Weighted Sobolev inequalities for spinors: symmetry and symmetry breaking

• Linear Instability

In[∘]:= $A = \frac{p}{2}\left(\frac{1}{2} - \alpha\right)^2;$

$B = \frac{p-2}{2}\,\mathrm{Abs}\left[\frac{1}{2} - \alpha\right];$

$\delta[k\_, m\_] := \mathrm{If}\left[m == \frac{1}{2}, \frac{k+m+\frac{1}{2}}{2k+1}, \frac{1}{2}\,\frac{k+m+\frac{1}{2}}{2k+1}\right]$

$\mathrm{Akm}[k\_, m\_] := (1 + (p-2)\,\delta[k, m])\,A$

$\mathrm{Qkm}[k\_, m\_] := 4\left(k + \frac{1}{2} - \alpha\right)^2 - \left(\sqrt{4\,\mathrm{Akm}[k, m] + B^2} - B\right)^2$

k = 1, m = 1/2, -1/2 < α < 0

In[∘]:= $\mathrm{Simplify}\left[\mathrm{Qkm}\left[1, \frac{1}{2}\right] /. \mathrm{Abs}\left[\frac{1}{2} - \alpha\right] \to \frac{1}{2} - \alpha\right]$

Out[∘]= $(3 - 2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha + \sqrt{3}\sqrt{(12 - 20p + 19p^2)(-1 + 2\alpha)^2} + p(-3 + 6\alpha)\right)^2$

In[∘]:= $Q[\alpha\_, p\_] := (3 - 2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha + (1 - 2\alpha)\sqrt{3}\sqrt{12 - 20p + 19p^2} + p(-3 + 6\alpha)\right)^2$

FullSimplify[p /. Solve[Q[α, p] == 0, p][[1]]]

$\mathrm{p1}[\alpha\_] := \frac{5 - 4\alpha + \sqrt{97 + 8\alpha(-11 + 2\alpha)}}{2 - 4\alpha}$

Out[∘]= $\frac{5 - 4\alpha + \sqrt{97 + 8\alpha(-11 + 2\alpha)}}{2 - 4\alpha}$

In[∘]:= Solve[p1[α] == 6, α][[1]];

{p1[α], %, N[α /. %]}

Out[∘]= $\left\{\frac{5 - 4\alpha + \sqrt{97 + 8\alpha(-11 + 2\alpha)}}{2 - 4\alpha}, \left\{\alpha \to \frac{1}{4}\left(1 - \sqrt{3}\right)\right\}, -0.183013\right\}$

k = -2, m = 1/2, -1/2 < α < 0

In[∘]:= $\mathrm{Simplify}\left[\mathrm{Qkm}\left[-2, \frac{1}{2}\right] /. \mathrm{Abs}\left[\frac{1}{2} - \alpha\right] \to \frac{1}{2} - \alpha\right]$

Out[∘]= $(3 + 2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha + \sqrt{3}\sqrt{(12 - 4p + 11p^2)(-1 + 2\alpha)^2} + p(-3 + 6\alpha)\right)^2$

In[∘]:= $Q[\alpha\_, p\_] := (3+2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha + \sqrt{3}\,(1-2\alpha)\sqrt{\left(12 - 4p + 11p^2\right)} + p\,(-3+6\alpha)\right)^2$

```
FullSimplify[p /. Solve[Q[α, p] == 0, p]][[4]]
```

$p2[\alpha\_] := \frac{2\left(2 - 2\alpha(1+2\alpha) + \sqrt{(1-2\alpha)^2\,(13+16\alpha(2+\alpha))}\right)}{(1-2\alpha)^2}$

Out[∘]= $\frac{2\left(2 - 2\alpha(1+2\alpha) + \sqrt{(1-2\alpha)^2\,(13+16\alpha(2+\alpha))}\right)}{(1-2\alpha)^2}$

```
In[∘]:= Solve[p2[α] == 6, α][[1]];
{%, N[α /. %]}
```

Out[∘]= $\left\{\left\{\alpha \to \frac{1}{2}\left(1-\sqrt{2}\right)\right\}, -0.207107\right\}$

k = -2, m = 1/2, 1/2 < α < 3/2

In[∘]:= $\text{Simplify}\left[\text{Qkm}\left[1, \frac{1}{2}\right] \;/.\; \text{Abs}\left[\frac{1}{2} - \alpha\right] \to \frac{1}{2} - \alpha\right]$

Out[∘]= $(3-2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha + \sqrt{3}\sqrt{\left(12 - 20p + 19p^2\right)(-1+2\alpha)^2} + p\,(-3+6\alpha)\right)^2$

In[∘]:= $Q[\alpha\_, p\_] := (3-2\alpha)^2 - \frac{1}{144}\left(6 - 12\alpha - (1-2\alpha)\sqrt{3}\sqrt{12 - 20p + 19p^2} + p\,(-3+6\alpha)\right)^2$

```
FullSimplify[p /. Solve[Q[α, p] == 0, p]][[4]]
```

$pp3[\alpha\_] := \frac{-2 + (5-2\alpha)\alpha - (1-2\alpha)\sqrt{40 + \alpha(-64+25\alpha)}}{(1-2\alpha)^2}$

Out[∘]= $\frac{-2 + (5-2\alpha)\alpha + \sqrt{(1-2\alpha)^2\,(40 + \alpha(-64+25\alpha))}}{(1-2\alpha)^2}$

```
In[∘]:= Simplify[Solve[pp3[α] == 6, α][[1]]];
{%, N[α /. %]}
p3[α_] := If[α < 1, pp3[α], 2]
```

Out[∘]= $\left\{\left\{\alpha \to \frac{1}{6}\left(3+\sqrt{3}\right)\right\}, 0.788675\right\}$

Figures (colours)

In[∘]:=
```
SymFilling = RGBColor[0.296875`, 0.59765625`, 0]
SymBreakingFilling = RGBColor[0.8, 0, 0]
SymThreshold = Blue
```

Out[∘]=

Out[∘]=

Out[∘]=

Symmetry breaking range: -1/2 < α < 0

In[∘]:=
```
P1a = Show[
    Plot[{6, p2[α]}, {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black, Filling → {1 → {2}},
     FillingStyle → SymBreakingFilling], Plot[{6, p1[α]}, {α, -0.5, 1/4 (1 - √3)},
     PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
    Plot[p2[α], {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black], PlotRange → All];
```

Symmetry breaking range: -1 < α < -1/2

In[∘]:=
```
P1b = Show[
    Plot[{6, p2[-α - 1]}, {α, -0.5, -1/2 (3 - √2)}, PlotStyle → Black,
     Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
    Plot[{6, p1[-α - 1]}, {α, -0.5, 1/4 (√3 - 5)}, PlotStyle → Black,
     Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
    Plot[p2[-α - 1], {α, -0.5, -1/2 (3 - √2)}, PlotStyle → Black], PlotRange → All];
```

Symmetry breaking range: 1/2 < α < 3/2

In[∘]:=
```
P2a = Show[
    Plot[{6, p3[α]}, {α, 1/6 (3 + √3), 3}, PlotStyle → Black,
     Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

Symmetric breaking range: -2 < α < -1

In[∘]:=
```
P2b = Show[
  Plot[{6, p3[-α - 1]}, {α, -1 - 1/6 (3 + √3), -4}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

• Symmetry range

Analysis of the condition $a_l^+ + b_l^+ < 1$

In[∘]:=
```
aa[α_] := 4 (1 - α) / ((l + 1) (l + 2 - 2 α) (l q + 2))
bb[α_] := l (l + 1 - 2 α) ((l + 2) q - 2) / ((l + 1) (l + 2 - 2 α) (l q + 2))
```

In[∘]:=
```
Res = Simplify[aa[α] + bb[α]];
{Simplify[D[Res, α]], Simplify[Res - 1 /. {q → 2, α → 1 / 2}]}
{Solve[Res == 1, q][[1]][[1]],
 Simplify[Solve[{q == 2 (2 + p)/(6 - p), Res == 1 /. l → 1}, {q, p}][[1]]]}
```

Out[∘]= $\left\{-\frac{2\, l\, (2+l)\, q}{(1+l)\,(2+l\, q)\,(2+l-2\,\alpha)^2},\ -\frac{l\,(3+2\, l)}{(1+l)^3}\right\}$

Out[∘]= $\left\{q \to \frac{2\,(-2-l+2\,\alpha)}{\alpha},\ \left\{q \to 4-\frac{6}{\alpha},\ p \to \frac{2\,(-9+5\,\alpha)}{3\,(-1+\alpha)}\right\}\right\}$

In[∘]:=
```
Simplify[{aa[α], bb[α]} /. l → 1/x];
Simplify[{%, D[%, x]} /. x → 0]
```

Out[∘]= $\left\{\{0,\ 1\},\ \left\{0,\ -\frac{4}{q}\right\}\right\}$

Analysis of the condition $a_l^- + b_l^- < 1$

In[∘]:=
```
aa[α_] := 4 (1 + 2 α) / ((l + 2) (l + 1 + 2 α) (l q + 2))
bb[α_] := (l + 1) (l + 2 α) ((l + 2) q - 2) / ((l + 2) (l + 1 + 2 α) (l q + 2))
```

In[∘]:=
```
Res = Simplify[aa[α] + bb[α]];
{Solve[Res == 1, q][[1]][[1]],
  Simplify[Solve[{q == 2 (2 + p)/(6 - p), Res == 1 /. l → 1}, {q, p}][[1]]]}
```

Out[∘]=
$$\left\{q\to\frac{2\left(2\,l+l^2+\alpha+2\,l\,\alpha\right)}{(2+l)\,\alpha},\ \left\{q\to 2+\frac{2}{\alpha},\ p\to\frac{6+4\,\alpha}{1+2\,\alpha}\right\}\right\}$$

In[∘]:=
```
Show[Plot[2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), {α, -0.5, 0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[{(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (6 + 4 α)/(1 + 2 α)]}, {α, 0, 1},
   PlotStyle → {{Thick, SymThreshold}, {Thick, SymThreshold}}], PlotRange → All];
Plot[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)],
  {α, -0.5, 1}, PlotStyle → {Thick, SymThreshold}];
P0a = Show[Plot[{2, If[0 < α < 1/2, 6,
      Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]}, {α, -1/2, 1},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], %];
```

In[∘]:=
```
Show[Plot[(6 + 8 α^2)/(7 + 8 α + 4 α^2), {α, -1.5, -0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[{(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), Min[(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]}, {α, -2, -1},
   PlotStyle → {{Thick, SymThreshold}, {Thick, SymThreshold}}], PlotRange → All];
Plot[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)],
  {α, -2, -0.5}, PlotStyle → {Thick, SymThreshold}];
P0b = Show[Plot[{2, If[-3/2 < α < -1, 6,
      Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]}, {α, -2, -1/2},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], %];
```

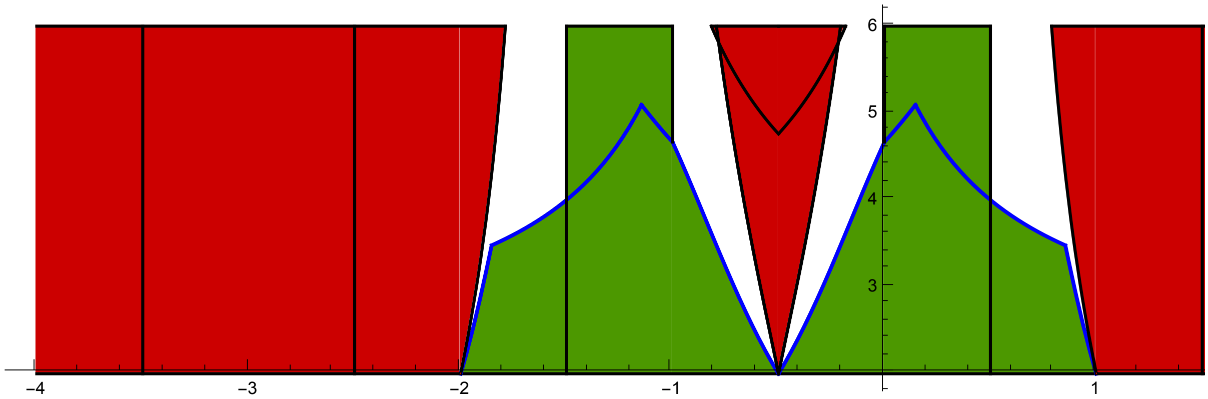
• Figures: Symmetry and symmetry breaking ranges

Separation lines

In[∘]:=
```
LP[α_] := ListLinePlot[{{α, 2}, {α, 6}}, PlotStyle → Black]
LPp[α_] := ListLinePlot[{{α, 14 / 3}, {α, 6}}, PlotStyle → Black]
LPh[h_] := ListLinePlot[{{-4.2, h}, {3.2, h}}, PlotStyle → Black]
P = Show[Table[LP[k - 0.5], {k, -3, 3}], LPp[-1], LPp[0], LPh[2], LPh[6]];
```

In[∘]:=
```
Show[Show[P0a, P0b, P1a, P1b, P2a, P2b, PlotRange → {{-4, 3}, Automatic},
  AspectRatio → 0.25], Show[Table[LP[k - 0.5], {k, -3, -1}],
  Table[LP[k - 0.5], {k, 1, 3}], LPp[-1], LPp[0], PlotRange → All]]
```

Out[∘]=



Plot: SCKNlog

In[∘]:= SymThreshold = Black;

$$\text{Show}\left[\text{Plot}\left[\frac{2\left(7+8\alpha+4\alpha^2\right)}{3+4\alpha^2}, \{\alpha, -0.5, 0.5\}, \text{PlotStyle}\to\{\text{Thick}, \text{SymThreshold}\}\right],\right.$$

$$\text{Plot}\left[\left\{\frac{14-20\alpha+8\alpha^2}{3-6\alpha+4\alpha^2}, \text{Min}\left[\frac{14-20\alpha+8\alpha^2}{3-6\alpha+4\alpha^2}, \frac{6+4\alpha}{1+2\alpha}\right]\right\}, \{\alpha, 0, 1\},\right.$$

$$\left.\left.\text{PlotStyle}\to\{\{\text{Thick}, \text{SymThreshold}\}, \{\text{Thick}, \text{SymThreshold}\}\}\right], \text{PlotRange}\to\text{All}\right];$$

$$\text{Plot}\left[\text{Min}\left[\frac{14-20\alpha+8\alpha^2}{3-6\alpha+4\alpha^2}, \frac{2\left(7+8\alpha+4\alpha^2\right)}{3+4\alpha^2}, \frac{2\,(-9+5\alpha)}{3\,(-1+\alpha)}, \frac{6+4\alpha}{1+2\alpha}\right],\right.$$

$$\left.\{\alpha, -0.5, 1\}, \text{PlotStyle}\to\{\text{Thick}, \text{SymThreshold}\}\right];$$

$$\text{P0a} = \text{Show}\left[\text{Plot}\left[\left\{2, \text{If}\left[0<\alpha<\frac{1}{2}, 6,\right.\right.\right.\right.$$

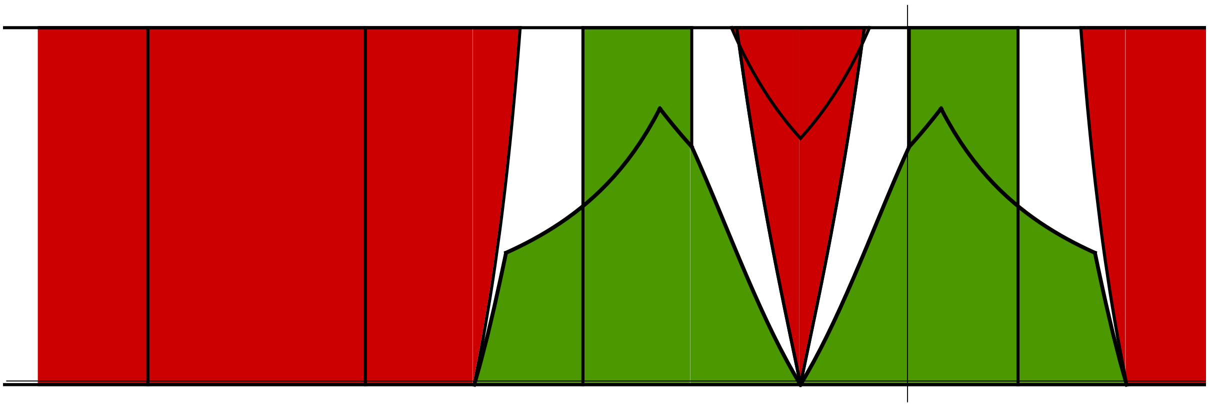
$$\left.\left.\left.\text{Min}\left[\frac{14-20\alpha+8\alpha^2}{3-6\alpha+4\alpha^2}, \frac{2\left(7+8\alpha+4\alpha^2\right)}{3+4\alpha^2}, \frac{2\,(-9+5\alpha)}{3\,(-1+\alpha)}, \frac{6+4\alpha}{1+2\alpha}\right]\right]\right\}, \left\{\alpha, -\frac{1}{2}, 1\right\},\right.$$

$$\left.\left.\text{PlotStyle}\to\text{Black}, \text{Filling}\to\{1\to\{2\}\}, \text{FillingStyle}\to\text{SymFilling}\right], \%\right];$$

In[•]:=
```
Show[Plot[(6 + 8 α^2)/(7 + 8 α + 4 α^2), {α, -1.5, -0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[{(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), Min[(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]}, {α, -2, -1},
   PlotStyle → {{Thick, SymThreshold}, {Thick, SymThreshold}}], PlotRange → All];
Plot[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)],
  {α, -2, -0.5}, PlotStyle → {Thick, SymThreshold}];
P0b = Show[Plot[{2, If[-3/2 < α < -1, 6,
      Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]}, {α, -2, -1/2},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], %];
```

In[•]:=
```
Show[Show[P0a, P0b, P1a, P1b, P2a, P2b,
   PlotRange → {{-4, 3}, Automatic}, AspectRatio → 0.25],
  Show[Table[LP[k - 0.5], {k, -3, -1}], Table[LP[k - 0.5], {k, 1, 3}],
   LPp[-1], LPp[0], LPh[2], LPh[6], PlotRange → All], Ticks → None]
```

Out[•]=



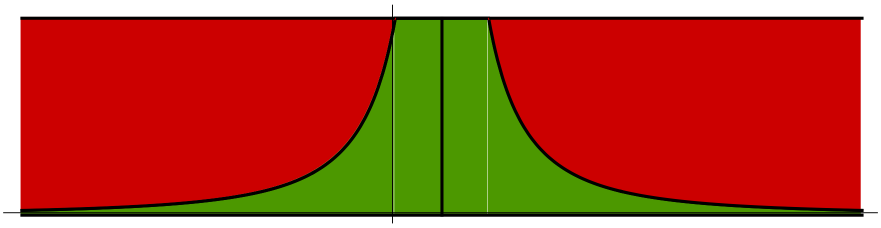
• Scalar CKN inequality: Symmetry and symmetry breaking ranges

In[•]:=
```
ϵ = 10^-14;
```

Plot: CKNlog

In[•]:=
```
Show[Plot[{2, Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]}, {a, -4, 5}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymFilling, PlotRange → All],
  Plot[{6, Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]}, {a, -4, 5}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling, PlotRange → All],
  ListLinePlot[{{0.5, 2}, {0.5, 6}}, PlotStyle → Black],
  AspectRatio → 0.25, Ticks → None]
```

Out[•]=



• Scalar CKN inequality: Symmetry and symmetry breaking ranges in (α,β) coordinates

In[•]:=
```
Solve[6/(1 + 2 b - 2 a) == p, b][[1]]
Ba[p_] := (6 - p + 2 a p)/(2 p)
```

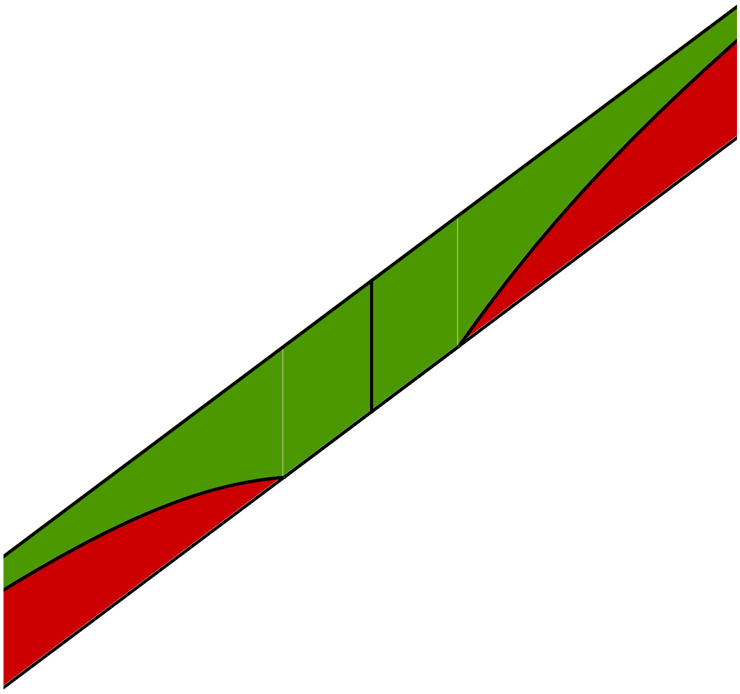
Out[•]=
$$\left\{b \to \frac{6 - p + 2\,a\,p}{2\,p}\right\}$$

In[•]:=
```
Show[Plot[{a, Ba[Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]]}, {a, -3, 4},
   PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[{a + 1, Ba[Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]]}, {a, -3, 4}, PlotPoints → 80,
   PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling],
  ListLinePlot[{{0.5, 0.5}, {0.5, 1.5}}, PlotStyle → Black]];
```

Plot: CKN

In[∘]:=
```
Show[%, PlotRange → {{-1.5, 2.5}, {-1.5, 3.5}},
 AspectRatio → 1, AxesOrigin → {0.0}, Ticks → None, Axes → None]
```

Out[∘]=



• SCKN inequality: Symmetry and symmetry breaking ranges in (α,β) coordinates

In[∘]:=
$$B\alpha[p\_] := \frac{6 - p + 2\,\alpha\,p}{2\,p}$$

```
In[]:= SymThreshold = Blue;
       Show[Plot[Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]],
          {α, -2, -0.5}, PlotStyle → {Thick, SymThreshold}],
         Plot[Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (2 (7 + 8 α + 4 α^2))/(3 + 4 α^2), (2 (-9 + 5 α))/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]],
          {α, -0.5, 1}, PlotStyle → {Thick, SymThreshold}], PlotRange → All];
       P0 = Show[Plot[{α + 1, If[-3/2 < α < -1, α,
              Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]]},
           {α, -2, -0.5}, PlotStyle → Black, Filling → {1 → {2}},
           FillingStyle → SymFilling], Plot[{α + 1, If[0 < α < 1/2, α,
              Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (2 (7 + 8 α + 4 α^2))/(3 + 4 α^2), (2 (-9 + 5 α))/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]]},
           {α, -0.5, 1}, PlotStyle → Black, Filling → {1 → {2}},
           FillingStyle → SymFilling], %, PlotRange → All];

In[]:= P1a = Show[Plot[{α, Bα[p1[α]]}, {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black,
           Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
         Plot[{α, Bα[p2[α]]}, {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black,
           Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
         Plot[Bα[p1[α]], {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black], PlotRange → All];

In[]:= P1b = Show[Plot[{α, Bα[p1[-α - 1]]}, {α, -0.5, 1/4 (√3 - 5)},
           PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
         Plot[{α, Bα[p2[-α - 1]]}, {α, -0.5, -1/2 (3 - √2)}, PlotStyle → Black,
           Filling → {1 → {2}}, FillingStyle → SymBreakingFilling], Plot[
           Bα[p1[-α - 1]], {α, -0.5, 1/4 (√3 - 5)}, PlotStyle → Black], PlotRange → All];

In[]:= P2a = Show[
         Plot[{α, Bα[p3[α]]}, {α, 1/6 (3 + √3), 2}, PlotStyle → Black,
           Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

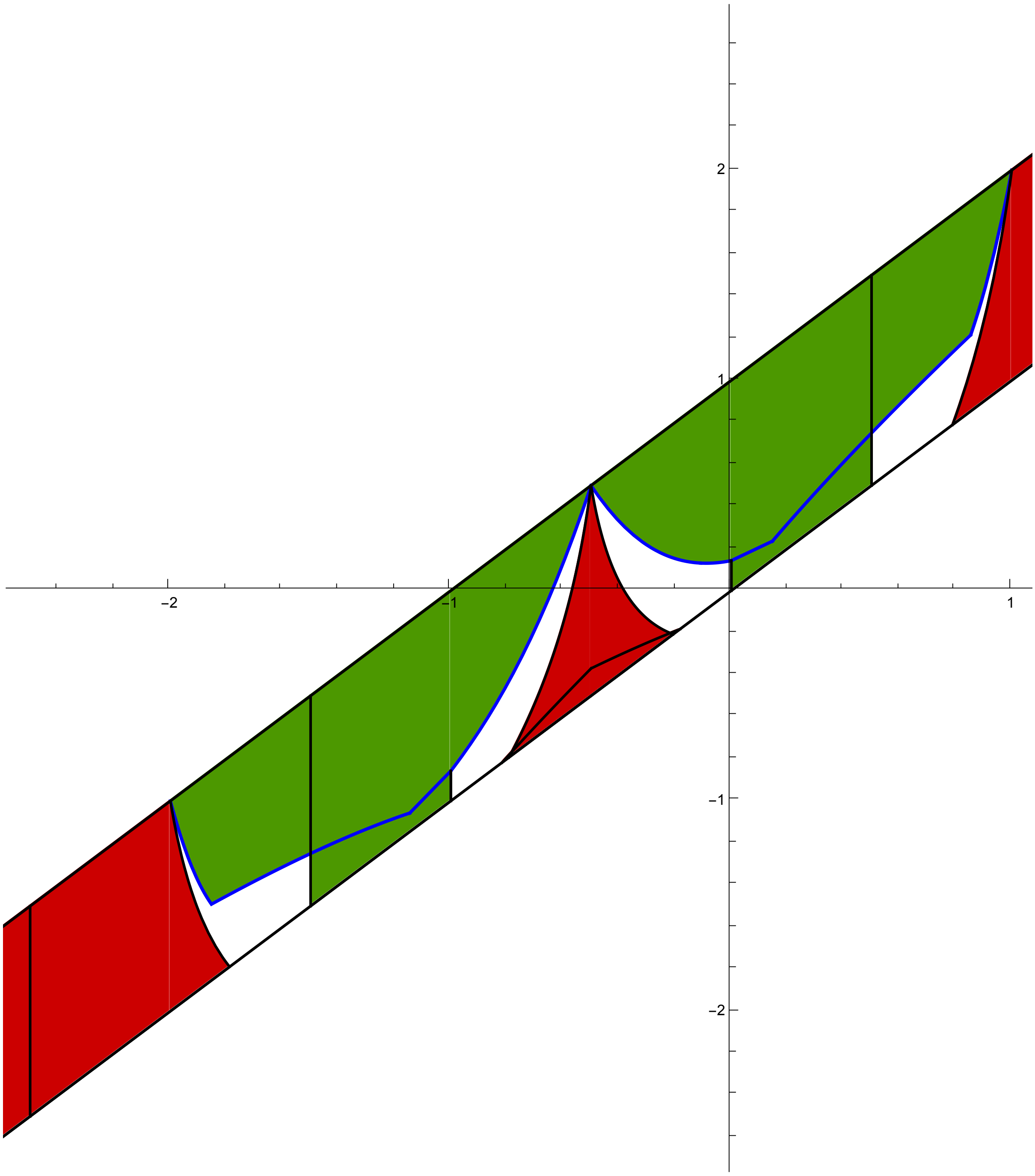
In[∘]:=
```
P2b = Show[
   Plot[{α, Bα[p3[-α - 1]]}, {α, -1 - 1/6 (3 + √3), -3}, PlotStyle → Black,
    Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

In[∘]:=
```
LP[α_] := ListLinePlot[{{α, α}, {α, α + 1}}, PlotStyle → Black]
LPp[α_] := ListLinePlot[{{α, 3/28 (4/3 + 28 α/3)}, {α, α}}, PlotStyle → Black]
LPh[h_] := ListLinePlot[{{-3.1, -3.1 + h}, {2.1, 2.1 + h}}, PlotStyle → Black]
P = Show[LP[-2.5], LP[-1.5], LP[0.5], LP[1.5], LPp[-1],
   LPp[0], LPh[0], LPh[1], LPp[-1], LPp[0], LPh[0], LPh[1]];
```

In[∘]:=
```
Show[P0, P1a, P1b, P2a, P2b, P, PlotRange → {{-3, 2}, All}];
```

In[•]:=
```
Show[%, PlotRange → {{-2.5, 1.5}, {-2.5, 2.5}}, AspectRatio → 1, AxesOrigin → {0.0}]
```

Out[•]=



Plot: SCKNlog

In[•]:=
```
SymThreshold = Black;
P0 = Show[Plot[{α + 1, If[-3/2 < α < -1, α,
      Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]]}, {α, -2, -0.5},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], Plot[{α + 1,
     If[0 < α < 1/2, α, Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (2 (7 + 8 α + 4 α^2))/(3 + 4 α^2), (2 (-9 + 5 α))/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]]},
    {α, -0.5, 1}, PlotStyle → Black, Filling → {1 → {2}},
    FillingStyle → SymFilling], PlotRange → All];
```

In[•]:=
```
P1a = Plot[{α, Max[Bα[p1[α]], Bα[p2[α]]]}, {α, -0.5, 1/4 (1 - √3)},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling];
```

In[•]:=
```
P1b = Plot[{α, Max[Bα[p1[-α - 1]], Bα[p2[-α - 1]]]}, {α, -0.5, 1/4 (√3 - 5)},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling];
```

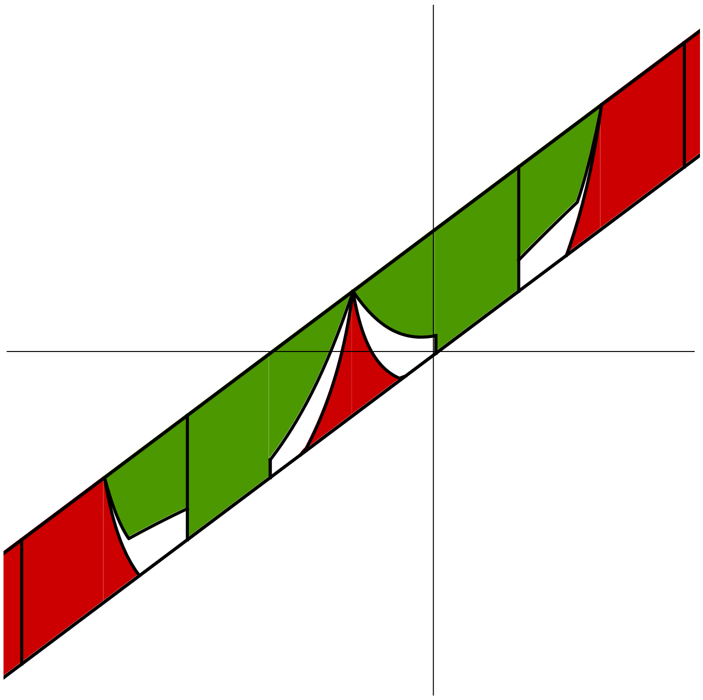
In[•]:=
```
P2a = Show[
    Plot[{α, Bα[p3[α]]}, {α, 1/6 (3 + √3), 2}, PlotStyle → Black,
     Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

In[•]:=
```
P2b = Show[
    Plot[{α, Bα[p3[-α - 1]]}, {α, -1 - 1/6 (3 + √3), -3}, PlotStyle → Black,
     Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

In[•]:=
```
LP[α_] := ListLinePlot[{{α, α}, {α, α + 1}}, PlotStyle → Black]
LPp[α_] := ListLinePlot[{{α, 3/28 (4/3 + (28 α)/3)}, {α, α}}, PlotStyle → Black]
LPh[h_] := ListLinePlot[{{-3.1, -3.1 + h}, {2.1, 2.1 + h}}, PlotStyle → Black]
P = Show[LP[-2.5], LP[-1.5], LP[0.5], LP[1.5], LPp[-1],
    LPp[0], LPh[0], LPh[1], LPp[-1], LPp[0], LPh[0], LPh[1]];
```

In[*]:=
```
Show[P0, P1a, P1b, P2a, P2b, P, PlotRange → {{-3, 2}, All}];
Show[%, PlotRange → {{-2.5, 1.5}, {-2.5, 2.5}},
 AspectRatio → 1, AxesOrigin → {0.0}, Ticks → None]
```

Out[*]=

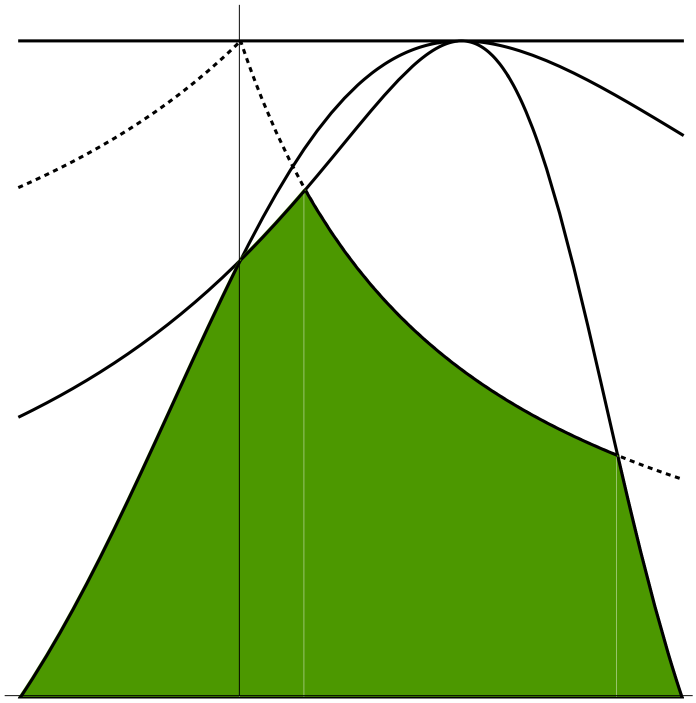


• Two additional figures

In[◦]:=

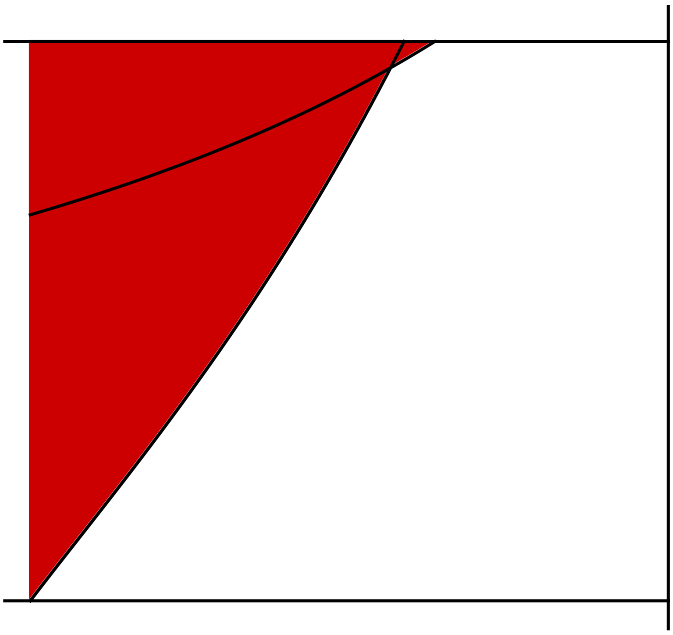
```
Plot[{Min[2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (6 + 4 α)/(1 + 2 α)], 2}, {α, -0.5, 1},
  PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling];
Show[Plot[{(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), Min[2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]}, {α, -0.5, 1},
  PlotStyle → {Black, Black, {Black, Dotted}}, PlotRange → {All, {2, 6.2}}],
 ListLinePlot[{{-0.5, 6}, {1, 6}}, PlotStyle → Black],
 %, AspectRatio → 1, Ticks → None]
```

Out[◦]=



In[∘]:=
```
P1a = Show[Plot[{2, 6}, {α, -0.52, 0}, PlotStyle → Black],
  ListLinePlot[{{0, 1.8}, {0, 6.25}}, PlotStyle → Black],
  Plot[{6, p2[α]}, {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[{6, p1[α]}, {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[p2[α], {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black],
  PlotRange → All, AspectRatio → 1, Ticks → None, Axes → None]
```

Out[∘]=



Figures (black-and-white)

In[∘]:=
```
SymFilling = GrayLevel[.5]
SymBreakingFilling = GrayLevel[.8]
SymThreshold = Black
```

Out[∘]= 

Out[∘]= 

Out[∘]= 

Symmetry breaking range: -1/2 < α < 0

In[∘]:= P1a = Show[

Plot[{6, p2[α]}, {α, -0.5, $\frac{1}{2}\left(1-\sqrt{2}\right)$}, PlotStyle → Black, Filling → {1 → {2}},

FillingStyle → SymBreakingFilling], Plot[{6, p1[α]}, {α, -0.5, $\frac{1}{4}\left(1-\sqrt{3}\right)$},

PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],

Plot[p2[α], {α, -0.5, $\frac{1}{2}\left(1-\sqrt{2}\right)$}, PlotStyle → Black], PlotRange → All];

Symmetry breaking range: -1 < α < -1/2

In[∘]:= P1b = Show[

Plot[{6, p2[-α - 1]}, {α, -0.5, $-\frac{1}{2}\left(3-\sqrt{2}\right)$}, PlotStyle → Black,

Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],

Plot[{6, p1[-α - 1]}, {α, -0.5, $\frac{1}{4}\left(\sqrt{3}-5\right)$}, PlotStyle → Black,

Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],

Plot[p2[-α - 1], {α, -0.5, $-\frac{1}{2}\left(3-\sqrt{2}\right)$}, PlotStyle → Black], PlotRange → All];

Symmetry breaking range: 1/2 < α < 3/2

In[∘]:= P2a = Show[

Plot[{6, p3[α]}, {α, $\frac{1}{6}\left(3+\sqrt{3}\right)$, 3}, PlotStyle → Black,

Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];

Symmetric breaking range: -2 < α < -1

In[∘]:= P2b = Show[

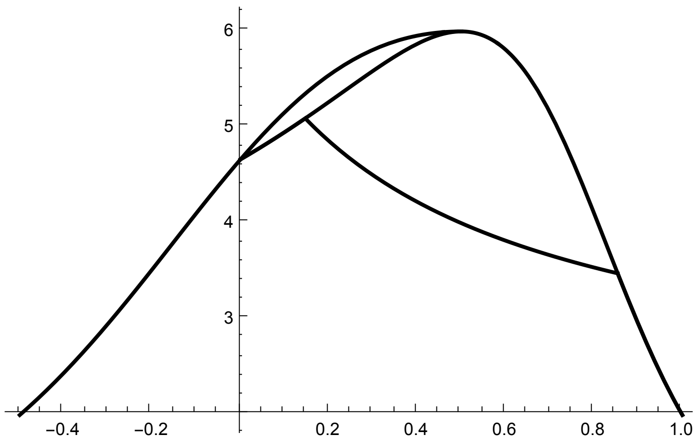
Plot[{6, p3[-α - 1]}, {α, $-1-\frac{1}{6}\left(3+\sqrt{3}\right)$, -4}, PlotStyle → Black,

Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];

• Symmetry range

In[∘]:=
```
Show[Plot[2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), {α, -0.5, 0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[{(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (6 + 4 α)/(1 + 2 α)]}, {α, 0, 1},
   PlotStyle → {{Thick, SymThreshold}, {Thick, SymThreshold}}], PlotRange → All]
Plot[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)],
  {α, -0.5, 1}, PlotStyle → {Thick, SymThreshold}];
P0a = Show[Plot[{2, If[0 < α < 1/2, 6,
      Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]}, {α, -1/2, 1},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], %];
```

Out[∘]=



In[∘]:=
```
Show[Plot[(6 + 8 α^2)/(7 + 8 α + 4 α^2), {α, -1.5, -0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[{(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), Min[(42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]}, {α, -2, -1},
   PlotStyle → {{Thick, SymThreshold}, {Thick, SymThreshold}}], PlotRange → All];
Plot[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)],
  {α, -2, -0.5}, PlotStyle → {Thick, SymThreshold}];
P0b = Show[Plot[{2, If[-3/2 < α < -1, 6,
      Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]}, {α, -2, -1/2},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling], %];
```

• Figures: Symmetry and symmetry breaking ranges

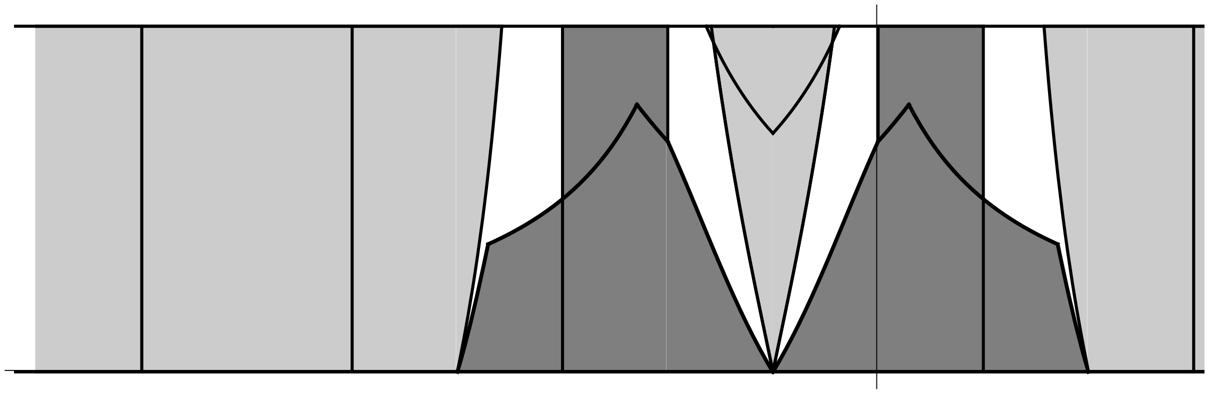
Separation lines

In[∘]:=
```
LP[α_] := ListLinePlot[{{α, 2}, {α, 6}}, PlotStyle → Black]
LPp[α_] := ListLinePlot[{{α, 14 / 3}, {α, 6}}, PlotStyle → Black]
LPh[h_] := ListLinePlot[{{-4.1, h}, {3.1, h}}, PlotStyle → Black]
P = Show[Table[LP[k - 0.5], {k, -3, 3}], LPp[-1], LPp[0], LPh[2], LPh[6]];
```

Plot: SCKNlog

In[∘]:=
```
Show[Show[P0a, P0b, P1a, P1b, P2a, P2b,
   PlotRange → {{-4, 3}, Automatic}, AspectRatio → 0.25],
  Show[Table[LP[k - 0.5], {k, -3, -1}], Table[LP[k - 0.5], {k, 1, 3}],
   LPp[-1], LPp[0], LPh[2], LPh[6], PlotRange → All], Ticks → None]
```

Out[∘]=

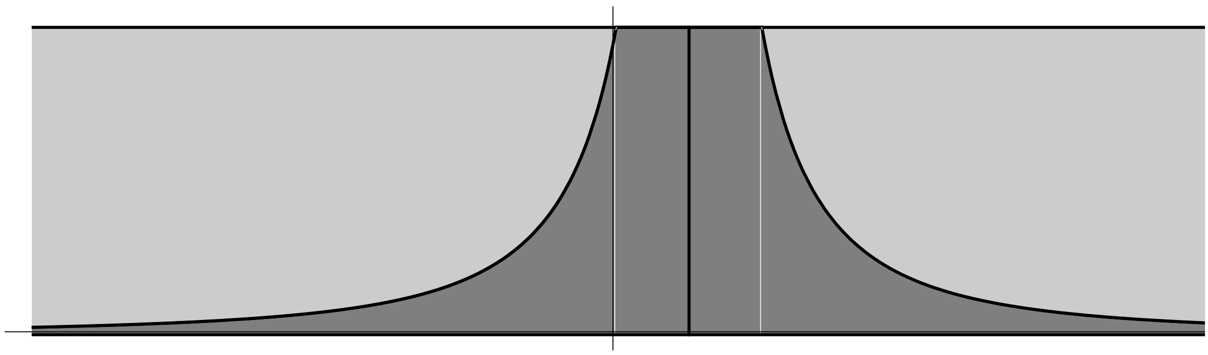


• Scalar CKN inequality: Symmetry and symmetry breaking ranges

Plot: CKNlog

In[•]:=
```
Show[Plot[{2, Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]}, {a, -4, 5}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymFilling, PlotRange → All],
  Plot[{6, Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]}, {a, -4, 5}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling, PlotRange → All],
  ListLinePlot[{{0.5, 2}, {0.5, 6}}, PlotStyle → Black],
  AspectRatio → 0.25, Ticks → None]
```

Out[•]=



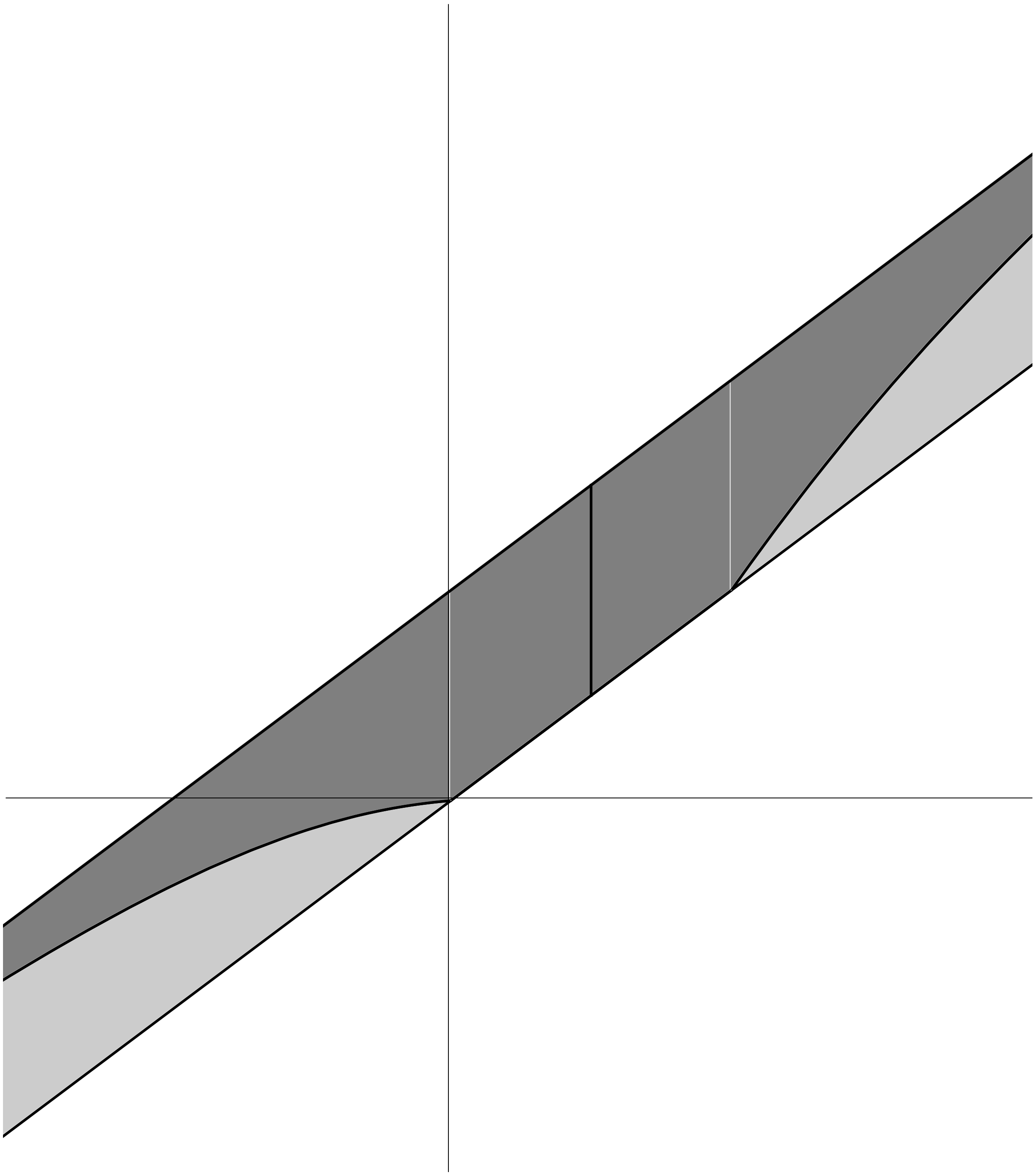
• Scalar CKN inequality: Symmetry and symmetry breaking ranges in (α,β) coordinates

In[•]:=
```
Show[Plot[{a, Ba[Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]]}, {a, -3, 4},
   PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[{a + 1, Ba[Min[(2 Sqrt[9 - 4 a + 4 a^2])/(ϵ + Abs[1 - 2 a]), 6]]}, {a, -3, 4}, PlotPoints → 80,
   PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling],
  ListLinePlot[{{0.5, 0.5}, {0.5, 1.5}}, PlotStyle → Black]];
```

Plot: CKN

In[ ]:=
```
Show[%, PlotRange → {{-1.5, 2.5}, {-1.5, 3.5}},
 AspectRatio → 1, AxesOrigin → {0.0}, Ticks → None]
```

Out[ ]=



• SCKN inequality: Symmetry and symmetry breaking ranges in (*α*,*β*) coordinates

In[•]:=

```
Show[Plot[Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]],
   {α, -2, -0.5}, PlotStyle → {Thick, SymThreshold}],
  Plot[Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (2 (7 + 8 α + 4 α^2))/(3 + 4 α^2), (2 (-9 + 5 α))/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]],
   {α, -0.5, 1}, PlotStyle → {Thick, SymThreshold}], PlotRange → All];
P0 = Show[Plot[{α + 1, If[-3/2 < α < -1, α,
      Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]]},
    {α, -2, -0.5}, PlotStyle → Black, Filling → {1 → {2}},
    FillingStyle → SymFilling], Plot[{α + 1, If[0 < α < 1/2, α,
      Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (2 (7 + 8 α + 4 α^2))/(3 + 4 α^2), (2 (-9 + 5 α))/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]]},
    {α, -0.5, 1}, PlotStyle → Black, Filling → {1 → {2}},
    FillingStyle → SymFilling], %, PlotRange → All];
```

In[•]:=

```
P1a = Show[Plot[{α, Bα[p1[α]]}, {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black,
    Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
   Plot[{α, Bα[p2[α]]}, {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black,
    Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
   Plot[Bα[p1[α]], {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black], PlotRange → All];
```

In[•]:=

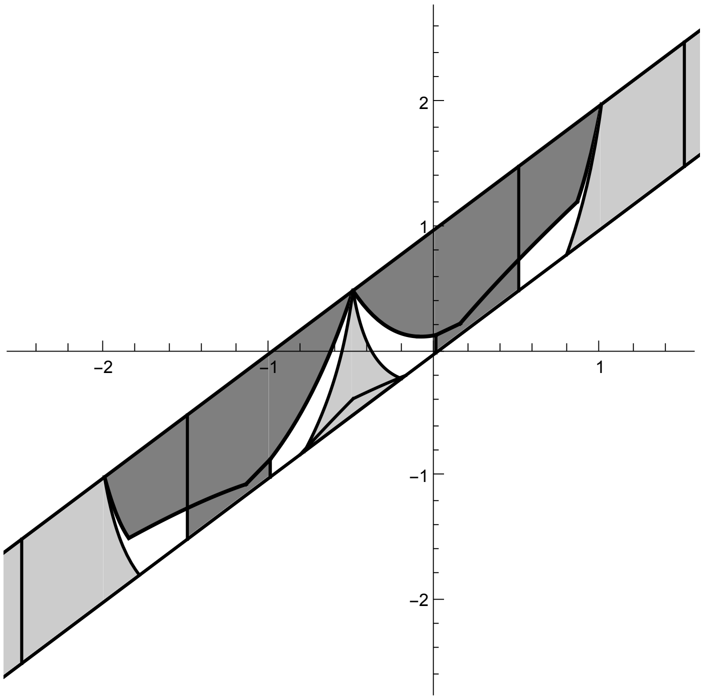
```
P1b = Show[Plot[{α, Bα[p1[-α - 1]]}, {α, -0.5, 1/4 (√3 - 5)},
    PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
   Plot[{α, Bα[p2[-α - 1]]}, {α, -0.5, -1/2 (3 - √2)}, PlotStyle → Black,
    Filling → {1 → {2}}, FillingStyle → SymBreakingFilling], Plot[
    Bα[p1[-α - 1]], {α, -0.5, 1/4 (√3 - 5)}, PlotStyle → Black], PlotRange → All];
```

In[•]:=

```
P2a = Show[
   Plot[{α, Bα[p3[α]]}, {α, 1/6 (3 + √3), 2}, PlotStyle → Black,
    Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];
```

```
In[•]:= P2b = Show[
          Plot[{α, Bα[p3[-α - 1]]}, {α, -1 - 1/6 (3 + √3), -3}, PlotStyle → Black,
           Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];

In[•]:= LP[α_] := ListLinePlot[{{α, α}, {α, α + 1}}, PlotStyle → Black]
        LPp[α_] := ListLinePlot[{{α, 3/28 (4/3 + 28 α/3)}, {α, α}}, PlotStyle → Black]
        LPh[h_] := ListLinePlot[{{-3.1, -3.1 + h}, {2.1, 2.1 + h}}, PlotStyle → Black]
        P = Show[LP[-2.5], LP[-1.5], LP[0.5], LP[1.5], LPp[-1],
            LPp[0], LPh[0], LPh[1], LPp[-1], LPp[0], LPh[0], LPh[1]];

In[•]:= Show[P0, P1a, P1b, P2a, P2b, P, PlotRange → {{-3, 2}, All}];

In[•]:= Show[%, PlotRange → {{-2.5, 1.5}, {-2.5, 2.5}}, AspectRatio → 1, AxesOrigin → {0.0}]
```

Out[•]=



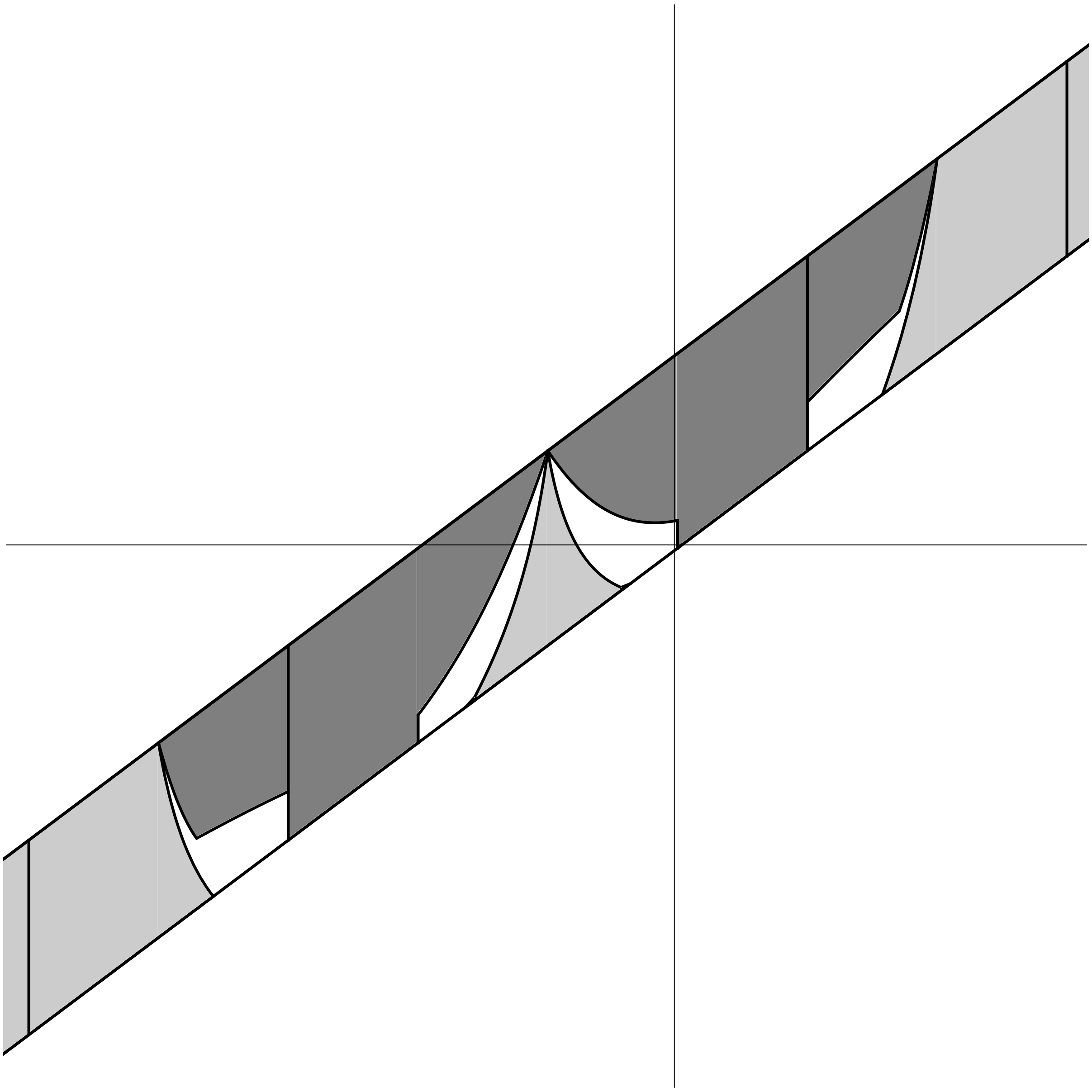
Plot: SCKNlog

```
In[•]:= P0 = Show[Plot[{α + 1, If[-3/2 < α < -1, α,
            Bα[Min[(6 + 8 α^2)/(7 + 8 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (42 + 36 α + 8 α^2)/(13 + 14 α + 4 α^2), (-2 + 4 α)/(1 + 2 α)]]]},
          {α, -2, -0.5}, PlotStyle → Black, Filling → {1 → {2}},
          FillingStyle → SymFilling], Plot[{α + 1, If[0 < α < 1/2, α,
            Bα[Min[(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]]]},
          {α, -0.5, 1}, PlotStyle → Black, Filling → {1 → {2}},
          FillingStyle → SymFilling], PlotRange → All];
```

```
In[•]:= P1a = Plot[{α, Max[Bα[p1[α]], Bα[p2[α]]]}, {α, -0.5, 1/4 (1 - √3)},
      PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling];

In[•]:= P1b = Plot[{α, Max[Bα[p1[-α - 1]], Bα[p2[-α - 1]]]}, {α, -0.5, 1/4 (√3 - 5)},
      PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymBreakingFilling];

In[•]:= P2a = Show[
      Plot[{α, Bα[p3[α]]}, {α, 1/6 (3 + √3), 2}, PlotStyle → Black,
       Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];

In[•]:= P2b = Show[
      Plot[{α, Bα[p3[-α - 1]]}, {α, -1 - 1/6 (3 + √3), -3}, PlotStyle → Black,
       Filling → {1 → {2}}, FillingStyle → SymBreakingFilling]];

In[•]:= LP[α_] := ListLinePlot[{{α, α}, {α, α + 1}}, PlotStyle → Black]
      LPp[α_] := ListLinePlot[{{α, 3/28 (4/3 + 28 α/3)}, {α, α}}, PlotStyle → Black]
      LPh[h_] := ListLinePlot[{{-3.1, -3.1 + h}, {2.1, 2.1 + h}}, PlotStyle → Black]
      P = Show[LP[-2.5], LP[-1.5], LP[0.5], LP[1.5], LPp[-1],
         LPp[0], LPh[0], LPh[1], LPp[-1], LPp[0], LPh[0], LPh[1]];
```

In[•]:=
```
Show[P0, P1a, P1b, P2a, P2b, P, PlotRange → {{-3, 2}, All}];
Show[%, PlotRange → {{-2.5, 1.5}, {-2.5, 2.5}},
 AspectRatio → 1, AxesOrigin → {0.0}, Ticks → None]
```

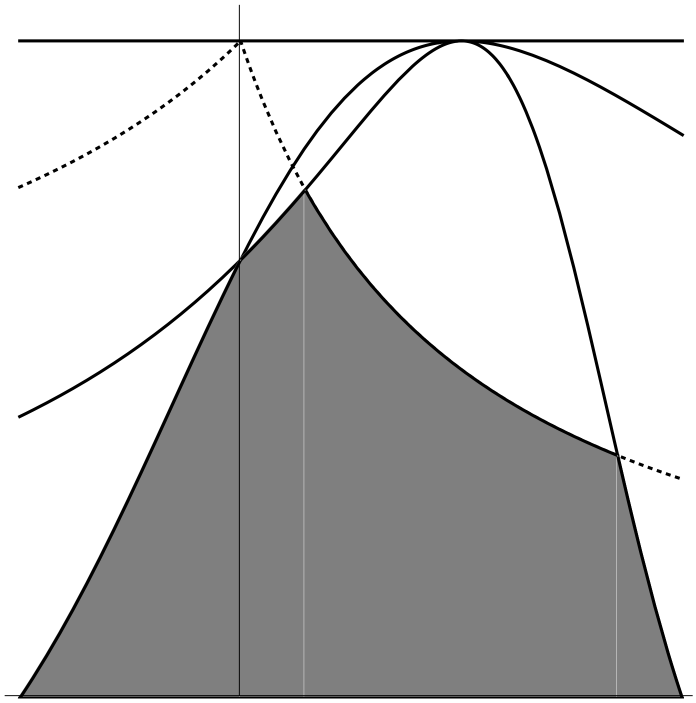
Out[•]=



• Two additional figures

In[∘]:=
```
Plot[{Min[2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), 2 (-9 + 5 α)/(3 (-1 + α)), (14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), (6 + 4 α)/(1 + 2 α)], 2}, {α, -0.5, 1},
  PlotStyle → Black, Filling → {1 → {2}}, FillingStyle → SymFilling];
Show[Plot[{(14 - 20 α + 8 α^2)/(3 - 6 α + 4 α^2), 2 (7 + 8 α + 4 α^2)/(3 + 4 α^2), Min[2 (-9 + 5 α)/(3 (-1 + α)), (6 + 4 α)/(1 + 2 α)]}, {α, -0.5, 1},
  PlotStyle → {Black, Black, {Black, Dotted}}, PlotRange → {All, {2, 6.2}}],
 ListLinePlot[{{-0.5, 6}, {1, 6}}, PlotStyle → Black],
 %, AspectRatio → 1, Ticks → None]
```

Out[∘]=



In[∘]:=
```
P1a = Show[Plot[{2, 6}, {α, -0.52, 0}, PlotStyle → Black],
  ListLinePlot[{{0, 1.8}, {0, 6.25}}, PlotStyle → Black],
  Plot[{6, p2[α]}, {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[{6, p1[α]}, {α, -0.5, 1/4 (1 - √3)}, PlotStyle → Black,
   Filling → {1 → {2}}, FillingStyle → SymBreakingFilling],
  Plot[p2[α], {α, -0.5, 1/2 (1 - √2)}, PlotStyle → Black],
  PlotRange → All, AspectRatio → 1, Ticks → None, Axes → None]
```

Out[∘]=