```
ϵ = 10^-7;
Off[NDSolve::ndinnt]
Off[NDSolve::ndnum]
Off[ReplaceAll::reps]
Off[FindRoot::cvmit]
Off[FindRoot::frmp]
Off[FindRoot::lstol]
Off[FindRoot::srect]
Off[FindRoot::nlnum]
Off[FindRoot::brmp]
Off[InterpolatingFunction::dmval]
Off[Plot::exclul]
```

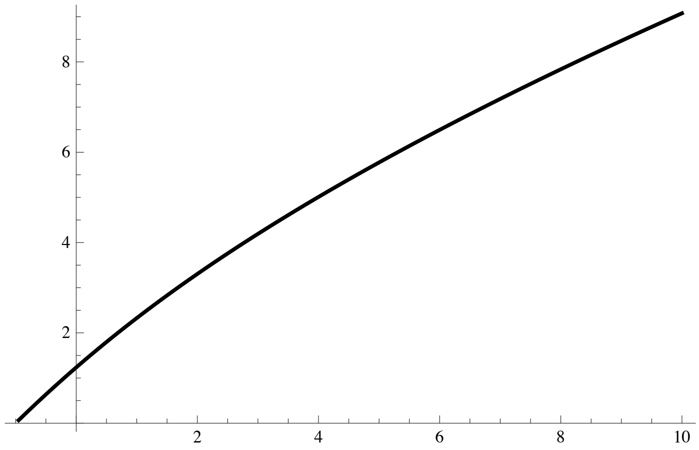
Black plots are reproduced in the article, while other plots are only intended for additional checks.

a) The case p>2: estimates of the optimal constant in the Keller-Lieb-Thirring inequality

The numerical computation of the optimal constant in the k=0 component (best uper estimate)

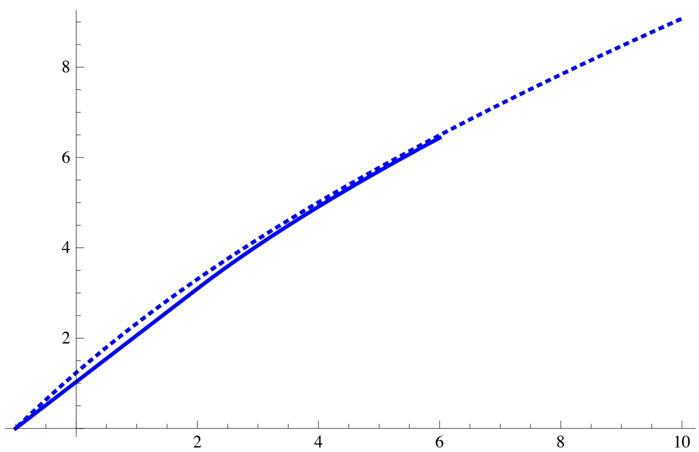
```
Fatir[amin_, amax_, λ_, p_, smax_] :=
 FindRoot[u[smax] /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 + λ) u[s] + Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {a, amin, amax}]
hh = 0.05;

LambdaaList = {{l, a} /. Fatir[0.01, 25, 10, 3, 6] /. l → 10}

For[j = 1, j < 220, j++, LambdaaList =
  Prepend[LambdaaList, {l, a} /. Fatir[0.01, LambdaaList[[1]][[2]], l, 3, 6] /.
    l → LambdaaList[[1]][[1]] - hh]]
{{10, 24.1391}}

Fmua[a_, λ_, p_, smax_] :=
 v[smax]^((p-2)/p) /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 + λ) u[s] + Abs[u[s]]^(p-2) u[s] == 0,
     v'[s] == s Abs[u[s]]^p, u[ϵ] == a, u'[ϵ] == 0, v[ϵ] == 0}, {u, u', v}, {s, ϵ, smax}][[1]]

LambdaaListVal = Table[
   {LambdaaList[[i]][[1]], Fmua[LambdaaList[[i]][[2]], LambdaaList[[i]][[1]], 3, 4]},
   {i, 1, Length[LambdaaList]}];
P0 = ListLinePlot[LambdaaListVal, PlotStyle → {Thick, Blue, Dotted}];
```

```
ListLinePlot[LambdaaListVal, PlotStyle → {Thick, Black}]
```

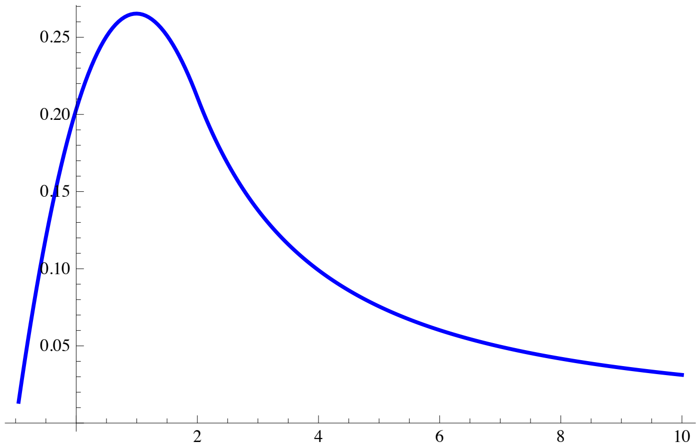


The lower estimate valid for an arbitrary magnetic field

```
FtirBNa[amin_, amax_, λ_, p_, smax_] :=
 FindRoot[u[smax] /. NDSolve[{u''[s] + u'[s]/s - λ u[s] + Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {a, amin, amax}]
FmuBNa[a_, λ_, p_, smax_] :=
 v[smax]^((p-2)/p) /. NDSolve[{u''[s] + u'[s]/s - λ u[s] + Abs[u[s]]^(p-2) u[s] == 0,
     v'[s] == s Abs[u[s]]^p, u[ϵ] == a, u'[ϵ] == 0, v[ϵ] == 0}, {u, u', v}, {s, ϵ, smax}][[1]]
Cp2a[amin_, amax_, λ_, p_, smax_] :=
 FmuBNa[a /. FtirBNa[amin, amax, λ, p, smax], λ, p, smax]

Sp2a[amin_, amax_, λ_, p_, smax_] := (2/p)^(2/p) (1 - 2/p)^(1-2/p) Cp2a[amin, amax, λ, p, smax]

EstimMina[amin_, amax_, p_, smax_, αmin_, αmax_, x_] :=
 Module[{K = {Sp2a[amin, amax, 1, p, smax], Cp2a[amin, amax, 1, p, smax] }},
  If[x < 2/(p - 2), K[[1]] (x + 1), K[[2]] x^(2/p)]]
Show[P0, Plot[EstimMina[1.2, 2.4, 3, 8, -1, 6, x], {x, -1, 6}, PlotStyle → {Thick, Blue}]]
```



```
Show[ListLinePlot[Table[{LambdaaListVal[[i]][[1]], LambdaaListVal[[i]][[2]] -
      EstimMina[1.2, 2.4, 3, 8, -1, 6, LambdaaListVal[[i]][[1]]]},
   {i, 1, Length[LambdaaListVal]}], PlotStyle → {Thick, Blue}, AxesOrigin → {0, 0}]]
```

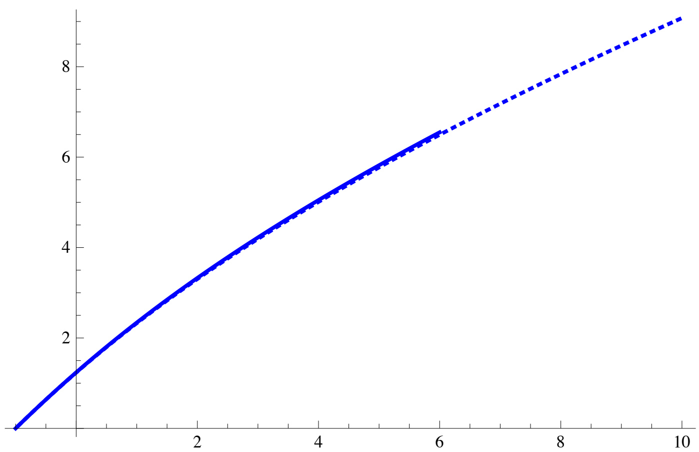


The upper estimate based on gaussian test functions (k=0 component)

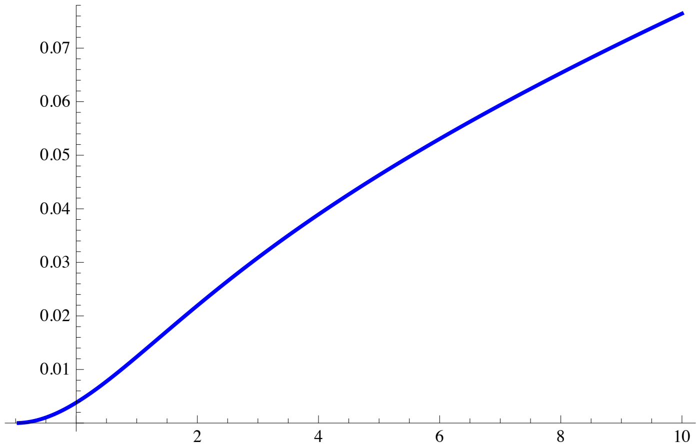
```
ha[a_, x_, p_] :=
```

$$ha[a\_, x\_, p\_] := \frac{\frac{1}{8}\left(4 + a^2\right) + x\,\frac{a}{2}}{\left(\frac{a}{p}\right)^{\frac{2}{p}}}$$

```
EstimMaxRad[x_, p_] :=
 Module[{M = FindRoot[FullSimplify[a^2 D[ha[a, x, p], a]] == 0, {a, 1.5}]}, ha[a, x, p] /. M]
Show[P0, Plot[EstimMaxRad[x, 3], {x, -1, 6}, PlotStyle → {Thick, Blue}]]
```



```
Show[ListLinePlot[Table[{LambdaaListVal[[i]][[1]],
    EstimMaxRad[LambdaaListVal[[i]][[1]], 3] - LambdaaListVal[[i]][[2]]},
   {i, 1, Length[LambdaaListVal]}], PlotStyle → {Thick, Blue}, AxesOrigin → {0, 0}]]
```

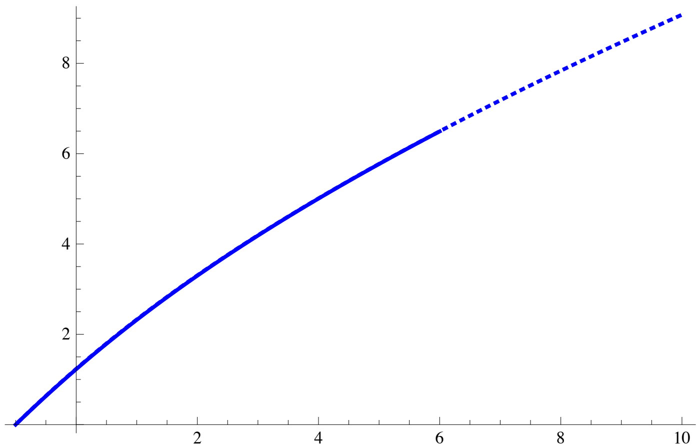


The lower estimate for a constant magnetic field based on the Loss-Thaller lemma

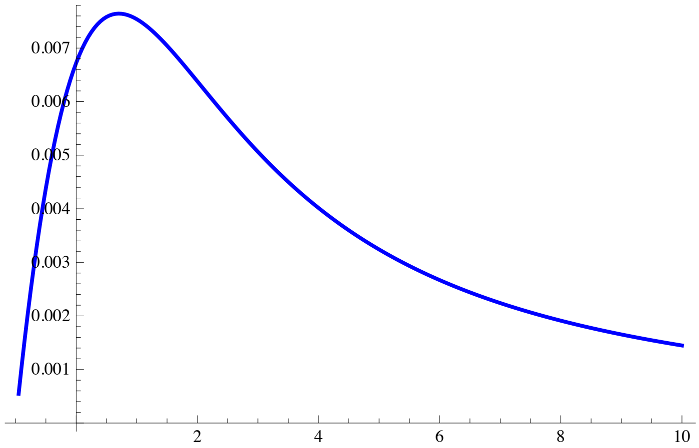
```
FLmua[amin_, amax_, p_, smax_, α_] := Cp2a[amin, amax, 1, p, smax]
```

$$\left(\alpha+\frac{\alpha-\frac{p\,\alpha}{2}+\sqrt{-1+p+\frac{1}{4}\,(-2+p)^2\,\alpha^2}}{-1+p}\right)^{2/p}\left(1-\frac{\left(\alpha-\frac{p\,\alpha}{2}+\sqrt{-1+p+\frac{1}{4}\,(-2+p)^2\,\alpha^2}\right)^2}{(-1+p)^2}\right)^{\frac{-2+p}{p}}$$

```
Show[P0, Plot[FLmua[1.2, 2.4, 3, 8, x], {x, -1, 6}, PlotStyle → {Thick, Blue}]]
```

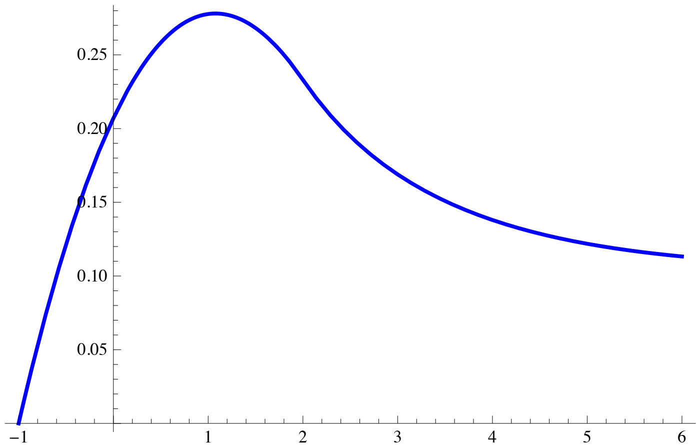


```
Show[ListLinePlot[Table[{LambdaaListVal[[i]][[1]],
    LambdaaListVal[[i]][[2]] - FLmua[1.2, 2.4, 3, 8, LambdaaListVal[[i]][[1]]]},
   {i, 1, Length[LambdaaListVal]}], PlotStyle → {Thick, Blue}, AxesOrigin → {0, 0}]]
```



Comparisons

```
Plot[EstimMaxRad[x, 3] - EstimMina[1.2, 2.4, 3, 8, -1, 6, x],
 {x, -1, 6}, PlotStyle → {Thick, Blue}]
```

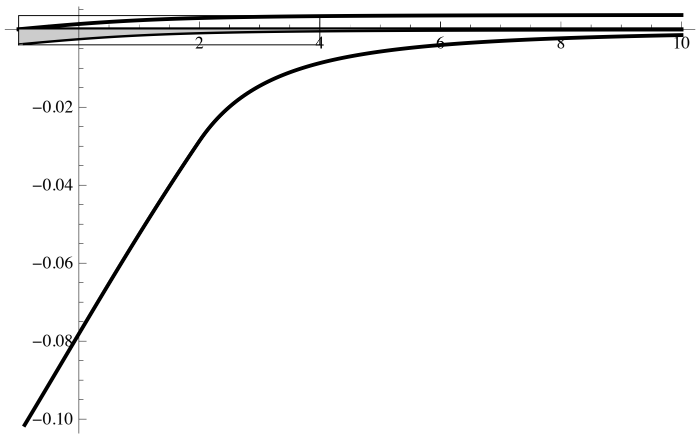



```
P1 = ListLinePlot[Prepend[Table[
    {LambdaaListVal[[i]][[1]], Log[10, EstimMaxRad[LambdaaListVal[[i]][[1]], 3] / LambdaaListVal[[i]][[2]]]},
    {i, 2, Length[LambdaaListVal]}], {-1, 0}], PlotStyle → {Thick, Black}];

P2 = ListLinePlot[Table[{LambdaaListVal[[i]][[1]],
    Log[10, EstimMina[1.2, 2.4, 3, 8, -1, 6, LambdaaListVal[[i]][[1]]] / LambdaaListVal[[i]][[2]]]},
   {i, 2, Length[LambdaaListVal]}], PlotStyle → {Thick, Black}];

P3 = ListLinePlot[Table[{LambdaaListVal[[i]][[1]],
    Log[10, FLmua[1.2, 2.4, 3, 8, LambdaaListVal[[i]][[1]]] / LambdaaListVal[[i]][[2]]]},
   {i, 2, Length[LambdaaListVal]}], PlotStyle → {Thick, Black}];
```

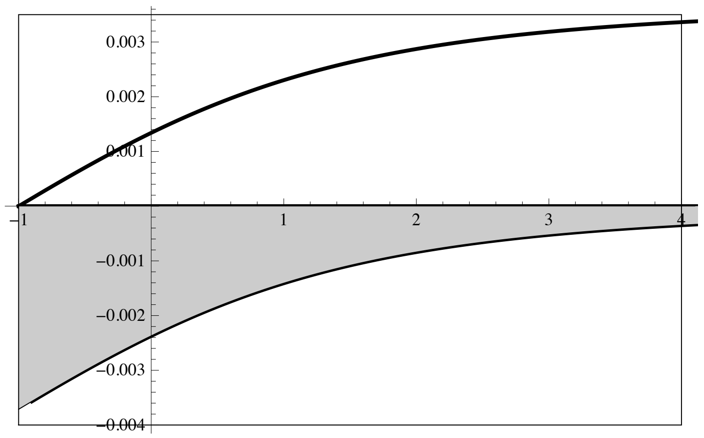
```
Interv = Module[{M = Table[{LambdaaListVal[[i]][[1]],
      Log[10, FLmua[1.2, 2.4, 3, 8, LambdaaListVal[[i]][[1]]] / LambdaaListVal[[i]][[2]]]},
     {i, 2, Length[LambdaaListVal]}]}, ListLinePlot[
   Prepend[M, {-1, 3 M[[1]][[2]] - 2 M[[2]][[2]]}], Filling → Axis,
   FillingStyle → GrayLevel[0.8], PlotStyle → Black]];
```

```
P4 = Plot[0, {x, -1, 10}, PlotStyle → {Thick, Black}];
PBox = ListLinePlot[{{-1, -0.004}, {4, -0.004},
    {4, 0.0035}, {-1, 0.0035}, {-1, -0.004}}, PlotStyle → Black];
```

```
Show[{P1, P2, P3, P4, Interv, PBox}, PlotRange → All]
```




```
Show[P1, P3, P4, Interv, PBox, PlotRange → {{-1, 4}, {-0.004, 0.0035}}]
```



b) The case p∈(1,2): estimates of the optimal constant in the Keller-Lieb-Thirring inequality

The computation of the optimal constant in the Gagliardo-Nirenberg inequality

```
FbPTirMinBN[a_, λ_, p_, smax_] :=
 FindMinimum[u[t]^2 + u'[t]^2 /. NDSolve[{u''[s] + u'[s]/s + λ u[s] - Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {t, 2, smax}]
```

```
FbPTirMinBN[a_, λ_, p_, smax_] :=
 FindMinimum[10 u[t]^2 + u'[t]^2 /. NDSolve[{u''[s] + u'[s]/s + λ u[s] - Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {t, 2, smax}]
```

```
η = 10^-8;
IterBN[λ_, p_, smax_, a_, h_, b_] := Module[{M = FbPTirMinBN[a + h, λ, p, smax]},
  If[Sqrt[M[[1]]] < η || Abs[h] < η, {λ, a, t /. M[[2]]}, If[M[[1]] < b,
    IterBN[λ, p, smax, a + h, h, M[[1]]], IterBN[λ, p, smax, a + h, -h / 2, M[[1]]]]]]
FbMinBN[λ_, p_, smax_, a_, h_] := IterBN[λ, p, smax, a, h,
  FbPTirMinBN[a, λ, p, smax][[1]]]
```

```
NbBN[λ_, p_, smax_, a_] :=
 v[smax] /. NDSolve[{u''[s] + u'[s]/s + λ u[s] - Abs[u[s]]^(p-2) u[s] == 0, u[ϵ] == a,
     u'[ϵ] == 0, v'[s] == s Abs[u[s]]^p, v[ϵ] == 0}, {u, u', v}, {s, ϵ, smax}][[1]]
Cp2b[λ_, p_, rmax_, a_, h_] := Module[{M = FbMinBN[λ, p, rmax, a, h]},
  λ NbBN[λ, p, M[[3]], M[[2]]]^(1-p/2)]
```

Computation of the solution in the k=0 component, with compact support

```
η = 10^-8;
```

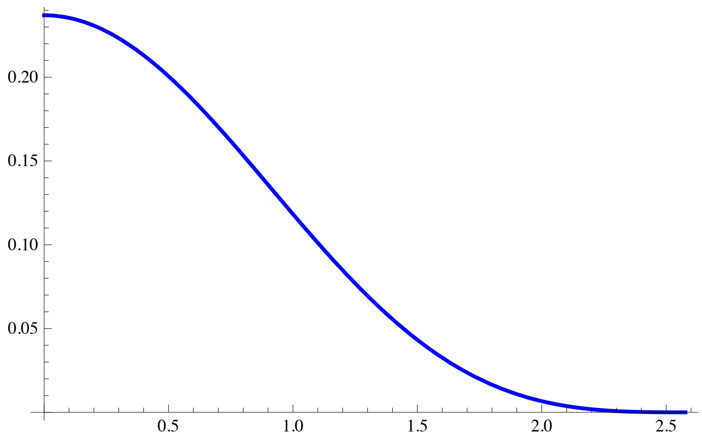
```
FbPTirMin[a_, λ_, p_, smax_] := FindMinimum[
  10 u[t]^2 + u'[t]^2 /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 - λ) u[s] - Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {t, 2, smax}]
```

```
Iter[λ_, p_, smax_, a_, h_, b_] := Module[{M = FbPTirMin[a + h, λ, p, smax]},
  If[Sqrt[M[[1]]] < η || Abs[h] < η, {λ, a, t /. M[[2]]}, If[M[[1]] < b,
    Iter[λ, p, smax, a + h, h, M[[1]]], Iter[λ, p, smax, a + h, -h / 2, M[[1]]]]]]
FbMin[λ_, p_, smax_, a_, h_] := Iter[λ, p, smax, a, h, FbPTirMin[a, λ, p, smax][[1]]]
```

```
FbPlot[a_, λ_, p_, smax_] :=
 Plot[u[t] /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 - λ) u[s] - Abs[u[s]]^(p-2) u[s] == 0, u[ϵ] == a,
     u'[ϵ] == 0}, {u, u'}, {s, ϵ, smax}], {t, ϵ, smax}, PlotStyle → {Thick, Blue}]
```

```
FbMin[5, 1.4, 5, 0.1, 0.1]
FbPlot[%[[2]], 5, 1.4, %[[3]]]
```

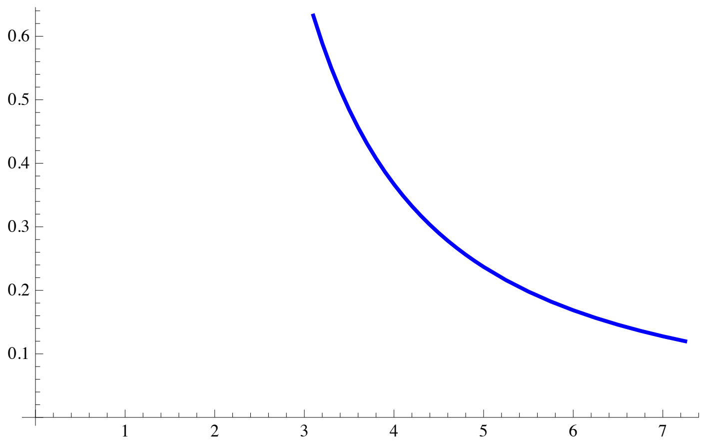
{5, 0.236972, 2.5774}



```
jmax = 20;
ResList = {FbMin[5, 1.4, 5, 0.1, 0.1]};
For[j = 1, j < jmax, j++, ResList =
  Join[{FbMin[5 - 0.1 j, 1.4, 5, ResList[[Length[ResList]]][[2]], -0.01]}, ResList]]
```

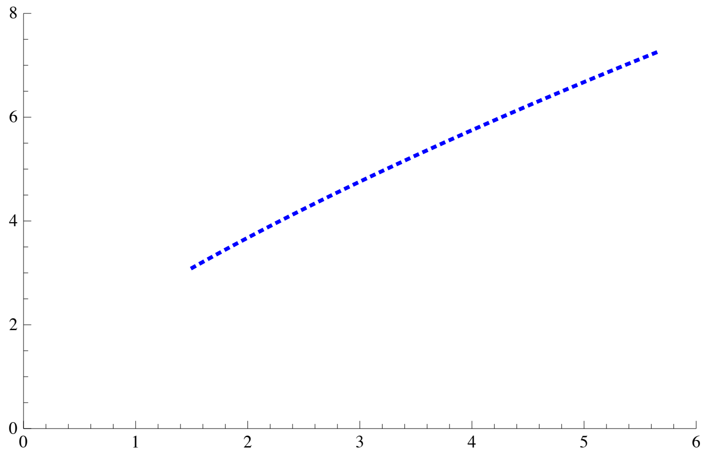
```
jmax = 10;
For[j = 1, j < jmax, j++, ResList =
   Append[ResList, FbMin[5 + 0.25 j, 1.4, 5, ResList[[Length[ResList]]][[2]], 0.01]]];
```

```
ParamTirb = ListLinePlot[Table[Take[ResList[[j]], 2], {j, 1, Length[ResList]}],
  AxesOrigin → {0, 0}, PlotStyle → {Blue, Thick}]
```

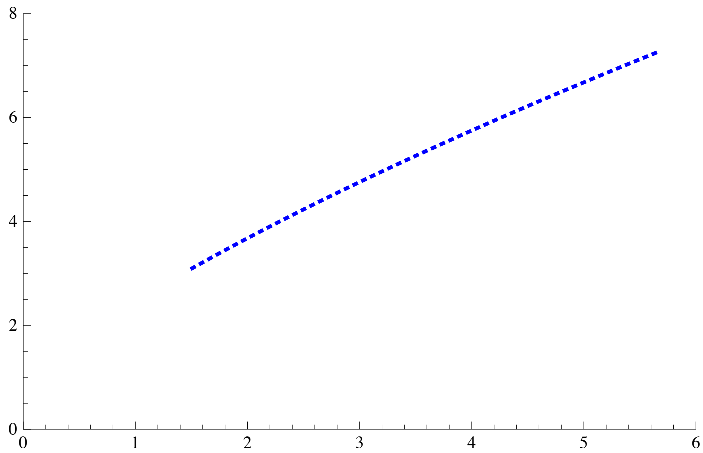


```
QuotientFbMinBeta[λ_, a_, smax_, p_] :=
 {w2[smax]^(1-2/p), λ} /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 - λ) u[s] - Abs[u[s]]^(p-2) u[s] == 0,
     u[ϵ] == a, u'[ϵ] == 0, w1'[s] == s u'[s]^2, w2'[s] == s Abs[u[s]]^p, w3'[s] == s u[s]^2,
     w1[ϵ] == 0, w2[ϵ] == 0, w3[ϵ] == 0}, {u, u', w1, w2, w3}, {s, ϵ, smax}][[1]]
```

```
QuotientTirbBlue =
 ListLinePlot[Table[QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]],
    ResList[[j]][[3]], 1.4], {j, 1, Length[ResList]}], AxesOrigin → {0, 0},
  PlotStyle → {Blue, Thick, Dotted}, PlotRange → {{0, 6}, {0, 8}}]
```




```
Show[QuotientTirbBlue,
 Plot[estimb[5, 1.4, 5, 0.1, 0.1, x], {x, 0, 6}, PlotStyle → {Blue, Thick}]]
```



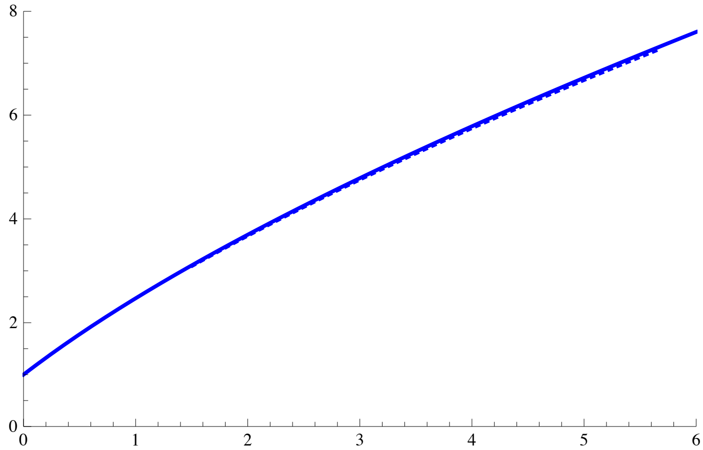
An estimate based on Gaussian functions

```
h[a_, x_, p_] := (1/8 (4 + a^2) + x (a/p)^(2/p)) / (a/2)
```

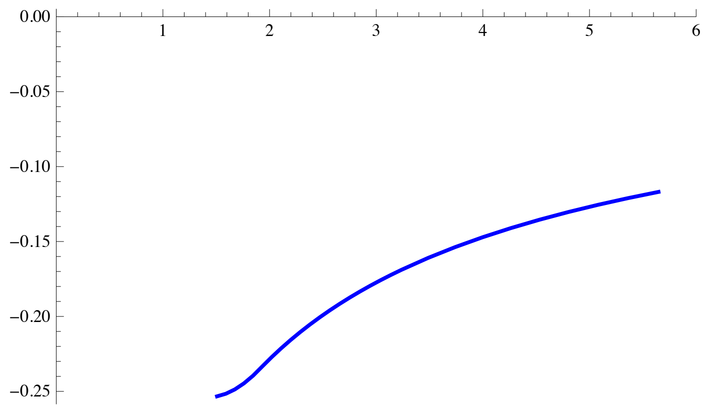
```
FullSimplify[a^2 D[h[a, x, p], a]]
GaussianEstm[x_, p_] :=
 Module[{M = FindRoot[-1 + a^2/4 - (2 (-2 + p) (a/p)^(2/p) x)/p == 0, {a, 1.5}]}, h[a, x, p] /. M]
```

$$-1+\frac{a^2}{4}-\frac{2\,(-2+p)\left(\frac{a}{p}\right)^{2/p}x}{p}$$

```
Show[QuotientTirbBlue, Plot[GaussianEstm[x, 1.4], {x, 0, 6}, PlotStyle → {Blue, Thick}]]
```




```
Elt1[j_] := Module[{M =
     QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], GaussianEstm[M[[1]], 1.4] - M[[2]]}]; ListLinePlot[
  Table[Elt2[j], {j, 1, Length[ResList]}], AxesOrigin → {0, 0},
  PlotStyle → {Blue, Thick}, PlotRange → {{0, 6}, Automatic}]
```

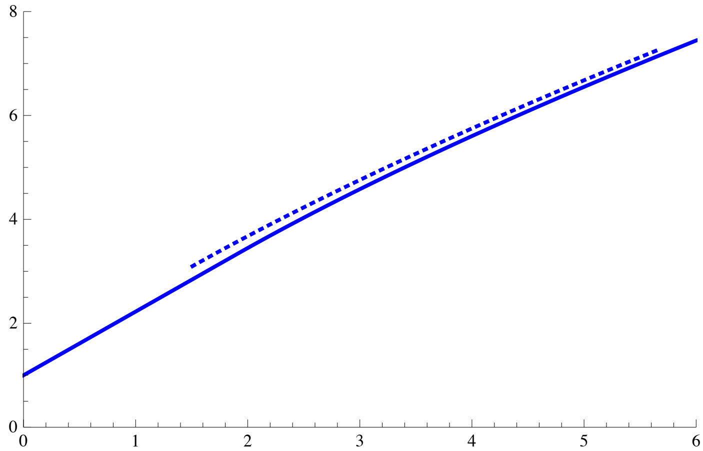



```
LogElt1[j_] := Module[{M =
     QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], Log[10, GaussianEstm[M[[1]], 1.4]/M[[2]]]}];
```

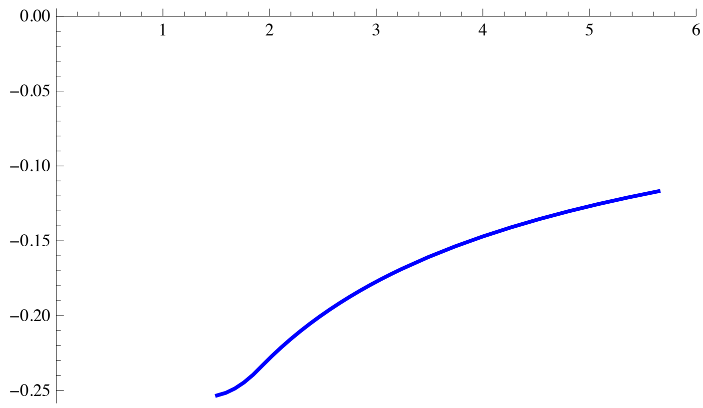
The general lower estimate

```
Pkb[λ_, p_, rmax_, a_, h_, βmax_, Color_] :=
 Module[{M = Cp2b[λ, p, rmax, a, h]}, Plot[If[x < (2/(2 - p))^(2/p) M^(-2/p), 1 + p/2 (1 - p/2)^(2/p - 1) M^(2/p) x, M x^(p/2)],
   {x, 0, βmax}, PlotStyle → {Color, Thick}, AxesOrigin → {0, 0}]]
```

```
Show[QuotientTirbBlue, Pkb[5, 1.4, 5, 0.1, 0.1, 6, Blue]]
```



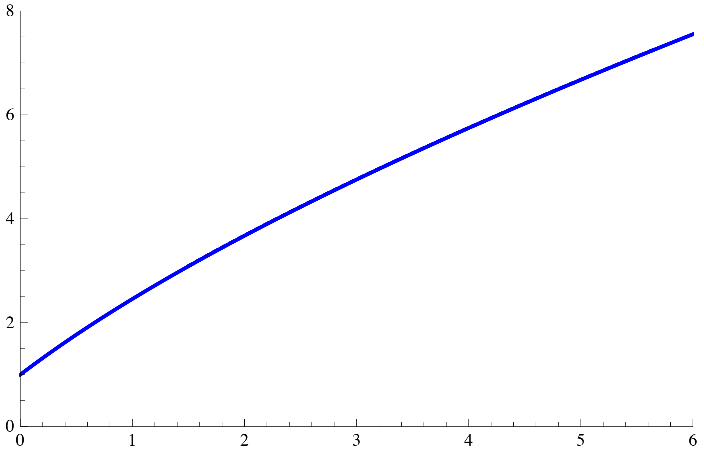

```
estim1b[λ_, p_, rmax_, a_, h_, x_] := 1 + p/2 (1 - p/2)^(2/p - 1) Cp2b[λ, p, rmax, a, h]^(2/p) x

estim2b[λ_, p_, rmax_, a_, h_, x_] := Cp2b[λ, p, rmax, a, h] x^(p/2)

estimb[λ_, p_, rmax_, a_, h_, x_] := Module[{M = (2/(2 - p))^(2/p) Cp2b[λ, p, rmax, a, h]^(-2/p)},
  If[x < M, estim1b[λ, p, rmax, a, h, x], estim2b[λ, p, rmax, a, h, x]]]

Elt2[j_] := Module[{M =
     QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], estimb[5, 1.4, 5, 0.1, 0.1, M[[1]]] - M[[2]]}];
ListLinePlot[Table[Elt2[j], {j, 1, Length[ResList]}], AxesOrigin → {0, 0},
 PlotStyle → {Blue, Thick}, PlotRange → {{0, 6}, Automatic}]
```



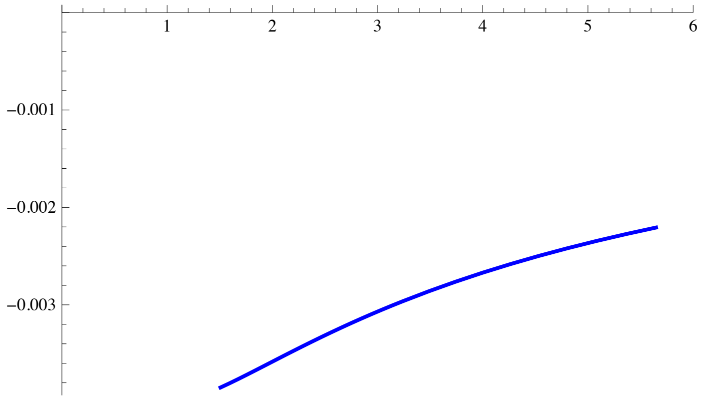

```
LogElt2[j_] := Module[{M =
     QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], Log[10, estimb[5, 1.4, 5, 0.1, 0.1, M[[1]]] / M[[2]]]}];
```

The lower estimate for a constant magnetic field based on the Loss-Thaller lemma

```
Fn2[p_, β_, Cpvar_] :=
  c + (1 - c^2)^(1-p/2) Cpvar β^(p/2) /. FindRoot[c (1 - c^2)^(-p/2) == 1/((2 - p) Cpvar β^(p/2)), {c, 0.5}]
Pb2[λ_, p_, rmax_, a_, h_, βmax_, Color_] := Module[{CpCp = Cp2b[λ, p, rmax, a, h]},
   Plot[Fn2[p, β, CpCp], {β, 0, βmax}, PlotStyle → {Thick, Color}]]
Show[QuotientTirbBlue, Pb2[5, 1.4, 5, 0.1, 0.1, 6, Blue]]
```




```
Elt3[j_] := Module[{M =
     QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], Fn2[1.4, M[[1]], Cp2b[5, 1.4, 5, 0.1, 0.1]] - M[[2]]}];
ListLinePlot[Table[Elt3[j], {j, 1, Length[ResList]}], AxesOrigin → {0, 0},
 PlotStyle → {Blue, Thick}, PlotRange → {{0, 6}, Automatic}]
```



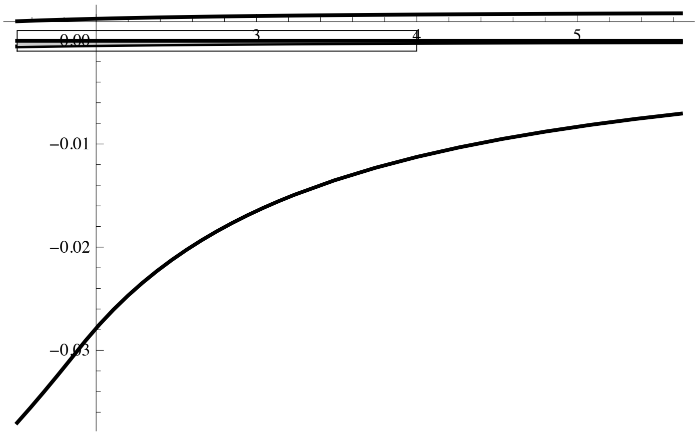

```
LogElt3[j_] := Module[{M =
      QuotientFbMinBeta[ResList[[j]][[1]], ResList[[j]][[2]], ResList[[j]][[3]], 1.4]},
   {M[[1]], Log[10, Fn2[1.4, M[[1]], Cp2b[5, 1.4, 5, 0.1, 0.1]] / M[[2]]]}];
```

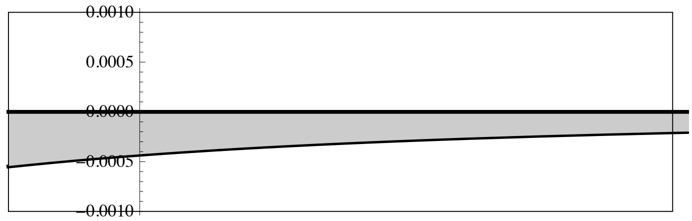
Comparisons

```
P1 =
  ListLinePlot[Table[LogElt1[i], {i, 1, Length[ResList]}], PlotStyle → {Thick, Black}];
P2 =
  ListLinePlot[Table[LogElt2[i], {i, 1, Length[ResList]}], PlotStyle → {Thick, Black}];
P3 =
  ListLinePlot[Table[LogElt3[i], {i, 1, Length[ResList]}], PlotStyle → {Thick, Black}];
Interv = ListLinePlot[Table[LogElt3[i], {i, 1, Length[ResList]}],
   Filling → Axis, FillingStyle → GrayLevel[0.8], PlotStyle → Black];
```

```
P4 = Plot[0, {x, LogElt1[1][[1]], LogElt1[Length[ResList]][[1]]},
   PlotStyle → {Thick, Black}];
PBox = ListLinePlot[{{LogElt1[1][[1]], -0.001}, {4, -0.001}, {4, 0.001},
    {LogElt1[1][[1]], 0.001}, {LogElt1[1][[1]], -0.001}}, PlotStyle → Black];
```

```
Show[{P1, P2, P3, Interv, P4, PBox}, PlotRange → All]
Show[%, PlotRange → {{LogElt1[1][[1]], 4}, {-0.001, 0.001}}, AspectRatio → 0.3]
```

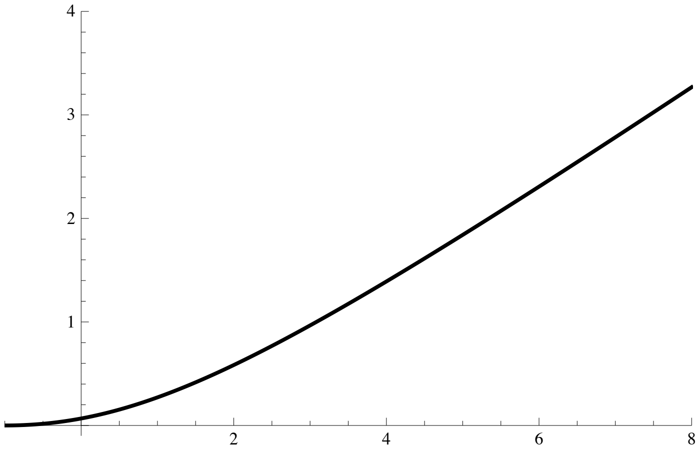




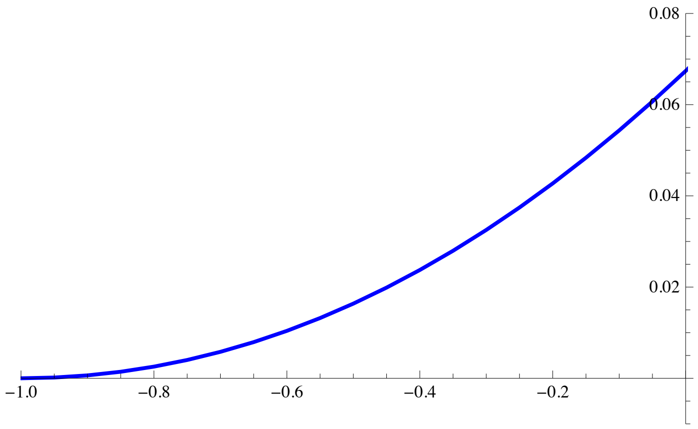
c) Computation of perturbation corresponding to the excited states (k=1 or higher)

```
FtirEv[numin_, numax_, j_, k_, p_, smax_] :=
 Module[{λ = LambdaaList[[j]][[1]], a = LambdaaList[[j]][[2]]},
  FindRoot[w[smax] /. NDSolve[{u''[s] + u'[s]/s - (s^2/4 + λ) u[s] + Abs[u[s]]^(p-2) u[s] == 0,
       w''[s] + w'[s]/s - (s^2/4 + k^2/s^2 - k + λ - ν) w[s] + p/2 Abs[u[s]]^(p-2) w[s] == 0, u[ϵ] == a, u'[ϵ] ==
        0, w[ϵ] == ϵ^k, w'[ϵ] == k ϵ^(k-1)}, {u, u', w, w'}, {s, ϵ, smax}], {ν, numin, numax}]]
```

```
ListLinePlot[Prepend[
   Table[{LambdaaList[[j]][[1]], ν /. FtirEv[-0.5, 0, j, 1, 3, 6]}, {j, 1, 180}], {-1, 0}],
  PlotStyle → {Black, Thick}, PlotRange → {{-1, 8}, {-0.1, 4}}]
```




```
Show[ListLinePlot[Prepend[
    Table[{LambdaaList[[j]][[1]], ν /. FtirEv[-0.5, 0, j, 1, 3, 6]}, {j, 1, 21}], {-1, 0}],
   PlotStyle → {Blue, Thick}, PlotRange → {{-1, 8}, {-0.1, 4}}],
  PlotRange → {{-1, 0}, {-0.01, 0.08}}]
```




```
Show[%, PlotRange → {{-1, -0.8}, {-0.001, 0.005}}]
```