionPlot[A[β, p, 2] < 0 && f[β, p, 2] < 0, {p, 9, 30},
  {β, -4, 8}, PlotStyle → GrayLevel[0.3], PlotPoints → 50],
 Plot[Betapm[p, 2], {p, 1, 30}, PlotPoints → 500,
  PlotStyle → {Thickness[0.005], Black}], Pg[2.001, 30], Pg[2.001, 14],
 ListLinePlot[{{1, 1}, {30, 1}}, PlotStyle → {Thickness[0.005], Black}],
 PlotRange → {{0, 28}, {-2.5, 2.5}}]
```



Positivity of A (theoretical)

```
A[β_, p_, d_] :=
```

$$(\alpha+\beta-1)^2 - 2\,\frac{d-1}{d+2}\,(p-1)\,(\alpha+\beta-1)\,\beta + \frac{d}{d+2}\,(p-1)\,\beta^2 \;/.\; \alpha \to \frac{d-1}{d+2}\,\beta\,(p-1)$$

```
B = FullSimplify[β /. Solve[A[β, p, d] == 0, β], Assumptions → d ≥ 2]
```

$$\left\{\frac{2+d}{2+d-\sqrt{(-1+p)\left(-1+d^2(-2+p)+p-2\,d\,p\right)}},\ \frac{2+d}{2+d+\sqrt{(-1+p)\left(-1+d^2(-2+p)+p-2\,d\,p\right)}}\right\}$$

```
b = Simplify[FullSimplify[
```

$$\beta \;/.\; \text{Solve}\left[\left(\frac{d-1}{d+2}\right)^2 (\beta\,(p-1))^2 - (\beta\,(p-2)+1)\,(\beta-1) - \frac{d}{d+2}\,\beta\,(p-1) == 0,\ \beta\right]\Big],$$

```
  Assumptions → d + 2 > 0]
```

$$\left\{\frac{2+d}{3+d+\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}-p},\ \frac{2+d}{3+d-\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}-p}\right\}$$

```
b = Simplify[FullSimplify[
   β /. Solve[((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1) == 0, β]],
  Assumptions → d + 2 > 0]
```

$$\left\{\frac{2+d}{3+d+\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}-p},\ \frac{2+d}{3+d-\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}-p}\right\}$$

```
R1 = FullSimplify[(2 + d) (1/B[[1]] - 1/b[[1]])]
Solve[R1 == 0, p]
Limit[Sqrt[p - (2 d^2 + 1)/(d - 1)^2] D[R1, p], p → (2 d^2 + 1)/(d - 1)^2]
```

$$-1-\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}+p-\sqrt{(-1+p)\,\left(-1+d^2\,(-2+p)+p-2\,d\,p\right)}$$

$$\left\{\{p\to 1\},\ \left\{p\to\frac{2\,d}{-2+d}\right\},\ \left\{p\to\frac{1+2\,d^2}{(-1+d)^2}\right\}\right\}$$

$$-\frac{1}{2}\sqrt{d\,(2+d)}$$

```
R2 = FullSimplify[(2 + d) (1/B[[2]] - 1/b[[1]])]
Solve[R2 == 0, p]
Limit[Sqrt[p - (2 d^2 + 1)/(d - 1)^2] D[R2, p], p → (2 d^2 + 1)/(d - 1)^2]
```

$$-1-\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}+p+\sqrt{(-1+p)\,\left(-1+d^2\,(-2+p)+p-2\,d\,p\right)}$$

$$\left\{\{p\to 1\},\ \left\{p\to\frac{2\,d}{-2+d}\right\},\ \left\{p\to\frac{1+2\,d^2}{(-1+d)^2}\right\}\right\}$$

$$\frac{1}{2}\sqrt{d\,(2+d)}$$