Logarithmic Sobolev and interpolation inequalities on the sphere : stability results

Section 1: Linear flow

In[∘]:= $\phi[s\_] := \dfrac{1-(p-2)\,s-(1-(p-2)\,s)^{-\frac{\gamma}{p-2}}}{2-p-\gamma}$

In[∘]:= `Simplify[{ϕ[0], ϕ'[0], ϕ''[0]}]`

Out[∘]= `{0, 1, γ}`

In[∘]:= $Zd = \text{Integrate}\left[\left(1-z^2\right)^{\frac{d}{2}-1}, \{z, -1, 1\}, \text{Assumptions} \to d \geq 1\right]$

$dmu[z\_] := \dfrac{1}{Zd}\left(1-z^2\right)^{\frac{d}{2}-1}$

Out[∘]= $\dfrac{\sqrt{\pi}\ \text{Gamma}\left[\frac{d}{2}\right]}{\text{Gamma}\left[\frac{1+d}{2}\right]}$

In[∘]:= $h[t\_] := t - \dfrac{1}{\gamma}\,\text{Log}[1+\gamma\, t]$

$hh[t\_] := \dfrac{t^2}{1+t\,\gamma}$

$\text{Simplify}\left[D\left[h[t] - \dfrac{\gamma}{2}\,hh[t], t\right]\right]$

Out[∘]= $\dfrac{t^2\,\gamma^2}{2\,(1+t\,\gamma)^2}$

In[∘]:= $u[z\_] := 1 + \epsilon\, z$

$\text{Integrate}\left[u[z]^q\,\dfrac{1}{Zd}\left(1-z^2\right)^{\frac{d}{2}-1}, \{z, -1, 1\}, \text{Assumptions} \to d \geq 1\ \&\&\ 0 < \epsilon < 1\right]$

Out[∘]= $\text{Hypergeometric2F1}\left[\dfrac{1-q}{2}, -\dfrac{q}{2}, \dfrac{1+d}{2}, \epsilon^2\right]$

Section 2: Coefficients

Main computations

In[∘]:= $\lambda[k\_] := k\,(k+d-1)$

In[∘]:= $h[k\_] := \dfrac{\lambda[k]}{\lambda[k+1]}\;\dfrac{k+d-x}{k+x}$

In[∘]:= $A1 = \text{Simplify}\left[\frac{1}{1-h[k]}\left(\frac{h[k]}{\lambda[k]}-\frac{1}{\lambda[k+1]}\right)\right]$

Out[∘]= $$\frac{-d+2x}{d^2 k-2k^2 (1+x)+d\left(k^2-x-2k(1+x)\right)}$$

In[∘]:= $A2 = \frac{d-2x}{k(k+d)(2+2x-d)+dx}$

Out[∘]= $$\frac{d-2x}{dx+k(d+k)(2-d+2x)}$$

In[∘]:= $\text{Simplify}\left[\frac{A1}{A2}\right]$

Out[∘]= 1

In[∘]:= `B1 = Simplify[A2 /. k → 2]`

Out[∘]= $$\frac{d-2x}{8-2d^2+8x+5dx}$$

In[∘]:= $B2 = \text{Simplify}\left[\frac{\frac{(d+1-x)(d-x)}{(x+1)x}-1}{2(d+1)}\right]$

Out[∘]= $$\frac{d-2x}{2x+2x^2}$$

In[∘]:= `Simplify[B2 - B1]`

Out[∘]= $$(d-2x)\left(\frac{1}{2x+2x^2}+\frac{1}{2d^2-5dx-8(1+x)}\right)$$

In[∘]:= `Solve[% == 0, x]`

Out[∘]= $$\left\{\left\{x\to\frac{1}{2}(-2+d)\right\},\left\{x\to\frac{d}{2}\right\},\{x\to 2(2+d)\}\right\}$$

In[∘]:= $\text{Simplify}\left[\frac{1}{2x+2x^2}+\frac{1}{2d^2-5dx-8(1+x)}\ /.\ x\to d/2\right]$

Out[∘]= $$\frac{4(8+3d)}{d(2+d)(4+d)^2}$$

In[∘]:= $\text{Simplify}\left[d\,\frac{\frac{(d+1-x)(d-x)}{(x+1)x}-1}{2(d+1)(q-2)}-1\ /.\ x\to\frac{d}{q}\right]$

Out[∘]= $$\frac{d(-2+q)-2q}{2(d+q)}$$

Other computations

In[∘]:= `FullSimplify[Gamma'[x] / Gamma[x]]`

Out[∘]= `PolyGamma[0, x]`

In[∘]:=
```
eta[k_] := FullSimplify[PolyGamma[0, k + d / 2] - PolyGamma[0, d / 2]]
Table[{k, FullSimplify[(k + d / 2)^2 - lambda[k] / (2 eta[k])]}, {k, 1, 5}]
g[k_] := lambda[k + 1] / (2 (k + 1 + d / 2)^2) - lambda[k] / (2 (k + d / 2)^2)
Simplify[g[k]]
```

Section 2

Stability constants

In[∘]:=
```
Zd = Integrate[(1 - z^2)^(d/2 - 1), {z, -1, 1}, Assumptions → d ≥ 1]
I2 = Integrate[z^2/Zd (1 - z^2)^(d/2 - 1), {z, -1, 1}, Assumptions → d ≥ 1]
I4 = Integrate[z^4/Zd (1 - z^2)^(d/2 - 1), {z, -1, 1}, Assumptions → d ≥ 1]
```

Out[∘]= $\dfrac{\sqrt{\pi}\ \mathrm{Gamma}\left[\frac{d}{2}\right]}{\mathrm{Gamma}\left[\frac{1+d}{2}\right]}$

Out[∘]= $\dfrac{1}{1+d}$

Out[∘]= $\dfrac{3}{3+4d+d^2}$

In[∘]:=
```
u[z_] := 1 + ϵ z
```

In[∘]:=
```
Normal[Series[u[z]^p, {ϵ, 0, 4}]]
```

Out[∘]= $1 + p z \epsilon + \frac{1}{2}(-1+p)\,p z^2 \epsilon^2 + \frac{1}{6}(-2+p)(-1+p)\,p z^3 \epsilon^3 + \frac{1}{24}(-3+p)(-2+p)(-1+p)\,p z^4 \epsilon^4$

In[∘]:=
```
Res = 1 + 1/2 (-1 + p) p z^2 ϵ^2 + 1/24 (-3 + p) (-2 + p) (-1 + p) p z^4 ϵ^4 /. {z^4 → I4, z^2 → I2}
```

Out[∘]= $1 + \dfrac{(-1+p)\,p\,\epsilon^2}{2\,(1+d)} + \dfrac{(-3+p)(-2+p)(-1+p)\,p\,\epsilon^4}{8\left(3+4d+d^2\right)}$

In[∘]:=
```
Resp = Normal[Series[Res^(2/p), {ϵ, 0, 4}]]
Res2 = Resp /. p → 2
```

Out[∘]= $1 + \dfrac{(-1+p)\,\epsilon^2}{1+d} - \dfrac{(-2+p)(-1+p)(d+p)\,\epsilon^4}{2\,(1+d)^2\,(3+d)}$

Out[∘]= $1 + \dfrac{\epsilon^2}{1+d}$

In[∘]:=
```
LHS = Normal[Series[d (Res2 - 1) - d (Resp - Res2)/(p - 2), {ϵ, 0, 4}]]
```

Out[∘]=
$$\frac{d\,(-1+p)\,(d+p)\,\epsilon^4}{2\,(1+d)^2\,(3+d)}$$

In[∘]:=
```
RHS = Normal[Series[((d + 1) (Res2 - 1))^2/(d (Res2 - 1) + d/(p-2) Res2), {ϵ, 0, 4}]];
LHS/RHS
```

Out[∘]=
$$\frac{d^2\,(-1+p)\,(d+p)}{2\,(1+d)^2\,(3+d)\,(-2+p)}$$

In[∘]:=
```
RHS = Normal[Series[(d (Res2 - 1))^2/(d (Res2 - 1) + d/(p-2) Res2), {ϵ, 0, 4}]];
LHS/RHS
```

Out[∘]=
$$\frac{(-1+p)\,(d+p)}{2\,(3+d)\,(-2+p)}$$

Section 3

Improved Bakry-Emery estimates

```
phis = FullSimplify[t /. Solve[t - cc (t + d s)^2/(t + d s + 1 - (p - 2) s) == 0, t]][[2]]
Series[%, {s, 0, 2}]
```

In[∘]:=
```
Resx = Simplify[x /. Solve[y - d x == cc y^2/(y + d/(p-2) - d x), x]]
Limit[Resx/y, y → ∞]
```

Out[∘]=
$$\left\{\frac{d-4\,y+2\,p\,y-\sqrt{d^2+4\,cc\,(-2+p)^2\,y^2}}{2\,d\,(-2+p)},\ \frac{d-4\,y+2\,p\,y+\sqrt{d^2+4\,cc\,(-2+p)^2\,y^2}}{2\,d\,(-2+p)}\right\}$$

Out[∘]=
$$\left\{\frac{2+\sqrt{cc\,(-2+p)^2}-p}{2\,d-d\,p},\ \frac{-2+\sqrt{cc\,(-2+p)^2}+p}{d\,(-2+p)}\right\}$$

Section 4

Series expansions

In[∘]:=
```
Simplify[Normal[Series[(1 + s)^(2/p), {s, 0, 3}]]]
```

Out[∘]= $1+\frac{2\,s}{p}-\frac{(-2+p)\,s^2}{p^2}+\frac{2\,(-2+p)\,(-1+p)\,s^3}{3\,p^3}$

In[∘]:=
```
Normal[Series[(1 + x)^p, {x, 0, 6}]]
```

Out[∘]= $1+p\,x+\frac{1}{2}\,(-1+p)\,p\,x^2+\frac{1}{6}\,(-2+p)\,(-1+p)\,p\,x^3+\frac{1}{24}\,(-3+p)\,(-2+p)\,(-1+p)\,p\,x^4+\frac{1}{120}\,(-4+p)\,(-3+p)\,(-2+p)\,(-1+p)\,p\,x^5+\frac{1}{720}\,(-5+p)\,(-4+p)\,(-3+p)\,(-2+p)\,(-1+p)\,p\,x^6$

In[∘]:=
```
Normal[Series[(1 + s)^(2/p), {s, 0, 3}]]
% /. s → a x + b x^2 + c x^3
Normal[Series[%, {x, 0, 9}]];
Simplify[(% - (1 + 2 a x/p + (-a^2 (-2+p)/p^2 + 2 b/p) x^2))/x^3 /. x → 1/2]
```

Out[∘]= $1+\frac{2\,s}{p}+\frac{(2-p)\,s^2}{p^2}+\frac{2\,(-2+p)\,(-1+p)\,s^3}{3\,p^3}$

Out[∘]= $1+\frac{2\,\left(a\,x+b\,x^2+c\,x^3\right)}{p}+\frac{(2-p)\,\left(a\,x+b\,x^2+c\,x^3\right)^2}{p^2}+\frac{2\,(-2+p)\,(-1+p)\,\left(a\,x+b\,x^2+c\,x^3\right)^3}{3\,p^3}$

Out[∘]= $\frac{1}{96\,p^3}\left(6\,b\,c\,(c\,(-1+p)-8\,p)\,(-2+p)+12\,a\,(2\,b+c)\,(2\,b\,(-1+p)+c\,(-1+p)-8\,p)\,(-2+p)+12\,b^2\,(c\,(-1+p)-4\,p)\,(-2+p)+64\,a^3\,\left(2-3\,p+p^2\right)+8\,b^3\,\left(2-3\,p+p^2\right)+48\,a^2\,(2\,b+c)\,\left(2-3\,p+p^2\right)+c\,\left(-12\,c\,(-2+p)\,p+192\,p^2+c^2\,\left(2-3\,p+p^2\right)\right)\right)$

Spherical harmonics

In[∘]:=
```
Y[z_] := Sqrt[(d + 1)/d] z
Simplify[(1 - z^2) Y''[z] - d z Y'[z] + d Y[z]]
```

Out[∘]= 0

In[∘]:=
```
Integrate[Y'[z]^2 (1 - z^2) dmu[z], {z, -1, 1}, Assumptions → d ≥ 1]
Integrate[Y[z]^2 dmu[z], {z, -1, 1}, Assumptions → d ≥ 1]
Integrate[Y[z]^4 dmu[z], {z, -1, 1}, Assumptions → d ≥ 1]
Integrate[Y[z]^6 dmu[z], {z, -1, 1}, Assumptions → d ≥ 1]
```

Out[∘]= 1

Out[∘]= $\frac{1}{d}$

Out[∘]= $\frac{3\,(1+d)}{d^2\,(3+d)}$

Out[∘]= $\frac{15\,(1+d)^2}{d^3\,(3+d)\,(5+d)}$

In[∘]:=
```
Y2[z_] := Y[z]^2 - 1/d
Simplify[(1 - z^2) Y2''[z] - d z Y2'[z] + 2 (d + 1) Y2[z]]
Integrate[{Y2[z]^2 dmu[z], Y2'[z]^2 (1 - z^2) dmu[z]}, {z, -1, 1}, Assumptions → d ≥ 1]
FullSimplify[PowerExpand[%[[1]]/Sqrt[%[[2]]]]]
```

Out[∘]= 0

Out[∘]= $\left\{\frac{2}{3\,d+d^2}, \frac{4\,(1+d)}{d\,(3+d)}\right\}$

Out[∘]= $\frac{1}{\sqrt{d}\,\sqrt{1+d}\,\sqrt{3+d}}$

Out[∘]= $\left\{\frac{2}{3\,d+d^2}, \frac{4\,(1+d)}{d\,(3+d)}\right\}$

```
{Integrate[Y2[z]^2 dmu[z], {z, -1, 1}, Assumptions → d ≥ 1],
 Integrate[Y2'[z]^2 (1 - z^2) dmu[z], {z, -1, 1}, Assumptions → d ≥ 1]}
```

In[•]:=
```
Y3[z_] := Y[z]^3 - (3 (d + 1))/(d (d + 3)) Y[z]
Simplify[(1 - z^2) Y3''[z] - d z Y3'[z] + 3 (d + 2) Y3[z]]
Integrate[{Y3[z]^2 dmu[z], Y3'[z]^2 (1 - z^2) dmu[z]}, {z, -1, 1}, Assumptions → d ≥ 1]
FullSimplify[PowerExpand[%[[1]]/Sqrt[%[[2]]]]]
```

Out[•]= 0

Out[•]= $\left\{\frac{6\,(1+d)^2}{d^2\,(3+d)^2\,(5+d)},\ \frac{18\,(1+d)^2\,(2+d)}{d^2\,(3+d)^2\,(5+d)}\right\}$

Out[•]= $\frac{\sqrt{2}\,(1+d)}{d\,\sqrt{2+d}\,(3+d)\,\sqrt{5+d}}$

Out[•]= $\frac{6\,(1+d)^2}{d^2\,(3+d)^2\,(5+d)}$

Out[•]= $\frac{18\,(1+d)^2\,(2+d)}{d^2\,(3+d)^2\,(5+d)}$

Section 4

Global stability constant

In[•]:=
```
apd = (p (p - 1))/(2 d);
bpd = 1/24 p (p - 1) (p - 2) (p - 3) (3 (d + 1))/(d^2 (d + 3));
Simplify[bpd - ((p - 2) (p - 3))/4 apd (d + 1)/(d (d + 3))]
```

Out[•]= 0

In[•]:=
```
Simplify[((d (p - 1))/Sqrt[d (d + 1) (d + 3)])^2 - 4 ((-1 + p) (d + p))/(2 d (3 + d)) (d + 2)/(2 (d + 1))]
```

Out[•]= $\frac{(d\,(-2+p) - 2\,p)\,(-1+p)}{d\,(3+d)}$

In[•]:=
```
Solve[B^2 - 4 (A - x) (C - x) == 0, x]
```

Out[•]= $\left\{\left\{x \to \frac{1}{2}\left(A + C - \sqrt{A^2 + B^2 - 2\,A\,C + C^2}\right)\right\},\ \left\{x \to \frac{1}{2}\left(A + C + \sqrt{A^2 + B^2 - 2\,A\,C + C^2}\right)\right\}\right\}$

Appendix B: Carré du champ, range

Main computations

In[∘]:= $\text{Res}\beta = \text{Solve}\left[m == 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right), \beta\right][[1]]$

Out[∘]= $\left\{\beta \to \frac{2}{2 - p + m\,p}\right\}$

In[∘]:= $f[m\_] := \text{Simplify}\left[\left(\frac{d-1}{d+2}\right)^2 (\beta\,(p-1))^2 - (\beta\,(p-2)+1)\,(\beta-1) - \frac{d}{d+2}\,\beta\,(p-1)\ /.\ \text{Res}\beta\right]$

In[∘]:= $\text{Res} = \text{Normal}\left[\text{Series}\left[(2+d)^2\,(2+(-1+m)\,p)^2\,f[m], \{m, 0, 2\}\right]\right]$

$\text{Simplify}[\text{Solve}[\text{Res} == 0, m]];$

$\text{Simplify}\left[\%\ /.\ \sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)\,p^2} \to (2+d)\,p\,\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}\right]$

Out[∘]= $4 + 4\,d\,(3-2\,p)\,p + d^2\left(8 - 12\,p + 5\,p^2\right) + m^2\left(4\,p^2 + 4\,d\,p^2 + d^2\,p^2\right) + m\left(-8\,p - 2\,d^2\,p^2 - 4\,d\,p\,(1+p)\right)$

Out[∘]= $\left\{\left\{m \to \frac{2 - 2\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)} + d\,p}{(2+d)\,p}\right\}, \left\{m \to \frac{2 + 2\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)} + d\,p}{(2+d)\,p}\right\}\right\}$

Verifications

In[∘]:= $\text{Res}\beta = \text{Solve}\left[m == 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right), \beta\right][[1]]$

Out[∘]= $\left\{\beta \to \frac{2}{2 - p + m\,p}\right\}$

In[∘]:= $f[m\_] := \text{Simplify}\left[\left(\frac{d-1}{d+2}\right)^2 (\beta\,(p-1))^2 - (\beta\,(p-2)+1)\,(\beta-1) - \frac{d}{d+2}\,\beta\,(p-1)\ /.\ \text{Res}\beta\right]$

In[∘]:= $(2+d)^2\,(2+(-1+m)\,p)^2\,f[m]$

Out[∘]= $4\,(-1+m\,p)^2 + d^2\left(8 - 12\,p + \left(5 - 2\,m + m^2\right)p^2\right) + 4\,d\,p\left(3 - 2\,p + m^2\,p - m\,(1+p)\right)$

In[∘]:= $\text{Simplify}\left[\text{Series}\left[(2+d)^2\,(2+(-1+m)\,p)^2\,f[m], \{m, 0, 2\}\right]\right]$

Out[∘]= $\left(4 + 4\,d\,(3-2\,p)\,p + d^2\left(8 - 12\,p + 5\,p^2\right)\right) - 2\,((2+d)\,p\,(2+d\,p))\,m + (2+d)^2\,p^2\,m^2 + O[m]^3$

In[•]:= Res = Normal[Series[(2 + d)^2 (2 + (-1 + m) p)^2 f[m], {m, 0, 2}]]
Simplify[Solve[Res == 0, m]];
Simplify[% /. $\sqrt{-d\,(2+d)^2\,(d\,(-2+p)-2\,p)\,(-1+p)\,p^2} \to (2+d)\,p\,\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}$]

Out[•]= $4+4\,d\,(3-2\,p)\,p+d^2\left(8-12\,p+5\,p^2\right)+m^2\left(4\,p^2+4\,d\,p^2+d^2\,p^2\right)+m\left(-8\,p-2\,d^2\,p^2-4\,d\,p\,(1+p)\right)$

Out[•]= $\left\{\left\{m\to\frac{2-2\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}+d\,p}{(2+d)\,p}\right\},\left\{m\to\frac{2+2\sqrt{-d\,(d\,(-2+p)-2\,p)\,(-1+p)}+d\,p}{(2+d)\,p}\right\}\right\}$

In[•]:= Simplify[Res]

Out[•]= $4+4\,d\,(3-2\,p)\,p+(2+d)^2\,m^2\,p^2-2\,(2+d)\,m\,p\,(2+d\,p)+d^2\left(8-12\,p+5\,p^2\right)$

In[•]:= Simplify[Res /. m → 0]

Out[•]= $4+4\,d\,(3-2\,p)\,p+d^2\left(8-12\,p+5\,p^2\right)$

In[•]:= Solve[β == 1 /. $\beta \to \frac{2}{2-p+m\,p}$, m][[1]]
Simplify[$4+4\,d\,(3-2\,p)\,p+(2+d)^2\,m^2\,p^2-2\,(2+d)\,m\,p\,(2+d\,p)+d^2\left(8-12\,p+5\,p^2\right)$ /. %]
Simplify[% /. $p\to\frac{2\,d^2+1}{(d-1)^2}$]

Out[•]= {m → 1}

Out[•]= $4\,(-1+p)\left(-1+d^2\,(-2+p)+p-2\,d\,p\right)$

Out[•]= 0

In[•]:= Solve[$\beta == \frac{2}{4-p}$ /. $\beta\to\frac{2}{2-p+m\,p}$, m][[1]]
Simplify[$4+4\,d\,(3-2\,p)\,p+(2+d)^2\,m^2\,p^2-2\,(2+d)\,m\,p\,(2+d\,p)+d^2\left(8-12\,p+5\,p^2\right)$ /. %]

Out[•]= $\left\{m\to\frac{2}{p}\right\}$

Out[•]= $4+d\left(8+4\,p-8\,p^2\right)+d^2\left(12-16\,p+5\,p^2\right)$

In[•]:= Solve[$2\,\beta-2 == -1+\frac{p\,(\beta-1)}{(p-2)}+2\,\beta\,\delta$, δ][[1]]

Out[•]= $\left\{\delta\to\frac{2-4\,\beta+p\,\beta}{2\,(-2+p)\,\beta}\right\}$

In[•]:= Simplify[$2\,\beta-2+1-\frac{p\,(\beta-1)}{(p-2)}-2\,\beta\,\frac{2-(4-p)\,\beta}{2\,\beta\,(p-2)}$]

Out[•]= 0

In[ ]:= Solve$\left[\left\{\beta == \frac{4}{6-p}, m == 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)\right\}, \{m, \beta\}\right][[1]]$

Out[ ]= $\left\{m \to \frac{2+p}{2\,p}, \beta \to -\frac{4}{-6+p}\right\}$

In[ ]:= Simplify$\left[\frac{2 - 4\,\beta + p\,\beta}{2\,(-2+p)\,\beta} \;/.\; \beta \to \frac{4}{6-p}\right]$

Out[ ]= $\frac{1}{4}$