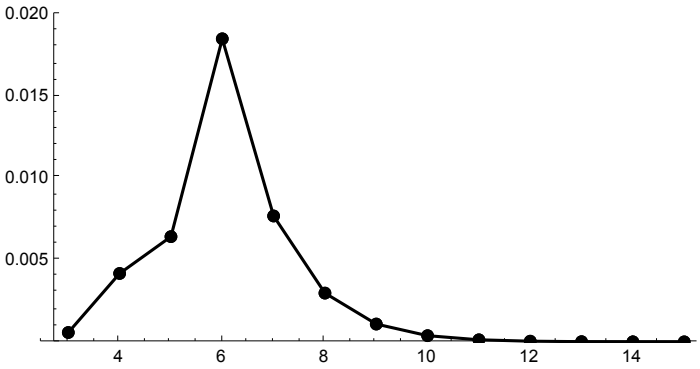
The computation for Table 1 and Figure 1

```
In[•]:= Off[Solve::ifun]

In[•]:= q[d_] := 2 d / (d - 2)

In[•]:= m3[x_] := 4/7 - 20/3 x - 5 x^2 - 2 x^3 - 1/3 x^4

        m45[x_, d_] := 4/(d + 4) - 1/3 (q[d] - 1) (q[d] - 2) x - 2/q[d] x^(q[d]-2)

        m6plus[x_, d_] := 4/(d + 4) - 2/q[d] x^(q[d]-2)

In[•]:= Res = Module[{r = FindMaximum[x^2/(1 + x^2) m3[x], {x, 0, 0.1}]}, {{3, r[[1]], r[[2]]}}]

        Res = Append[Res,
          Module[{r = FindMaximum[x^2/(1 + x^2) m45[x, 4], {x, 0, 0.1}]}, {4, r[[1]], r[[2]]}]];

        Res = Append[Res, Module[{r = FindMaximum[x^2/(1 + x^2) m45[x, 5], {x, 0, 0.2}]},
          {5, r[[1]], r[[2]]}]]

        Res = Join[Res, Table[Module[
            {r = FindMaximum[x^2/(1 + x^2) m6plus[x, d], {x, 0, Min[0.5, 150 d^-3]}]},
            {d, r[[1]], r[[2]]}], {d, 6, 15}]]

Out[•]= {{3, 0.000571422, {x → 0.0540815}}}

Out[•]= {{3, 0.000571422, {x → 0.0540815}},
         {4, 0.00417795, {x → 0.157052}}, {5, 0.00642591, {x → 0.204254}}}

Out[•]= {{3, 0.000571422, {x → 0.0540815}},
         {4, 0.00417795, {x → 0.157052}}, {5, 0.00642591, {x → 0.204254}},
         {6, 0.0185071, {x → 0.381493}}, {7, 0.00768756, {x → 0.274938}},
         {8, 0.00297761, {x → 0.189877}}, {9, 0.00108741, {x → 0.126323}},
         {10, 0.00037828, {x → 0.081417}}, {11, 0.00012628, {x → 0.0510467}},
         {12, 0.0000406504, {x → 0.0312378}}, {13, 0.0000126592, {x → 0.0187025}},
         {14, 3.82263 × 10^-6, {x → 0.0109757}}, {15, 1.12129 × 10^-6, {x → 0.00632092}}}
```

In[•]:=
```
Tbl = Table[{Res[[k]][[1]], Res[[k]][[2]]}, {k, 1, Length[Res]}];
P1 = ListLinePlot[Tbl, PlotRange → {All, {0, 0.02}},
  Mesh → Full, PlotStyle → Black, AspectRatio → 0.5]
```

Out[•]=



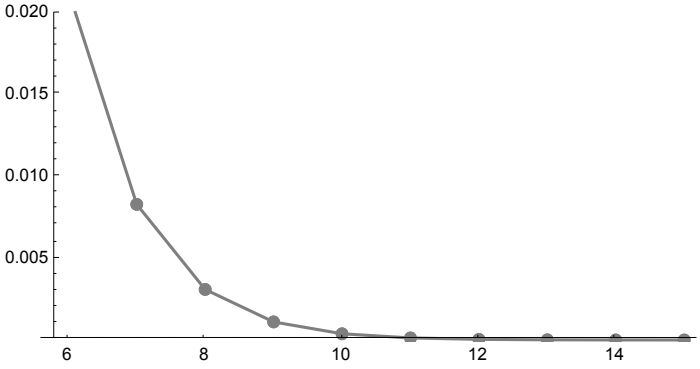
In[•]:=
```
Table[Tbl[[k]][[1]], {k, 1, Length[Tbl]}]
TeXForm[Table[NumberForm[Tbl[[k]][[2]], 5], {k, 1, Length[Tbl]}]]
```

Out[•]=
```
{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}
```

Out[•]//TeXForm=
```
\left\{0.00057142,0.0041779,0.0064259,
   0.018507,0.0076876,0.0029776,0.0010
   874,0.00037828,0.00012628,0.0000406
   5,0.000012659,3.8226\times
   10^{-6},1.1213\times
   10^{-6}\right\}
```

In[•]:=
```
TblLim = Table[{d, 2^(d+1) / (d (d+4)^(d/2))}, {d, 6, 15}];
P2 = ListLinePlot[TblLim, PlotRange → {All, {0, 0.02}},
  Mesh → Full, PlotStyle → Gray, AspectRatio → 0.5, PlotRange → All]
```

Out[•]=



In[•]:=
$$\text{TblLim} = \text{Table}\left[\left\{d, \frac{2^{d+1}}{d\,(d+4)^{\frac{d}{2}}}\right\}, \{d, 3, 15\}\right];$$

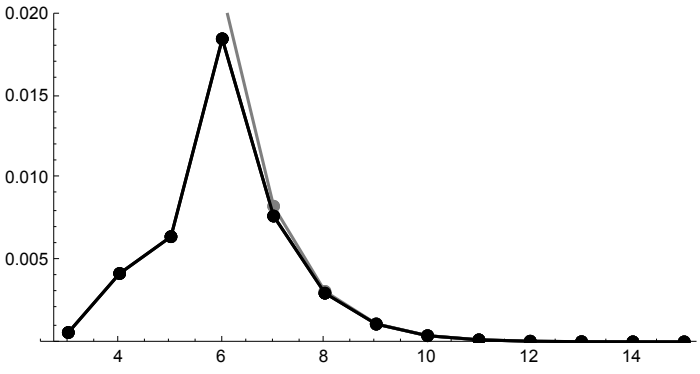
```
Table[TblLim[[k]][[1]], {k, 1, Length[Tbl]}]
TeXForm[Table[N[NumberForm[TblLim[[k]][[2]], 5]], {k, 1, Length[Tbl]}]]
```

Out[•]= {3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15}

Out[•]//TeXForm=
```
\left\{0.28797,0.125,0.052675,0.021333
   ,0.0082845,0.0030864,0.0011049,0.00
   038079,0.00012661,0.00004069,0.0000
   12664,3.8231\times
   10^{-6},1.1213\times
   10^{-6}\right\}
```

In[•]:=
```
Show[P1, P2, P1]
```

Out[•]=



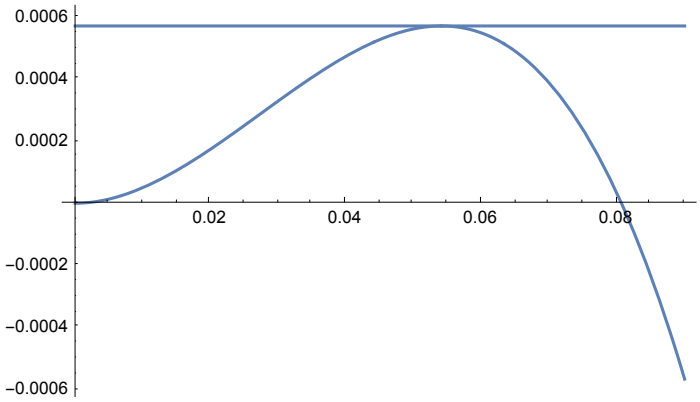
In[•]:=

In[•]:=

Checking the values (graphically)

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m3[x], {x, 0, 0.09}],
 ListLinePlot[{{0, Res[[1]][[2]]}, {0.09, Res[[1]][[2]]}}]]
Res[[
 1]]
```

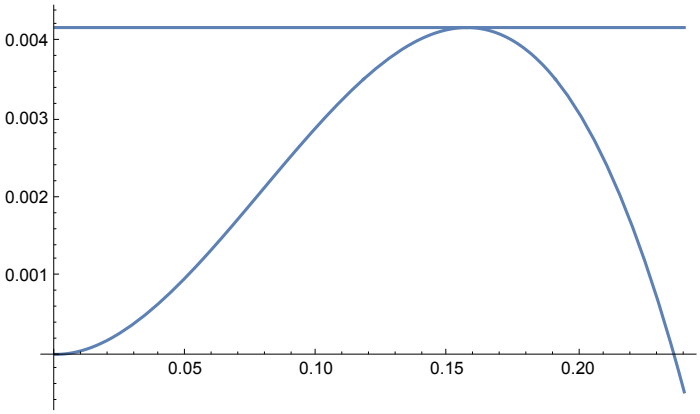
Out[•]=



Out[•]= {3, 0.000571422, {x → 0.0540815}}

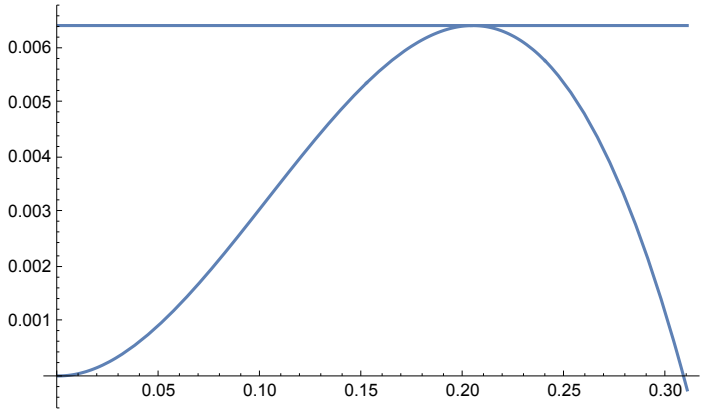
In[•]:=
```
Show[Plot[x^2/(1 + x^2) m45[x, 4], {x, 0, 0.24}],
 ListLinePlot[{{0, Res[[2]][[2]]}, {0.24, Res[[2]][[2]]}}]]
Res[[
 2]]
```

Out[•]=



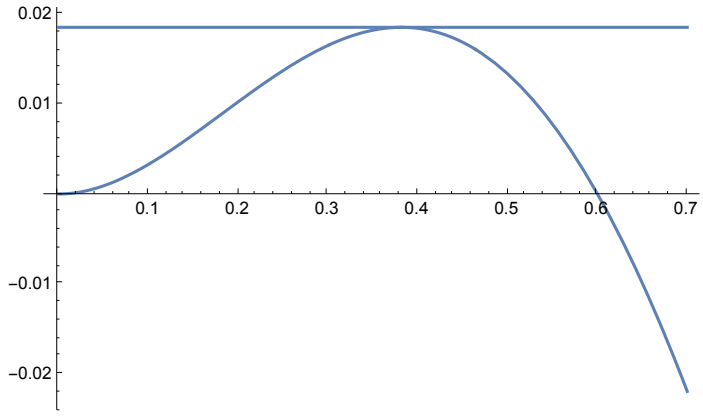
Out[•]= {4, 0.00417795, {x → 0.157052}}

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m45[x, 5], {x, 0, 0.31}],
 ListLinePlot[{{0, Res[[3]][[2]]}, {0.31, Res[[3]][[2]]}}]]
Res[[
 3]]
```



Out[•]= {5, 0.00642591, {x → 0.204254}}

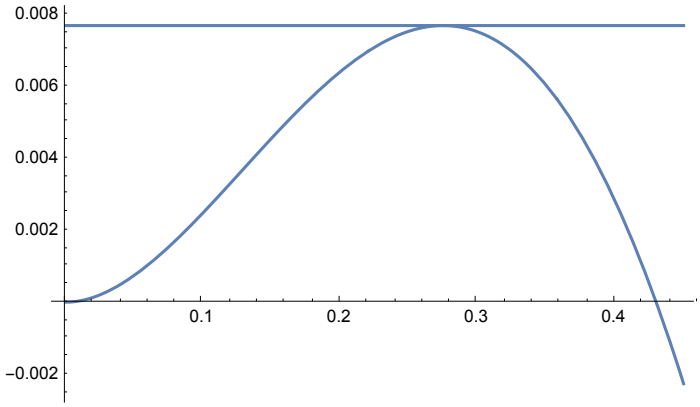
In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 6], {x, 0, 0.7}],
 ListLinePlot[{{0, Res[[4]][[2]]}, {0.7, Res[[4]][[2]]}}]]
Res[[
 4]]
```



Out[•]= {6, 0.0185071, {x → 0.381493}}

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 7], {x, 0, 0.45}],
 ListLinePlot[{{0, Res[[5]][[2]]}, {0.45, Res[[5]][[2]]}}]]
Res[[
 5]]
```

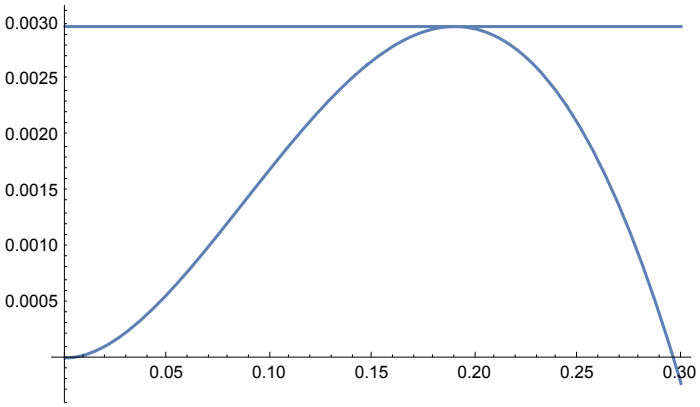
Out[•]=



Out[•]= {7, 0.00768756, {x → 0.274938}}

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 8], {x, 0, 0.3}],
 ListLinePlot[{{0, Res[[6]][[2]]}, {0.3, Res[[6]][[2]]}}]]
FindMaximum[x^2/(1 + x^2) m6plus[x, 8], {x, 0, 0.3}]
Res[[6]]
```

Out[•]=

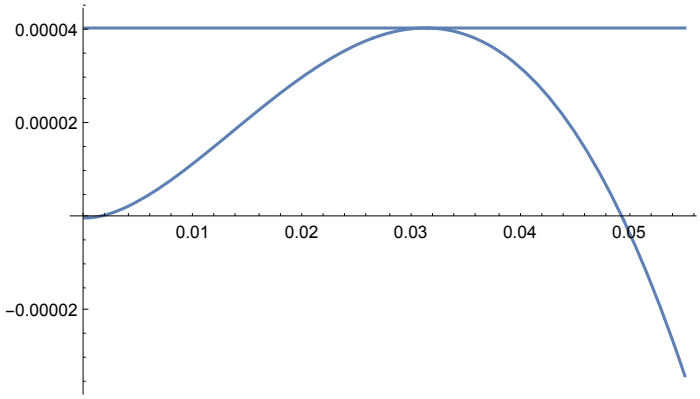


Out[•]= {0.00297761, {x → 0.189877}}

Out[•]= {8, 0.00297761, {x → 0.189877}}

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 12], {x, 0, 0.055}],
 ListLinePlot[{{0, Res[[10]][[2]]}, {0.055, Res[[10]][[2]]}}]]
FindMaximum[x^2/(1 + x^2) m6plus[x, 12], {x, 0, 0.06}]
Res[[10]]
```

Out[•]=

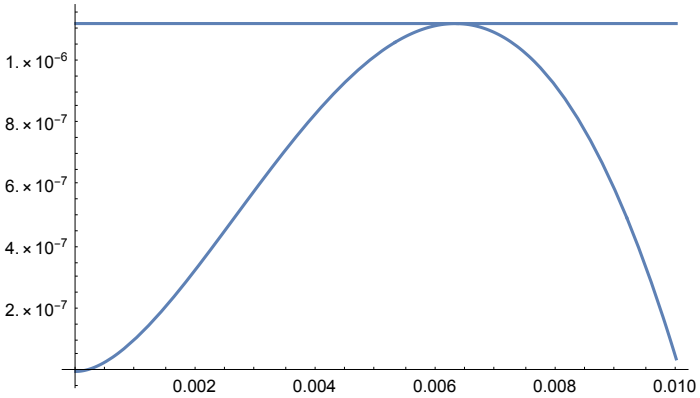


Out[•]= {0.0000406504, {x → 0.0312373}}

Out[•]= {12, 0.0000406504, {x → 0.0312378}}

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 15], {x, 0, 0.01}],
 ListLinePlot[{{0, Res[[13]][[2]]}, {0.01, Res[[13]][[2]]}}]]
FindMaximum[x^2/(1 + x^2) m6plus[x, 15], {x, 0, 0.01}]
Res[[13]]
```

Out[•]=

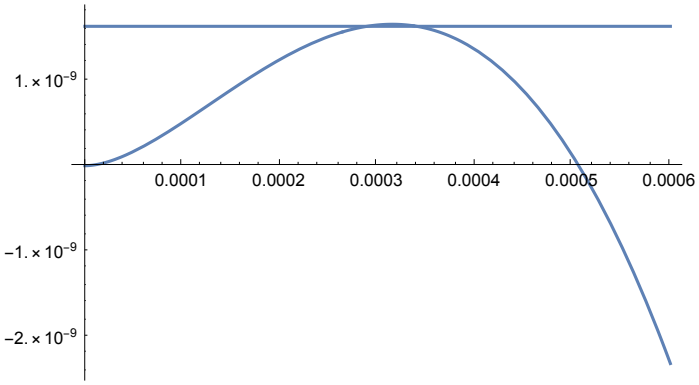


Out[•]= $\{1.12129 \times 10^{-6}, \{x \to 0.00631096\}\}$

Out[•]= $\{15, 1.12129 \times 10^{-6}, \{x \to 0.00632092\}\}$

In[•]:=
```
Show[Plot[x^2/(1 + x^2) m6plus[x, 20], {x, 0, 0.0006}], ListLinePlot[
  {{0, 1.6297293583994528`*^-9}, {0.0006, 1.6297293583994528`*^-9}}]]
FindMaximum[x^2/(1 + x^2) m6plus[x, 20], {x, 0, 0.0006}]
```

Out[•]=



Out[•]=
$$\{1.62973 \times 10^{-9}, \{x \to 0.000289058\}\}$$

Sign-changing solutions

In[•]:=
```
h[m_] := m^α + (1 - m)^α - 1
```

In[•]:=
```
Sd = 1/4 d (d - 2) (2 π^((d+1)/2)/Gamma[(d+1)/2])^(2/d);
α = 1 - 2/d;
hd = h[1/2];
Simplify[hd/(1 + hd)]
```

Out[•]=
$$1 - 4^{-1/d}$$

In[•]:=
```
gammad = 1/(2 h[1/2] Sd);
xid = 2 (1 - 2^-α);
zetad = Simplify[1 + xid gammad Sd];
etad = Simplify[1 + xid gammad Sd];
Simplify[zetad - (1 + (1 - 2/d) (2^(2/d) - 1)^-1)]
```

Out[•]=
$$-\frac{-4 + 4^{\frac{1}{d}}\, d}{2\left(-1 + 4^{\frac{1}{d}}\right) d}$$

In[•]:=
```
Simplify[hd / (2 hd + xid)]
```
Out[•]=
$$1 - 4^{-1/d}$$

In[•]:=
```
cd = Simplify[1 / (2 (zetad + etad))]
```
Out[•]=
$$\frac{1}{2}\left(1 - 4^{-1/d}\right)$$

In[•]:=
```
Table[{d, cd}, {d, 3, 15}];
N[%]
```
Out[•]=
```
{{3., 0.18502}, {4., 0.146447}, {5., 0.121071}, {6., 0.10315}, {7., 0.0898323},
 {8., 0.0795518}, {9., 0.071378}, {10., 0.0647247}, {11., 0.0592044},
 {12., 0.0545506}, {13., 0.0505745}, {14., 0.0471382}, {15., 0.0441388}}
```

In[•]:=
```
mestim[δ_] := 4 δ / (d + 4) - (1 - 2/d) δ^(d/(d-2))
```

In[•]:=
```
FullSimplify[Solve[mestim[δ] == 0, δ][[1]]]
```
Out[•]=
$$\left\{\delta \to 2^{-2+d}\left(2 - \frac{8}{d} + d\right)^{1-\frac{d}{2}}\right\}$$

In[•]:=
```
kappaestim = 2^(d+1) / (d (d + 4)^(d/2));
FullSimplify[D[mestim[δ], δ]]
FullSimplify[Solve[% == 0, δ][[1]]]
FullSimplify[{δ, mestim[δ] - kappaestim} /. %]
FullSimplify[PowerExpand[d (d + 4)^(d/2) %[[2]]]]
```
Out[•]=
$$\frac{4}{4+d} - \delta^{\frac{2}{-2+d}}$$

Out[•]=
$$\left\{\delta \to 2^{-2+d}\left(\frac{1}{4+d}\right)^{-1+\frac{d}{2}}\right\}$$

Out[•]=
$$\left\{2^{-2+d}\left(\frac{1}{4+d}\right)^{-1+\frac{d}{2}}, \frac{-(-2+d)\left(2^{-2+d}\left(\frac{1}{4+d}\right)^{-1+\frac{d}{2}}\right)^{\frac{d}{-2+d}} + 2^d\, d\left(\frac{1}{4+d}\right)^{d/2} - 2^{1+d}(4+d)^{-d/2}}{d}\right\}$$

Out[•]=
$$0$$

In[•]:=
```
Limit[{(1 - 2^(-2/d))/2 d/Log[2], kappaestim E^2 d^(1+d/2)/2^(d+1)}, d → ∞]
```
Out[•]=
$$\{1, 1\}$$