The range of the solutions

In[∘]:=

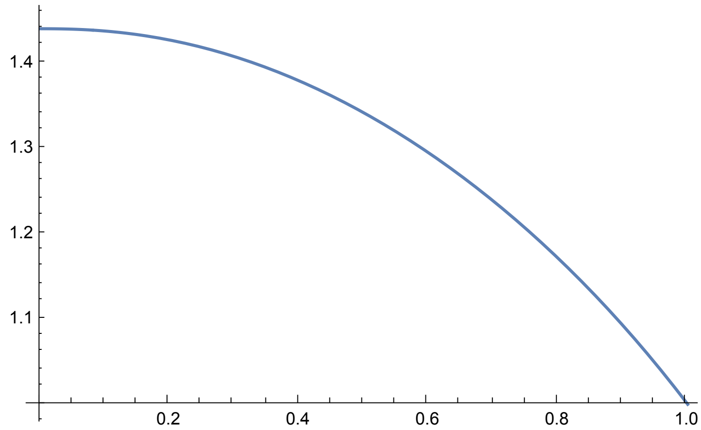
$$ac[p\_, q\_] := \left(\frac{q}{p}\right)^{\frac{1}{q-p}}$$

$$En[a\_, p\_, q\_] := \frac{a^q}{q} - \frac{a^p}{p}$$

```
b[a_, p_, q_] := b /. FindRoot[En[b, p, q] - En[a, p, q], {b, 1, ac[p, q]}]

Plot[b[a, 2.5, 3], {a, 0, 1}]
```

Out[∘]=



Period and integral quantities

In[∘]:=

$$df[a\_, f\_, p\_, q\_] := \left(\frac{p}{p-1}\,(En[a, p, q] - En[f, p, q])\right)^{\frac{1}{p}}$$

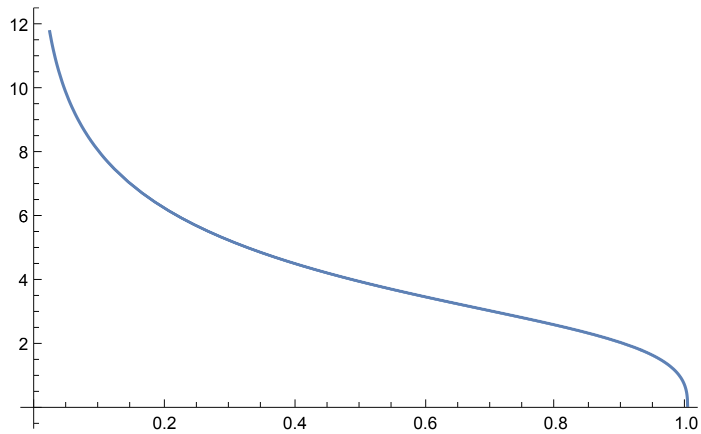
$$F[a\_, p\_, q\_] :=$$

$$2\,NIntegrate\left[\left\{\frac{1}{df[a, f, p, q]},\ df[a, f, p, q]^{p-1},\ \frac{f^p}{df[a, f, p, q]},\ \frac{f^q}{df[a, f, p, q]}\right\},\ \{f, a, b[a, p, q]\}\right]$$

In[∘]:=

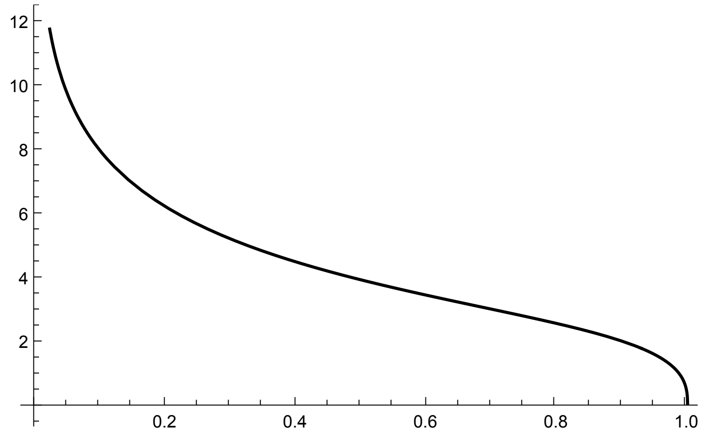
```
Plot[F[a, 3, 5][[1]], {a, 0, 1}]
```

Out[∘]=



In[•]:=
```
Plot[F[a, 3, 5][[1]], {a, 0, 1}, PlotStyle → Black]
```

Out[•]=



The constants λ_1 and λ_1^*

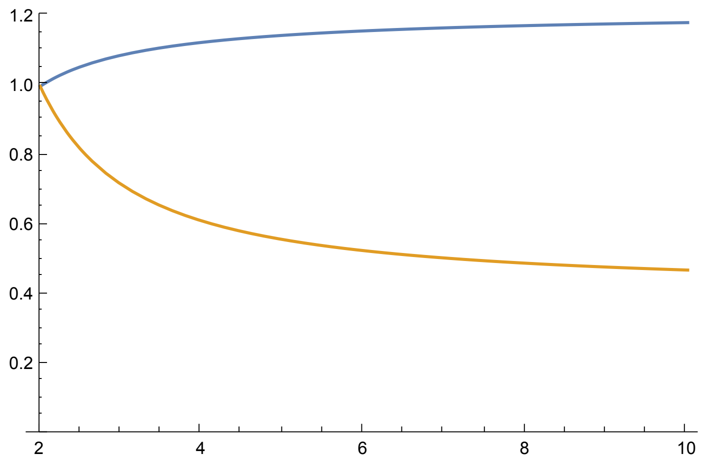
In[•]:=
$$\texttt{λ1[p\_] := NIntegrate}\left[\frac{2}{\pi}\left(\frac{2}{p}\,\frac{p-1}{1-x^p}\right)^{\frac{1}{p}},\ \{x, 0, 1\}\right]^2$$

$$\texttt{λstar[p\_] :=}$$

$$\frac{4}{\pi^2}\,\texttt{NIntegrate}\left[\left(\frac{2}{p}\,\frac{p-1}{1-x^2}\right)^{\frac{1}{p}-1},\ \{x, 0, 1\}\right]^{\frac{2}{p}-1}\texttt{NIntegrate}\left[\left(\frac{2}{p}\,\frac{p-1}{1-x^2}\right)^{\frac{1}{p}},\ \{x, 0, 1\}\right]^{3-\frac{2}{p}}$$

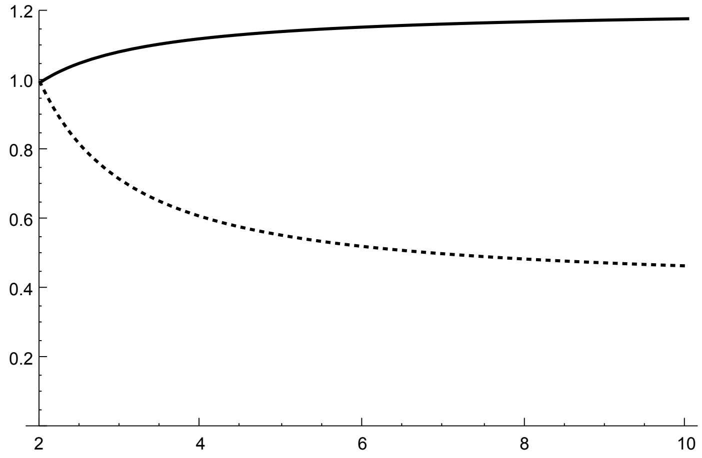
In[•]:=
```
Plot[{λstar[p], λ1[p]}, {p, 2, 10}, PlotRange → {All, {0, 1.2}}]
```

Out[•]=



In[•]:=
```
Plot[{λstar[p], λ1[p]}, {p, 2, 10},
 PlotRange → {All, {0, 1.2}}, PlotStyle → {Black, {Black, Dotted}}]
```

Out[•]=



The phase diagram

In[•]:=
$$\text{XYPlot[p\_, q\_, a\_, Tmax\_, PS\_] :=}$$
$$\text{ParametricPlot}\left[\{X[s], Y[s]\} \,/.\, \text{NDSolve}\left[\left\{X'[t] == \text{Abs}[Y[t]]^{\frac{p}{p-1}-1}\,\text{Sign}[Y[t]],\ Y'[t] == \text{Abs}[X[t]]^{p-2}\,X[t] - \text{Abs}[X[t]]^{q-2}\,X[t],\ X[0] == a,\ Y[0] == 0\right\}, \{X, Y\}, \{t, 0, \text{Tmax}\}\right], \{s, 0, \text{Tmax}\}, \text{PlotStyle} \to \text{PS}\right]$$

$$\text{V[p\_, q\_, PS\_] := VectorPlot}\left[\left\{\text{Abs}[y]^{\frac{p}{p-1}-1}\,\text{Sign}[y],\ \text{Abs}[x]^{p-2}\,x - \text{Abs}[x]^{q-2}\,x\right\}, \{x, -2, 2\}, \{y, -1, 1\}, \text{VectorStyle} \to \text{PS}\right]$$

$$\text{ZeroEnergy[p\_, q\_] := Plot}\left[\left\{\left(\frac{1}{p-1}\left(\text{Abs}[x]^{p} - \frac{p}{q}\,\text{Abs}[x]^{q}\right)\right)^{1-\frac{1}{p}},\ -\left(\frac{1}{p-1}\left(\text{Abs}[x]^{p} - \frac{p}{q}\,\text{Abs}[x]^{q}\right)\right)^{1-\frac{1}{p}}\right\}, \{x, -\text{ac}[p, q], \text{ac}[p, q]\}, \text{PlotStyle} \to \text{Black}\right]$$

In[•]:=
$$\text{bpos[Y0\_, p\_, q\_] := b /. FindRoot}\left[\text{En}[b, p, q] - \frac{p-1}{p}\,\text{Y0}^{p}, \{b, \text{ac}[p, q], 10\}\right]$$

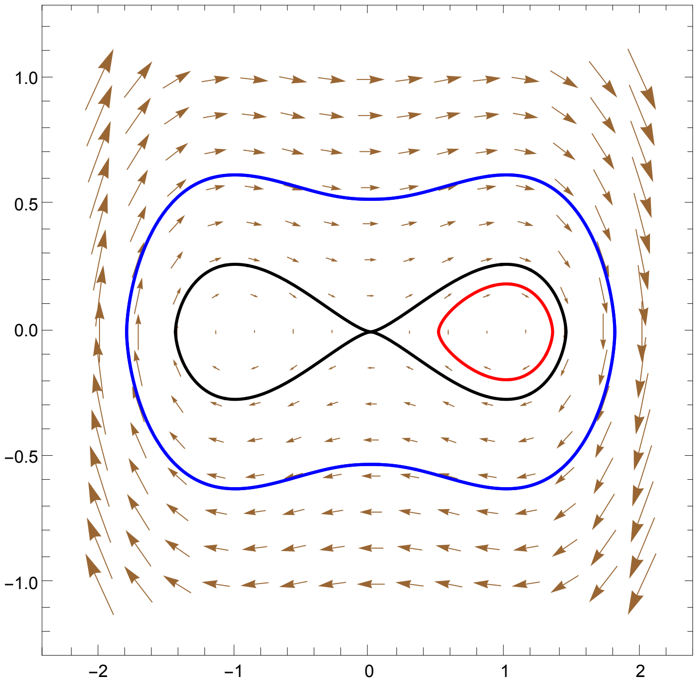
$$\text{dfpos[Y0\_, f\_, p\_, q\_] :=}\left(\text{Y0}^{p} - \frac{p}{p-1}\,\text{En}[f, p, q]\right)^{\frac{1}{p}}$$

$$\text{QuarterPeriod[p\_, q\_, Y0\_] :=}$$
$$2\,\text{NIntegrate}\left[\frac{1}{\text{dfpos}[\text{Y0}, f, p, q]}, \{f, 0, \text{bpos}[\text{Y0}, p, q]\}\right]$$

In[∘]:= $\left(\frac{p}{p-1}\text{En}[a, p, q]\right)^{\frac{1}{p}}$ /. {a → 1.8, p → 2.5, q → 3};

```
Show[V[2.5, 3, Brown], XYPlot[2.5, 3, 1.8, 4 QuarterPeriod[2.5, 3, %], Blue],
 XYPlot[2.5, 3, 0.5, F[0.5, 2.5, 3][[1]], Red], ZeroEnergy[2.5, 3]]
```

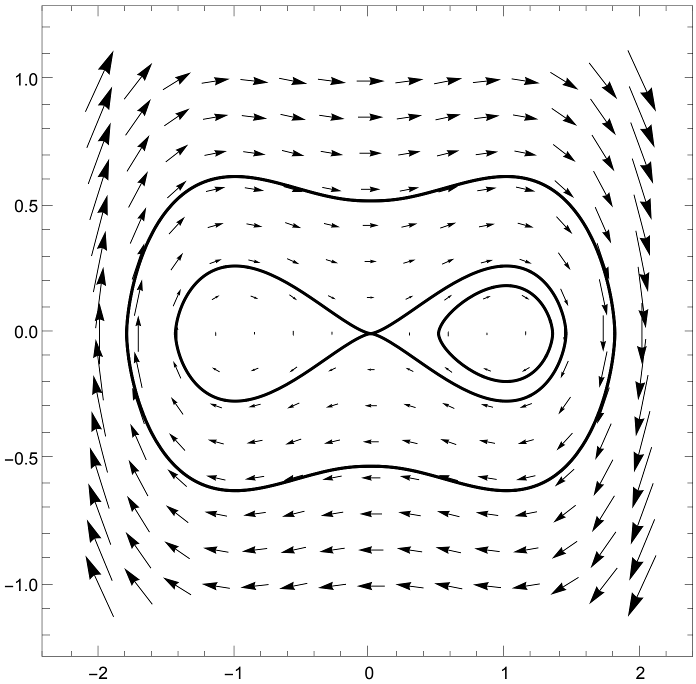
Out[∘]=



In[∘]:= $\left(\frac{p}{p-1}\text{En}[a, p, q]\right)^{\frac{1}{p}}$ /. {a → 1.8, p → 2.5, q → 3};

```
Show[V[2.5, 3, Black], XYPlot[2.5, 3, 1.8, 4 QuarterPeriod[2.5, 3, %], Black],
 XYPlot[2.5, 3, 0.5, F[0.5, 2.5, 3][[1]], Black], ZeroEnergy[2.5, 3]]
```

Out[∘]=



A parametric plot of the bifurcation diagram

In[•]:=

$$\text{Bifur[a\_, p\_, q\_] :=}$$

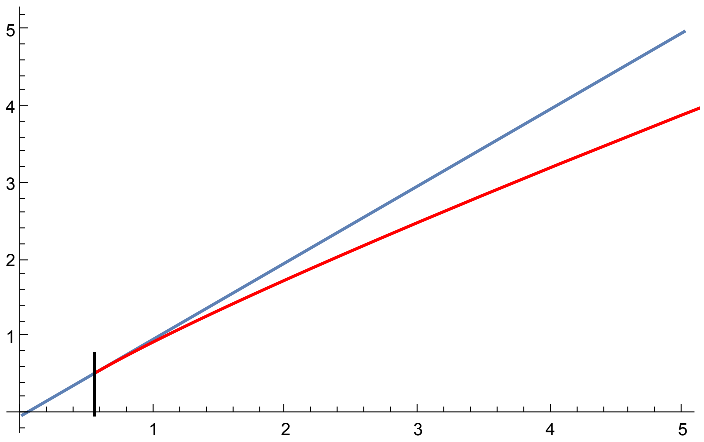
$$\text{Module}\left[\{\text{M = F[a, p, q]}\}, \left\{\left(\frac{\text{M[[1]]}}{2\,\pi}\right)^2 \left(\frac{\text{M[[3]]}}{\text{M[[2]]}}\right)^{\frac{p-2}{p}}, \left(\frac{\text{M[[1]]}}{2\,\pi}\right)^2 \text{M[[1]]}^{\frac{2}{q}-\frac{2}{p}} \frac{\text{M[[4]]}^{1-\frac{2}{q}}}{\text{M[[2]]}^{1-\frac{2}{p}}}\right\}\right]$$

```
BifurDiff[a_, p_, q_] := Module[{M = Bifur[a, p, q]}, {M[[1]], M[[1]] - M[[2]]}]
```

In[•]:=

$$\text{Show}\Big[\text{Plot}[\lambda, \{\lambda, 0, 5\}], \text{ParametricPlot[Bifur[a, 3, 5], \{a, 0, 1\}, PlotStyle} \to \text{Red]},$$
$$\text{ListLinePlot}\left[\left\{\left\{\frac{\lambda\text{star[3]}}{2}, -0.1\right\}, \left\{\frac{\lambda\text{star[3]}}{2}, 0.8\right\}\right\},\right.$$
$$\left.\text{PlotStyle} \to \text{Black, PlotRange} \to \{\{0, 5\}, \{0, 5\}\}\right]\Big]$$

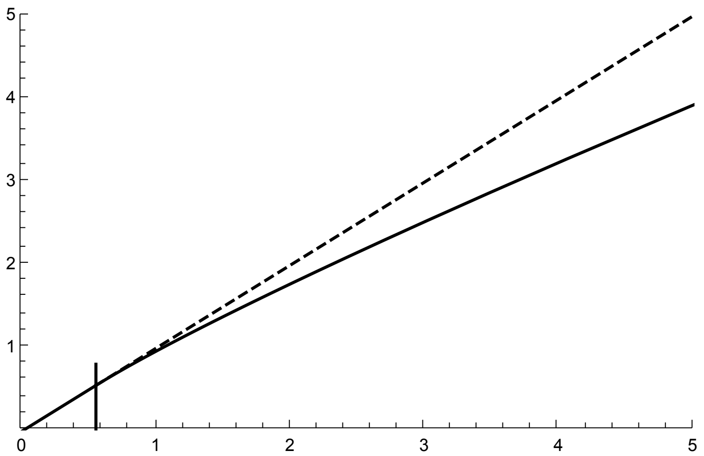
Out[•]=



In[•]:=

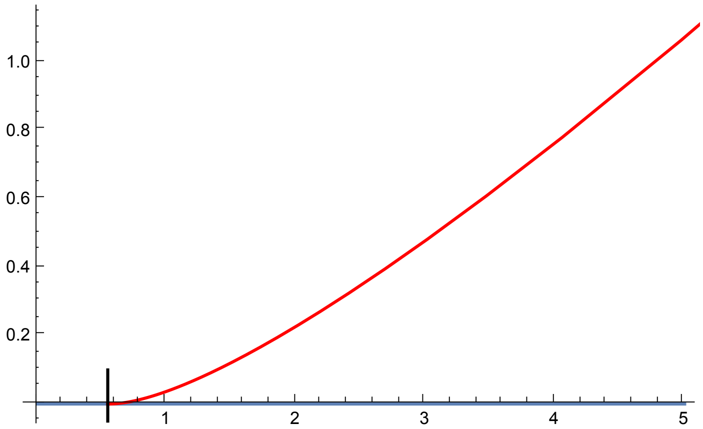
$$\text{Show}\Big[\text{Plot}\left[\lambda, \left\{\lambda, 0, \frac{\lambda\text{star[3]}}{2}\right\}, \text{PlotStyle} \to \text{Black, PlotRange} \to \{\{0, 5\}, \{0, 5\}\}\right],$$
$$\text{Plot}\left[\lambda, \left\{\lambda, \frac{\lambda\text{star[3]}}{2}, 5\right\}, \text{PlotStyle} \to \{\text{Black, Dashed}\}\right],$$
$$\text{ParametricPlot[Bifur[a, 3, 5], \{a, 0, 1\}, PlotStyle} \to \text{Black]},$$
$$\text{ListLinePlot}\left[\left\{\left\{\frac{\lambda\text{star[3]}}{2}, -0.1\right\}, \left\{\frac{\lambda\text{star[3]}}{2}, 0.8\right\}\right\},\right.$$
$$\left.\text{PlotStyle} \to \text{Black, PlotRange} \to \{\{0, 5\}, \{0, 5\}\}\right]\Big]$$

Out[•]=



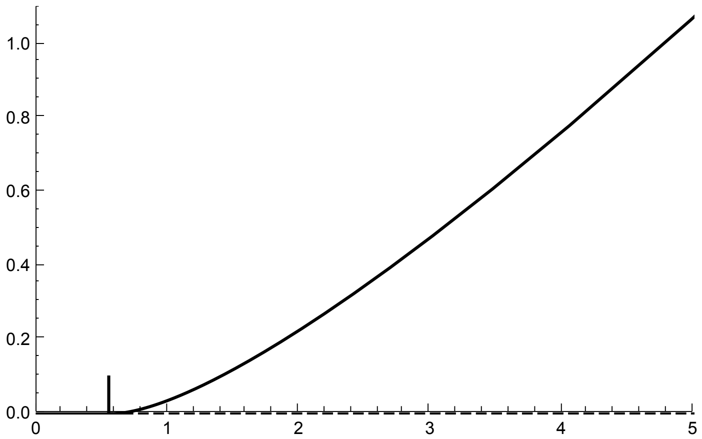
In[∘]:=
```
Show[Plot[0, {λ, 0, 5}],
 ParametricPlot[BifurDiff[a, 3, 5], {a, 0, 1}, PlotStyle → Red],
 ListLinePlot[{{λstar[3]/2, -0.05}, {λstar[3]/2, 0.1}}, PlotStyle → Black],
 PlotRange → {{0, 5}, {0, 1.1}}]
```

Out[∘]=



In[∘]:=
```
Show[Plot[0, {λ, 0, λstar[3]/2}, PlotStyle → Black, PlotRange → {{0, 5}, {0, 5}}],
 Plot[0, {λ, λstar[3]/2, 5}, PlotStyle → {Black, Dashed}],
 ParametricPlot[BifurDiff[a, 3, 5], {a, 0, 1}, PlotStyle → Black],
 ListLinePlot[{{λstar[3]/2, -0.05}, {λstar[3]/2, 0.1}}, PlotStyle → Black],
 PlotRange → {{0, 5}, {0, 1.1}}]
```

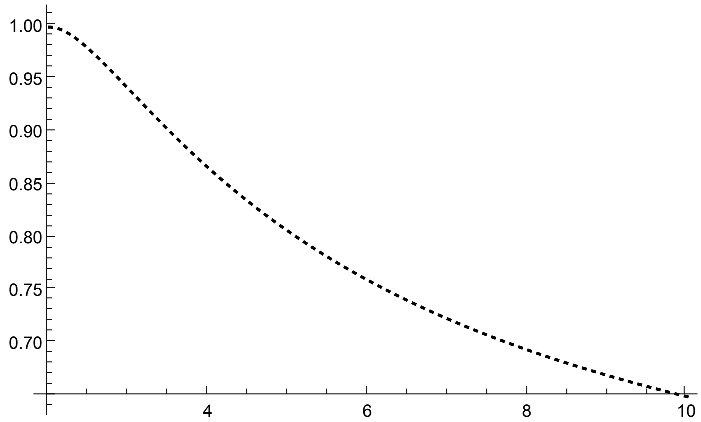
Out[∘]=



Erratum

In[∘]:=
```
Res = (2/π Integrate[((p - 1)/(1 - x^p))^(1/p), {x, 0, 1}, Assumptions → p > 2])^2
P = Plot[Res, {p, 2, 10}, PlotRange → All, PlotStyle → {Black, Dotted}]
Limit[Res, p → ∞];
{%, N[%]}
```

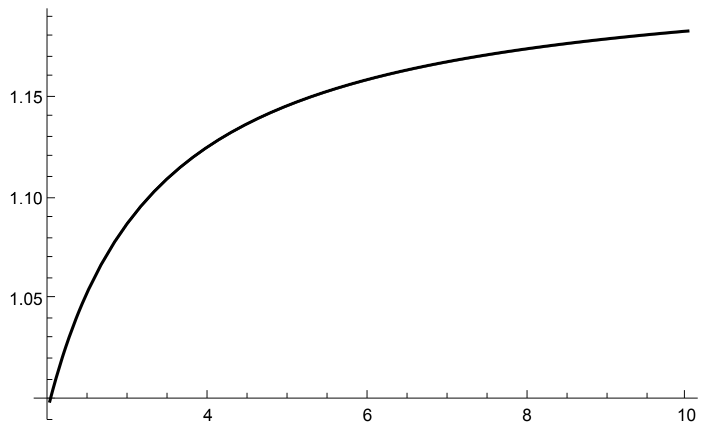
Out[∘]= $$\frac{4\,(-1+p)^{2/p}\,\mathrm{Csc}\left[\frac{\pi}{p}\right]^2}{p^2}$$

Out[∘]= 

Out[∘]= $\left\{\frac{4}{\pi^2},\ 0.405285\right\}$

In[∘]:=
```
ResStar = FullSimplify[PowerExpand[FullSimplify[
    PowerExpand[(2/π Integrate[(2/p (p - 1)/(1 - x^2))^(1/p - 1), {x, 0, 1}, Assumptions → p > 2])^(2/p - 1)
      (2/π Integrate[(2/p (p - 1)/(1 - x^2))^(1/p), {x, 0, 1}, Assumptions → p > 2])^(3 - 2/p)]] /.
   {Gamma[5/2 - 1/p] → (3/2 - 1/p) Z, Gamma[3/2 - 1/p] → Z}]] /.
  {Z → Gamma[3/2 - 1/p], Gamma[-1/p] → -p Gamma[1 - 1/p]}
PStar = Plot[ResStar, {p, 2, 10}, PlotRange → All, PlotStyle → Black]
Limit[ResStar, p → ∞];
{%, N[%]}
```

Out[∘]= $$\frac{4^{\frac{1}{p}}\,(-1+p)^{2/p}\,(-2+3\,p)^{1-\frac{2}{p}}\,\mathrm{Gamma}\left[1-\frac{1}{p}\right]^2}{p\,\pi\,\mathrm{Gamma}\left[\frac{3}{2}-\frac{1}{p}\right]^2}$$

Out[∘]=


Out[∘]= $$\left\{\frac{12}{\pi^2},\ 1.21585\right\}$$

In[∘]:=
```
Show[P, PStar, PlotRange → {All, {0, 1.2}}]
```

Out[∘]=