Figure 1: the bifurcation diagram

Initialization

In[1]:=
$$\epsilon = 10^{-7};$$
$$Z[d\_] := \frac{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{d}{2}\right]}{\mathrm{Gamma}\left[\frac{1+d}{2}\right]}$$

In[3]:=
```
Off[NDSolve::ndinnt]
Off[ReplaceAll::reps]
Off[FindRoot::cvmit]
Off[FindRoot::brmp]
Off[NDSolve::ndinnt]
```

In[8]:=
$$f[\beta\_] := \left(\frac{d-1}{d+2}\right)^2 (\beta\ (p-1))^2 - (\beta\ (p-2)+1)\ (\beta-1) - \frac{d}{d+2}\beta\ (p-1)$$

Shooting method

In[9]:=
$$G[\lambda\_, p\_, d\_] := a\ /.\ \mathrm{FindRoot}\Big[u'[\pi-\epsilon]\ /.$$
$$\mathrm{NDSolve}\Big[\Big\{u''[\theta] + (d-1)\ \mathrm{Cot}[\theta]\ u'[\theta] + \frac{d\,\lambda}{p-2}\left(\mathrm{Abs}[u[\theta]]^{p-2}\, u[\theta] - u[\theta]\right) == 0,$$
$$u'[\epsilon] == d\,\lambda\ \frac{a - \mathrm{Abs}\left[a^{p-2}\right] a}{d\ (p-2)}\,\epsilon,\ u[\epsilon] == a + d\,\lambda\ \frac{a - \mathrm{Abs}\left[a^{p-2}\right] a}{d\ (p-2)}\ \frac{\epsilon^2}{2}\Big\},$$
$$\{u, u'\}, \{\theta, \epsilon, \pi-\epsilon\}\Big], \{a, 0.001, 0.5\}\Big][[1]]$$

Computation of the optimal constant

In[10]:=
$$H[\lambda\_, p\_, d\_] :=$$
$$\mathrm{Module}\Big[\{a = G[\lambda, p, d]\}, \left(\frac{v[\pi-\epsilon]}{Z[d]}\right)^{\frac{p-2}{p}}\ /.$$
$$\mathrm{NDSolve}\Big[\Big\{u''[\theta] + (d-1)\ \mathrm{Cot}[\theta]\ u'[\theta] + \frac{d\,\lambda}{p-2}\left(\mathrm{Abs}[u[\theta]]^{p-2}\, u[\theta] - u[\theta]\right) == 0,$$
$$u'[\epsilon] == d\,\lambda\ \frac{a - \mathrm{Abs}\left[a^{p-2}\right] a}{d\ (p-2)}\,\epsilon,\ u[\epsilon] == a + d\,\lambda\ \frac{a - \mathrm{Abs}\left[a^{p-2}\right] a}{d\ (p-2)}\ \frac{\epsilon^2}{2},\ v'[\theta] ==$$
$$\mathrm{Sin}[\theta]^{d-1}\ \mathrm{Abs}[u[\theta]]^p,\ v[\epsilon] == \frac{\mathrm{Abs}\left[a^p\right]}{d}\,\epsilon^d\Big\}, \{u, u', v\}, \{\theta, \epsilon, \pi-\epsilon\}\Big]\Big][[1]]$$

Plot of the bifurcation diagram

In[11]:=
```
Show[Plot[λ , {λ, 3, 9}, PlotStyle → {Black, Thick, Dashed},
  PlotRange → {{0, 9}, {0, 9}}], Plot[λ , {λ, 0, 3.15}, PlotStyle → {Black, Thick}],
 Plot[λ H[λ / 3, 3, 3], {λ, 3.15, 9}, PlotStyle → {Black, Thick}],
 ListLinePlot[{{3, 0}, {3, 3.5}}, PlotStyle → {Black, Dotted}],
 ListLinePlot[{{0, 3}, {3.25, 3}}, PlotStyle → {Black, Dotted}],
 ListLinePlot[{{3, 2.5}, {3, 3.5}}, PlotStyle → Black],
 ListLinePlot[{{2.75, 3}, {3.25, 3}}, PlotStyle → Black]]
```

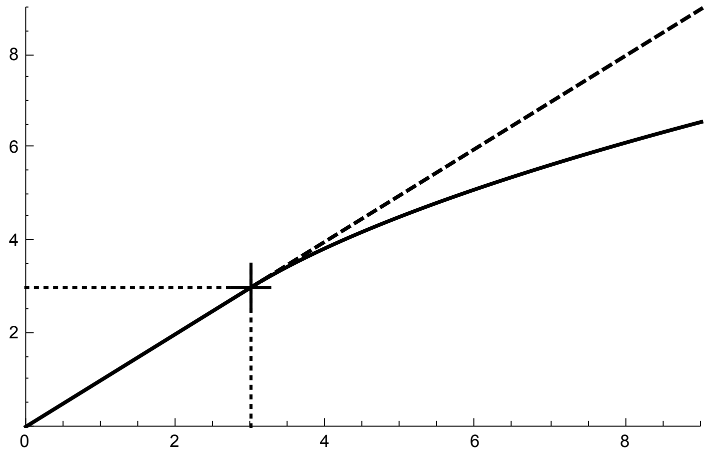
Out[•]=


Figure 2: the branches on the sphere

In[1]:=
```
δ = 10^-7;
λmax = 20.2;
Off[FindRoot::brmp]
Off[NDSolve::ndsz]
Off[NDSolve::ndnum]
Off[ReplaceAll::reps]
Off[NDSolve::ndinnt]
Off[ReplaceAll::reps]
Off[FindRoot::nlnum]
Off[FindRoot::jsing]
Off[FindRoot::lstol]
```

In[12]:= $Z[d\_] := \dfrac{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{d}{2}\right]}{\mathrm{Gamma}\left[\frac{1+d}{2}\right]}$

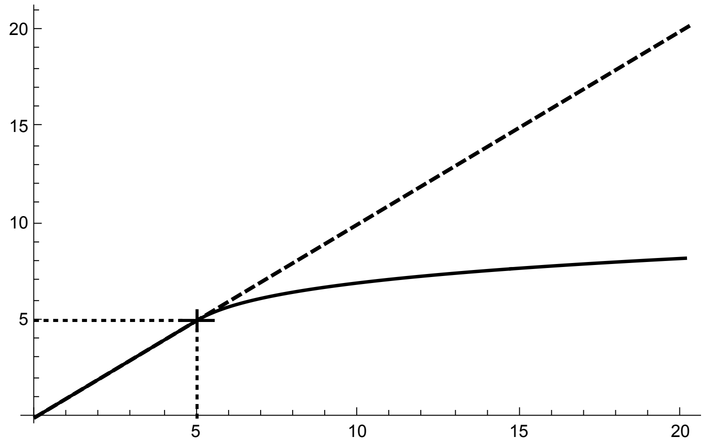
In[13]:=

First branch

In[14]:=
```
Tir[d_, p_, λ_, a_?NumericQ] := Tanh[u'[-1 + δ]] /.
  NDSolve[{-(1 - z^2) u''[z] + d z u'[z] + λ/(p - 2) u[z] - 1/(p - 2) Abs[u[z]]^(p-2) u[z] == 0,
     u[1 - δ] == a, u'[1 - δ] == (a^(p-1) - λ a)/(d (p - 2))}, {u, u'}, {z, -1 + δ, 1 - δ}][[1]]
Mu[d_, p_, λ_, a_?NumericQ] := v[-1 + δ]^((p-2)/p) /.
  NDSolve[{-(1 - z^2) u''[z] + d z u'[z] + λ/(p - 2) u[z] - 1/(p - 2) Abs[u[z]]^(p-2) u[z] == 0,
     u[1 - δ] == a - (a^(p-1) - λ a)/(d (1 - δ)) δ, u'[1 - δ] == (a^(p-1) - λ a)/(d (p - 2)), v'[z] == -Abs[u[z]]^p (1 - z^2)^((d-2)/2)/Z[d],
     v[1 - δ] == 0}, {u, u', v}, {z, -1 + δ, 1 - δ}][[1]]
FTir[d_, p_, λ_] := Module[{x = FindRoot[Tir[d, p, λ, a] == 0,
     {a, δ, 1/(λ - d + 0.2 + 0.02 (λ - d)^4)}]}, a /. x]
LL = Table[{λ, FTir[5, 3, λ]}, {λ, 5.01, λmax, 0.1}];
LLL = Join[{{0, 0}, {5, 5}},
   Table[{λ, Mu[5, 3, λ, FTir[5, 3, λ]]}, {λ, 5.01, λmax, 0.1}]];
```

In[30]:=
```
P1 = Show[ListLinePlot[{{0, 0}, {λmax, λmax}}, PlotStyle → {Black, Thick, Dashed}],
  ListLinePlot[LLL, PlotStyle → {Thickness[0.005], Black}],
  ListLinePlot[{{0, 5}, {4.5, 5}}, PlotStyle → {Black, Dotted}],
  ListLinePlot[{{4.5, 5}, {5.5, 5}}, PlotStyle → Black],
  ListLinePlot[{{5, 0}, {5, 4.5}}, PlotStyle → {Black, Dotted}],
  ListLinePlot[{{5, 4.5}, {5, 5.5}}, PlotStyle → Black]]
```

Out[•]=



In[31]:=

Second branch

In[19]:=
```
Tir[d_, p_, λ_, a_?NumericQ] := u'[0] /.
  NDSolve[{-(1 - z^2) u''[z] + d z u'[z] + λ/(p - 2) u[z] - 1/(p - 2) Abs[u[z]]^(p-2) u[z] == 0,
    u[1 - δ] == a, u'[1 - δ] == (a^(p-1) - λ a)/(d (p - 2))}, {u, u'}, {z, 0, 1 - δ}][[1]]
```

In[20]:=
```
Mutest[d_, p_, λ_, a_?NumericQ] := (2 v[0])^((p-2)/p) /.
  NDSolve[{-(1 - z^2) u''[z] + d z u'[z] + λ/(p - 2) u[z] - 1/(p - 2) Abs[u[z]]^(p-2) u[z] == 0,
    u[1 - δ] == a - (a^(p-1) - λ a)/(d (1 - δ)) δ, u'[1 - δ] == (a^(p-1) - λ a)/(d (p - 2)),
    v'[z] == -Abs[u[z]]^p (1 - z^2)^((d-2)/2)/Z[d], v[1 - δ] == 0}, {u, u', v}, {z, 0, 1 - δ}][[1]]
```

In[21]:=
```
Branch[λ1_, λ2_, h_, ainit_] :=
 {Res0 = {{λ1, a /. FindRoot[Tir[5, 3, λ1, a], {a, ainit}]}};
   Monitor[For[i = 1, Sign[λ2 - λ1] (λ1 + h * i * Sign[λ2 - λ1] - λ2) < 0,
     i++, Res0 = Append[Res0, {λ1 + h * i * Sign[λ2 - λ1],
        a /. FindRoot[Tir[5, 3, λ1 + h * i * Sign[λ2 - λ1], a],
          {a, Res0[[Length[Res0]]][[2]]}]}]], λ1 + h * i * Sign[λ2 - λ1]];
   Res0}[[1]]
```

In[22]:=
```
BifDiagram[λ1_, λ2_, h_, ainit_] :=
 {Res0 = {{λ1, a /. FindRoot[Tir[5, 3, λ1, a], {a, ainit}]}};
   ResBif = {{λ1, Mutest[5, 3, λ1, Res0[[Length[Res0]]][[2]]]}};
   Monitor[For[i = 1, Sign[λ2 - λ1] (λ1 + h * i * Sign[λ2 - λ1] - λ2) < 0, i++,
     {Res0 = Append[Res0, {λ1 + h * i * Sign[λ2 - λ1], a /. FindRoot[Tir[5, 3,
            λ1 + h * i * Sign[λ2 - λ1], a], {a, Res0[[Length[Res0]]][[2]]}]}],
      ResBif = Append[ResBif, {λ1 + h * i * Sign[λ2 - λ1], Mutest[5, 3, λ1 + h * i * Sign[
            λ2 - λ1], Res0[[Length[Res0]]][[2]]]}]}], λ1 + h * i * Sign[λ2 - λ1]];
   ListLinePlot[ResBif, PlotStyle → {Thickness[0.005], Black}]}[[1]]
```

In[23]:=
```
TableInit = {10, 13, 180};
Nbre = Length[TableInit];
Table[ FindRoot[Tir[5, 3, 11.5, a], {a, TableInit[[i]]}], {i, 1, Nbre}]
Branch[11.5, 7.6, 0.05, 180];
%[[Length[%]]]
Branch[11.5, 7.6, 0.05, 13.140709400649873`];
%[[Length[%]]]
TableInit = {10, 20, 220};
Nbre = Length[TableInit];
Table[ FindRoot[Tir[5, 3, 13, a], {a, TableInit[[i]]}], {i, 1, Nbre}]
```

Out[∘]=
```
{{a → 11.5}, {a → 13.1407}, {a → 180.307}}
```

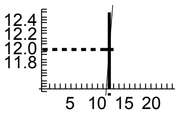
Out[∘]=
```
{7.65, 57.8833}
```

Out[∘]=
```
{7.65, 45.5676}
```

Out[∘]=
```
{{a → 10.1298}, {a → 13.}, {a → 220.366}}
```

In[33]:=
```
Show[BifDiagram[11.5, 7.6, 0.05, 180],
  BifDiagram[7.65, 7.617, 0.001, 57.883334045563444`],
  BifDiagram[11.5, 7.6, 0.05, 13.140709400649873`],
  BifDiagram[7.65, 7.617, 0.001, 45.56761914396513`],
  BifDiagram[11.5, 12, 0.001, 13.140709400649873`],
  BifDiagram[11.5, 14, 0.05, 180], PlotRange → All];
P2 = Show[ListLinePlot[{{12, 11.5}, {12, 12.5}}, PlotStyle → Black],
   ListLinePlot[{{11.5, 12}, {12.5, 12}}, PlotStyle → Black],
   ListLinePlot[{{12, 0}, {12, 11.5}}, PlotStyle → {Dotted, Black}],
   ListLinePlot[{{0, 12}, {11.5, 12}}, PlotStyle → {Dotted, Black}],
   BifDiagram[13, 12, 0.01, 10.129844993171785`],
   BifDiagram[13, 15, 0.05, 10.129844993171785`], %];
```

In[35]:=
```
Show[P1, P2, PlotRange → {{0, 13.5}, {0, 14}}]
```

Out[∘]= Show[P1, , PlotRange → {{0, 13.5}, {0, 14}}]

In[36]:=

Computation of the optimal constant: looking for the largest value of λ such that $\mu(\lambda) = \lambda$

In[50]:=

$$\text{Mutest[d\_, p\_, }\lambda\text{\_, a\_?NumericQ] := } (2\,v[0])^{\frac{p-2}{p}} \;/.$$

$$\text{NDSolve}\Big[\Big\{-\left(1-z^2\right) u''[z] + d\,z\,u'[z] + \frac{\lambda}{p-2}\,u[z] - \frac{1}{p-2}\,\text{Abs}[u[z]]^{p-2}\,u[z] == 0,$$

$$u[1-\delta] == a - \frac{a^{p-1} - \lambda a}{d\,(1-\delta)}\,\delta,\; u'[1-\delta] == \frac{a^{p-1} - \lambda a}{d\,(p-2)},$$

$$v'[z] == -\text{Abs}[u[z]]^p\,\frac{\left(1-z^2\right)^{\frac{d-2}{2}}}{Z[d]},\; v[1-\delta] == 0\Big\},\; \{u, u', v\},\; \{z, 0, 1-\delta\}\Big][[1]]$$

In[51]:=
```
h = 0.01;
Res1 = {{3.3, b, a /. FindRoot[Tir[5, 3.3, b, a], {a, 250}]}} /. FindRoot[
    b - Mutest[5, 3.3, b, a /. FindRoot[Tir[5, 3.3, b, a], {a, 250}]], {b, 7}];
Timing[Monitor[For[i = 1, 3.3 - h * i > 2.31, i++,
   Res1 = Append[Res1, {3.3 - h * i, b, aopt /. FindRoot[
         Tir[5, 3.3 - h * i, b, aopt], {aopt, Res1[[Length[Res1]]][[3]]}]} /.
      FindRoot[b - Mutest[5, 3.3 - h * i, b, a /. FindRoot[
           Tir[5, 3.3 - h * i, b, a], {a, Res1[[Length[Res1]]][[3]]}]],
       {b, Res1[[Length[Res1]]][[2]]}]]], 3.3 - h * i]]
Res1;
P3 = ListLinePlot[Table[{Res1[[i]][[1]], Res1[[i]][[2]]}, {i, 1, Length[Res1]}],
   PlotStyle → {Thickness[0.005], Black}];
```

Out[∘]= {42.524, Null}

In[56]:=
```
TT = Table[{Res1[[i]][[1]], Res1[[i]][[2]]}, {i, 1, Length[Res1]}];
P3 = ListLinePlot[TT, PlotStyle → {Thickness[0.005], Black}];
```

In[58]:=
```
h = 0.001;
Res2 = {{3.3, b, a /. FindRoot[Tir[5, 3.3, b, a], {a, 250}]}} /. FindRoot[
    b - Mutest[5, 3.3, b, a /. FindRoot[Tir[5, 3.3, b, a], {a, 250}]], {b, 7}];
Timing[Monitor[For[i = 1, 3.3 + h * i < 3.333, i++,
   Res2 = Append[Res2, {3.3 + h * i, b, aopt /. FindRoot[
         Tir[5, 3.3 + h * i, b, aopt], {aopt, Res2[[Length[Res2]]][[3]]}]} /.
      FindRoot[b - Mutest[5, 3.3 + h * i, b, a /. FindRoot[
           Tir[5, 3.3 + h * i, b, a], {a, Res2[[Length[Res2]]][[3]]}]],
       {b, Res2[[Length[Res2]]][[2]]}]]], 3.3 + h * i]]
Res2;
P4 = ListLinePlot[Table[{Res2[[i]][[1]], Res2[[i]][[2]]}, {i, 1, Length[Res2]}],
   PlotStyle → {Thickness[0.005], Black}];
```

Out[∘]= {13.8648, Null}

In[63]:=
```
Nbre = Length[TT];
Show[P3, P4, ListLinePlot[{{1, 12}, TT[[Nbre]]},
  PlotStyle → {Thickness[0.005], Black, Dashed}],
 PlotRange → {{1, 3.3333}, {6, 12}}, AxesOrigin → {2, 6}]
```

Out[∘]=