Reverse Hardy-Littlewood-Sobolev inequalities

```
Off[NIntegrate::nlim]
Off[NIntegrate::inumr]
```

Definitions

$$Z = \texttt{Integrate}\left[\texttt{Sin}[\phi]^{N-2}, \{\phi, 0, \pi\}, \texttt{Assumptions} \to N > 1\right]$$

$$\texttt{FullSimplify}\left[\frac{1}{Z} \texttt{Integrate}\left[(1 - t\,\texttt{Cos}[\phi])^{\frac{\lambda}{2}} \texttt{Sin}[\phi]^{N-2}, \{\phi, 0, \pi\}, \texttt{Assumptions} \to N > 1\,\&\&\,t > 0\,\&\&\,t < 1\right]\right]$$

$$\frac{\sqrt{\pi}\ \texttt{Gamma}\left[\frac{1}{2}(-1+N)\right]}{\texttt{Gamma}\left[\frac{N}{2}\right]}$$

$$(1+t)^{\lambda/2}\,\texttt{Hypergeometric2F1}\left[\frac{1}{2}(-1+N), -\frac{\lambda}{2}, -1+N, \frac{2t}{1+t}\right]$$

$$j[N\_, \lambda\_, t\_] := (1+t)^{\lambda/2}\,\texttt{Hypergeometric2F1}\left[\frac{1}{2}(-1+N), -\frac{\lambda}{2}, -1+N, \frac{2t}{1+t}\right]$$

$$F[N\_, \lambda\_] := \frac{N(N+\lambda)}{2}\texttt{NIntegrate}\left[r^{N-1} s^{N-1} \left(\left(r^2+s^2\right)\right)^{\frac{\lambda}{2}} j\left[N, \lambda, \frac{2 r s}{r^2+s^2}\right], \{r, 0, 1\}, \{s, 0, 1\}\right]$$

$$G[N\_, \lambda\_] := \texttt{Module}\left[\left\{M = \frac{1}{2 F[N, \lambda]}\right\}, \frac{2N(1-M)}{\lambda + 2N(1-M)}\right]$$

The optimal case is achieved for R=1

```
FR[N_, λ_, R_] :=
```

$$\frac{N\ (N+\lambda)}{R^{N}+R^{N+\lambda}}\ \texttt{NIntegrate}\left[r^{N-1}\ s^{N-1}\ \left(\left(r^{2}+s^{2}\right)\right)^{\frac{\lambda}{2}}\ j\left[N,\ \lambda,\ \frac{2\ r\ s}{r^{2}+s^{2}}\right],\ \{r,\ 0,\ 1\},\ \{s,\ 0,\ R\}\right]$$

```
Tbl = Table[{λ, FindMaximum[FR[4, λ, R], {R, 1, 5}]}, {λ, 2.25, 12, 0.25}]
```

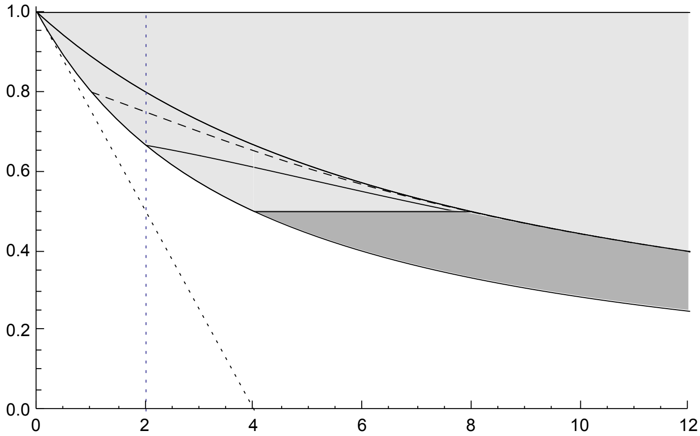
```
{{2.25, {1.10413, {R → 1.}}}, {2.5, {1.22177, {R → 1.}}},
 {2.75, {1.35474, {R → 1.}}}, {3., {1.50511, {R → 1.}}}, {3.25, {1.67523, {R → 1.}}},
 {3.5, {1.86783, {R → 1.}}}, {3.75, {2.08602, {R → 1.}}},
 {4., {2.33333, {R → 1.}}}, {4.25, {2.61386, {R → 1.}}}, {4.5, {2.93226, {R → 1.}}},
 {4.75, {3.29388, {R → 1.}}}, {5., {3.70487, {R → 1.}}}, {5.25, {4.17228, {R → 1.}}},
 {5.5, {4.70418, {R → 1.}}}, {5.75, {5.30986, {R → 1.}}},
 {6., {6., {R → 1.}}}, {6.25, {6.78686, {R → 1.}}}, {6.5, {7.68453, {R → 1.}}},
 {6.75, {8.70925, {R → 1.}}}, {7., {9.87966, {R → 1.}}}, {7.25, {11.2173, {R → 1.}}},
 {7.5, {12.7468, {R → 1.}}}, {7.75, {14.4968, {R → 1.}}},
 {8., {16.5, {R → 1.}}}, {8.25, {18.7944, {R → 1.}}}, {8.5, {21.4235, {R → 1.}}},
 {8.75, {24.4379, {R → 1.}}}, {9., {27.8955, {R → 1.}}}, {9.25, {31.8635, {R → 1.}}},
 {9.5, {36.4194, {R → 1.}}}, {9.75, {41.6527, {R → 1.}}},
 {10., {47.6667, {R → 1.}}}, {10.25, {54.5809, {R → 1.}}},
 {10.5, {62.5336, {R → 1.}}}, {10.75, {71.6845, {R → 1.}}},
 {11., {82.2184, {R → 1.}}}, {11.25, {94.3492, {R → 1.}}},
 {11.5, {108.324, {R → 1.}}}, {11.75, {124.431, {R → 1.}}}, {12., {143., {R → 1.}}}}
```

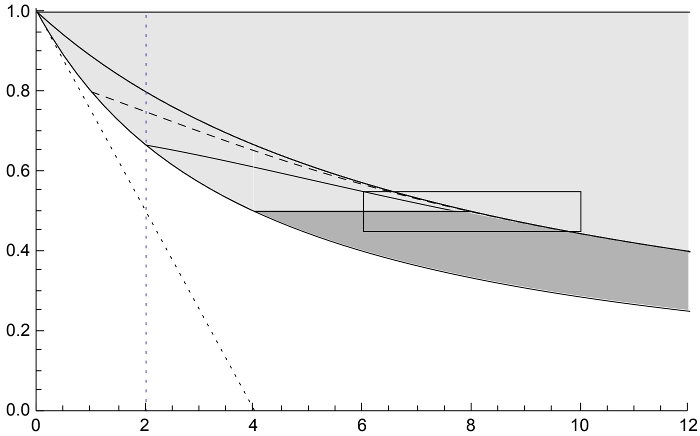
The figure and an enlargement in dimension N=4

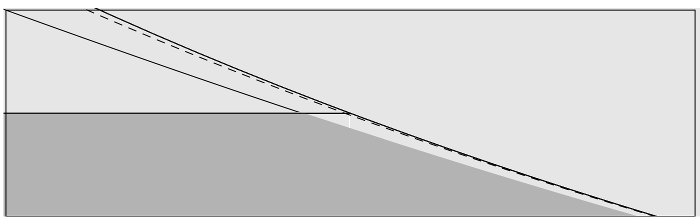
```
P[N_] := Show[Plot[1 - λ / N, {λ, 0, N},
   PlotStyle → {Black, Dotted}, PlotRange → {{0, 3 N}, {0, 1.01}}],
  Plot[{Min[Max[N/(λ + N), 1 - 2/N], 2 N/(λ + 2 N)], 2 N/(λ + 2 N)}, {λ, 0, 3 N},
   Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9], PlotStyle → Black],
  Plot[G[N, λ], {λ, N/2 + 1, 12}, PlotStyle → Black],

  Plot[{G[N, λ], Min[1 - 2/N, 2 N/(λ + 2 N)]}, {λ, 2 N/(N - 2), 3 N}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.9], PlotStyle → Black], Plot[{1, 2 N/(λ + 2 N)}, {λ, 0, 3 N},
   Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9], PlotStyle → Black],
  Plot[{Min[G[N, λ], 1 - 2/N], N/(λ + N)}, {λ, 2 N/(N - 2), 3 N}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.7], PlotStyle → None],
  Plot[{Min[G[N, λ], 1 - 2/N], N/(λ + N)}, {λ, 7.8, 8.2}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.7], PlotStyle → None],
  Plot[2 N (1 - 2^-λ)/(λ + 2 N (1 - 2^-λ)), {λ, 1, 3 N}, PlotStyle → {Black, Dashed}],
  Plot[{(N - 2)/N}, {λ, 2 N/(N - 2), 4 N/(N - 2)}, PlotStyle → Black],
  Plot[{N/(λ + N), 2 N/(λ + 2 N)}, {λ, 0, 3 N}, PlotStyle → Black],
  ListLinePlot[{{2, 0}, {2, 1}}, PlotStyle → Dotted],
  Plot[G[N, λ], {λ, 2, N/2 + 1}, PlotStyle → Black]]
```

```
P[4]
Show[%, ListLinePlot[
  {{6, 0.45}, {10, 0.45}, {10, 0.55}, {6, 0.55}, {6, 0.45}}, PlotStyle → Black]]
Show[%, PlotRange → {{6, 10.01}, {0.4495, 0.55}}, AspectRatio → 3 / 10]
```

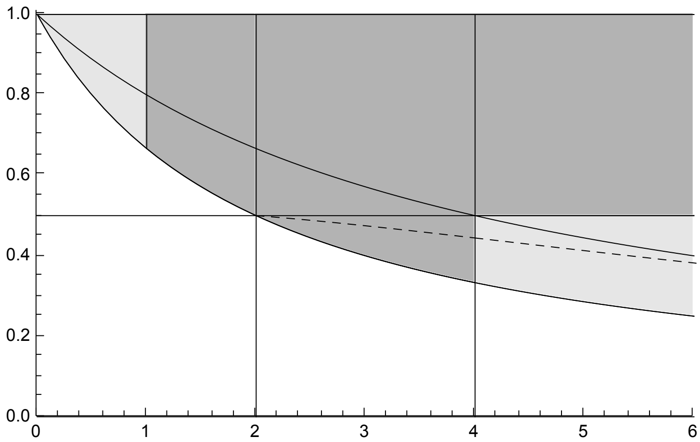




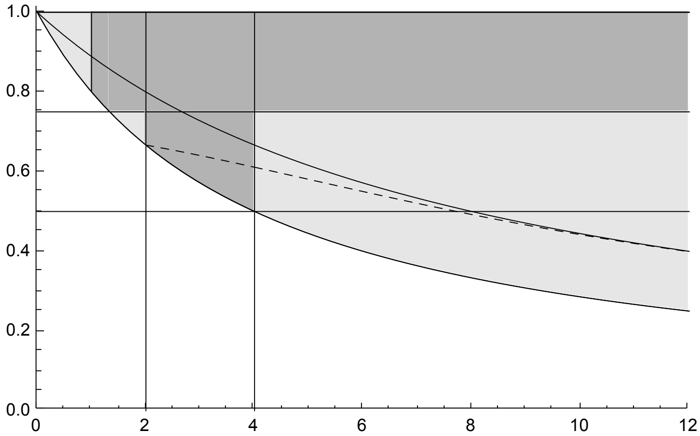


```
P[N_] := Show[
  Plot[{N/(λ + N), 1}, {λ, 0, 3 N}, PlotStyle → Black, PlotRange → {{0, 3 N}, {0, 1.01}},
   Filling → {1 → {2}}, FillingStyle → GrayLevel[0.9]],
  Plot[(N - 1)/N, {λ, 0, 3 N}, PlotStyle → Black], Plot[{1, Max[N/(λ + N), (N - 1)/N]},
   {λ, 1, 3 N}, PlotStyle → Black, PlotRange → {{0, 3 N}, {0, 1.01}},
   Filling → {1 → {2}}, FillingStyle → GrayLevel[0.7]],
  Plot[{N/(λ + N), (N - 1)/N}, {λ, 2, 4}, PlotStyle → Black,
   PlotRange → {{0, 3 N}, {0, 1.01}}, Filling → {1 → {2}},
   FillingStyle → GrayLevel[0.7]], Plot[{N/(λ + N), 2 N/(λ + 2 N), 1, (N - 2)/N},
   {λ, 0, 3 N}, PlotStyle → Black, PlotRange → {{0, 3 N}, {0, 1.01}}],
  ListLinePlot[{{1, N/(N + 1)}, {1, 1}}, PlotStyle → Black],
  ListLinePlot[{{2, 0}, {2, 1}}, PlotStyle → Black],
  ListLinePlot[{{4, 0}, {4, 1}}, PlotStyle → Black],
  Plot[G[N, λ], {λ, 2, 4}, PlotStyle → {Black, Dashed}],
  Plot[G[N, λ], {λ, 4, 12}, PlotStyle → {Black, Dashed}]]
```

```
P[2]
```



```
P[4]
```




```
Show[P[10], PlotRange → {{0, 12}, All}]
```