Figure 1: the bifurcation diagram

Initialization

```
ϵ = 10^-7;
Z[d_] := (Sqrt[π] Gamma[d/2]) / Gamma[(1+d)/2]
Off[NDSolve::ndinnt]
Off[ReplaceAll::reps]
Off[FindRoot::cvmit]
Off[FindRoot::brmp]
Off[NDSolve::ndinnt]
f[β_] := ((d - 1)/(d + 2))^2 (β (p - 1))^2 - (β (p - 2) + 1) (β - 1) - d/(d + 2) β (p - 1)
```

Plot of the shooting criterion

```
F[λ_, p_, d_] := Plot[u'[π - ϵ] /.
  NDSolve[{u''[θ] + (d - 1) Cot[θ] u'[θ] + (d λ)/(p - 2) (Abs[u[θ]]^(p-2) u[θ] -  u[θ]) == 0,
    u'[ϵ] == d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ, u[ϵ] == a + d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ^2/2},
   {u, u'}, {θ, ϵ, π - ϵ}], {a, 0, 1}]
```

Shooting method

```
G[λ_, p_, d_] := a /. FindRoot[u'[π - ϵ] /.
   NDSolve[{u''[θ] + (d - 1) Cot[θ] u'[θ] + (d λ)/(p - 2) (Abs[u[θ]]^(p-2) u[θ] -  u[θ]) == 0,
     u'[ϵ] == d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ, u[ϵ] == a + d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ^2/2},
    {u, u'}, {θ, ϵ, π - ϵ}], {a, 0.001, 0.5}][[1]]
```

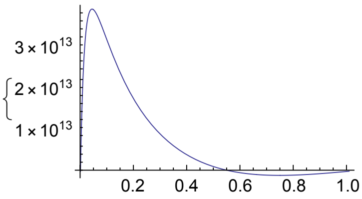
Computation of the optimal constant

```
H[λ_, p_, d_] :=
 Module[{a = G[λ, p, d]}, (v[π - ϵ]/Z[d])^((p-2)/p) /.
   NDSolve[{u''[θ] + (d - 1) Cot[θ] u'[θ] + (d λ)/(p - 2) (Abs[u[θ]]^(p-2) u[θ] -  u[θ]) == 0,
     u'[ϵ] == d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ, u[ϵ] == a + d λ (a - Abs[a^(p-2)] a)/(d (p - 2)) ϵ^2/2, v'[θ] ==
      Sin[θ]^(d-1) Abs[u[θ]]^p, v[ϵ] == Abs[a^p]/d ϵ^d}, {u, u', v}, {θ, ϵ, π - ϵ}]][[1]]
```

Range for searching the root

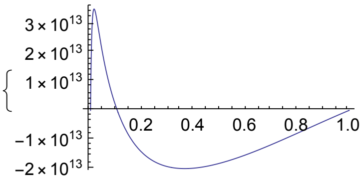
```
R[λ_, p_, d_] := {F[λ, p, d], G[λ, p, d], H[λ, p, d]}
```

```
R[1.05, 3, 3]
```

{

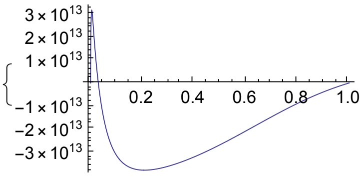
, 0.562701, 0.998605}

```
R[1.5, 3, 3]
```

{

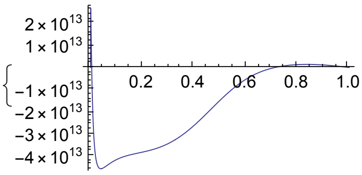
, 0.103396, 0.935047}

```
R[2, 3, 3]
```

{

, 0.0300427, 0.853182}

```
R[3, 3, 3]
```

{

, 0.00446763, 0.729349}

Estimates and plot of the bifurcation diagram

```
Estim1[p_, d_, λmin_, λmax_, PS_] :=
```

$$\text{Plot}\left[\frac{p-2}{d}\left(\frac{d\,(d-2)}{4}\right)^{d\,\frac{p-2}{2p}}\left(\frac{\lambda\, d}{p-2}\right)^{1-d\,\frac{p-2}{2p}},\ \{\lambda, \lambda\text{min}, \lambda\text{max}\},\ \text{PlotStyle} \to \text{PS}\right]$$

```
Simplify[f[1]]
```

$$\frac{(-1+p)\,\left(-1+d^2\,(-2+p)+p-2\,d\,p\right)}{(2+d)^2}$$

```
Estim2[p_, d_] :=
```

$$\text{Plot}\left[\left(\lambda+\frac{p-2}{\gamma}\,(\lambda-1)\right)^{\frac{\gamma}{\gamma+p-2}}\ /.\ \gamma \to -\frac{(-1+p)\,\left(-1+d^2\,(-2+p)+p-2\,d\,p\right)}{(2+d)^2},\right.$$
$$\left.\{\lambda, 1, 3\},\ \text{PlotStyle} \to \{\text{Black}, \text{Thick}, \text{Dotted}\}\right]$$

```
Show[Plot[λ , {λ, 0, 3}, PlotStyle → Black],
 Plot[λ , {λ, 0, 1.05}, PlotStyle → {Black, Thick}],
 Plot[λ H[λ, 3, 3], {λ, 1.05, 3}, PlotStyle → {Black, Thick}],
 Estim1[3, 3, 0, 1, {Black, Dashed}], Estim1[3, 3, 1, 3, {Black, Thick, Dashed}],
 Estim2[3, 3], ListLinePlot[{{1, 0}, {1, 1.25}}, PlotStyle → Black]]
```

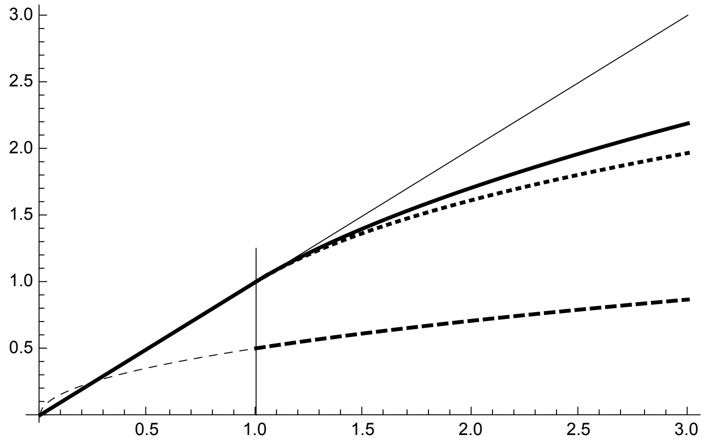


Figure 2: parameter range

$$\text{Simplify}\left[m\ /.\ \text{Solve}\left[\left\{\frac{1}{\beta}+\frac{p}{2}==1+m\,\frac{p}{2},\ f[\beta]==0\right\},\ \{\beta, m\}\right]\right]$$

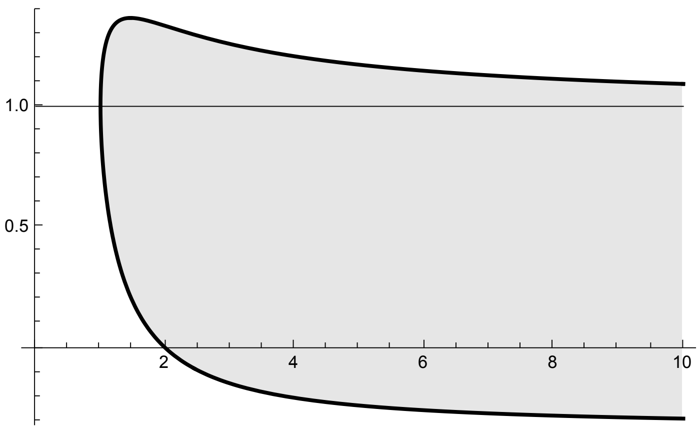
$$\left\{\frac{1}{(2+d)^2\,p}\left(2\left(2+\sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)}\right)+d^2\,p+2\,d\,(1+p)\right),\right.$$
$$\left.\frac{4+2\,d-2\,\sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)}+2\,d\,p+d^2\,p}{(2+d)^2\,p}\right\}$$

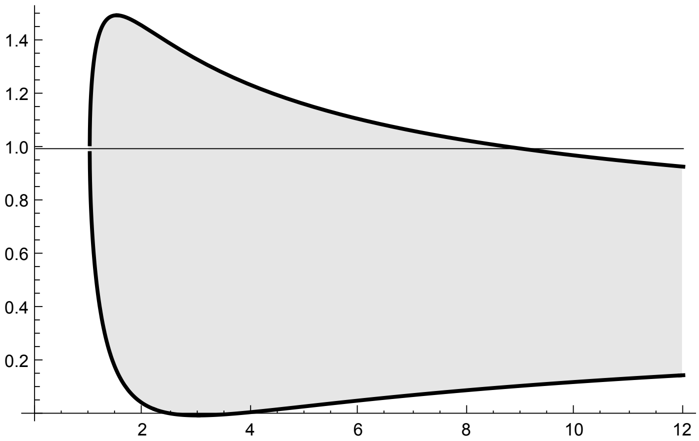
ResmFn[p_, d_] :=

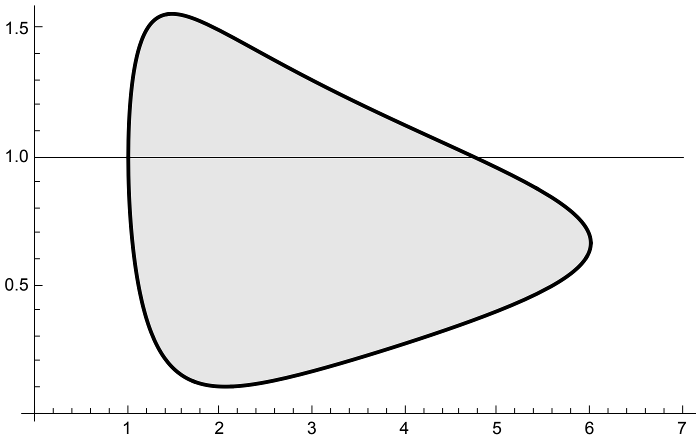
$$\left\{\frac{1}{(2+d)^2\,p}\left(2\left(2+\sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)}\right)+d^2\,p+2\,d\,(1+p)\right),\right.$$
$$\left.\frac{1}{(2+d)^2\,p}\left(4+2\,d-2\,\sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)}+2\,d\,p+d^2\,p\right)\right\}$$

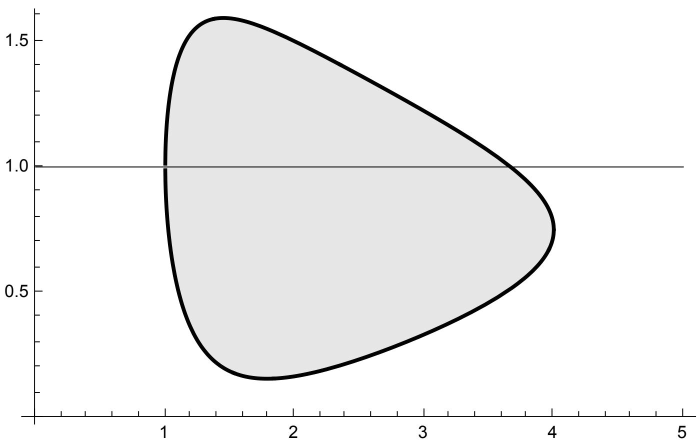
```
Pm[d_, pmax_] := Plot[{1, ResmFn[p, d][[1]], ResmFn[p, d][[2]]}, {p, 0, pmax},
  AxesOrigin → {0, 0}, PlotStyle → {Black, {Black, Thick}, {Black, Thick}},
  Filling → {2 → {3}}, FillingStyle → GrayLevel[0.9]]
```

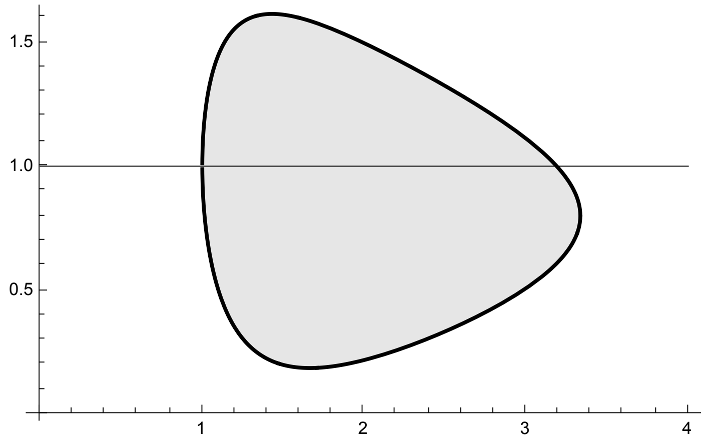
```
Pm[1, 10]
Pm[2, 12]
Pm[3, 7]
Pm[4, 5]
Pm[5, 4]
Pm[10, 3]
```

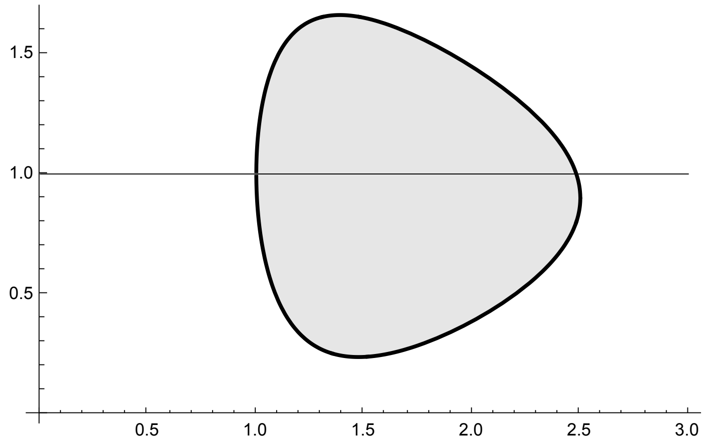

1.0
0.5
2
4
6
8
10


1.4
1.2
1.0
0.8
0.6
0.4
0.2
2
4
6
8
10
12


1.5
1.0
0.5
1
2
3
4
5
6
7


1.5
1.0
0.5
1
2
3
4
5


1.5
1.0
0.5
1
2
3
4


1.5
1.0
0.5
0.5
1.0
1.5
2.0
2.5
3.0