• Section 1

In[1]:=
$$\text{ResA}\eta = \text{Solve}\left[\left\{A-1 == d\,(1-m)\,(A+1)-2A,\ \eta == \frac{1}{1-A}\right\},\ \{\eta, A\}\right][[1]]$$

$$mc = \frac{d-2}{d};$$

$$m1 = \frac{d-1}{d};$$

$$m2 = \frac{d+1}{d};$$

Out[1]=
$$\left\{\eta \to \frac{3-d+d\,m}{2\,(1-d+d\,m)},\ A \to \frac{1+d-d\,m}{3-d+d\,m}\right\}$$

In[4]:=
$$\text{P0} = \text{Simplify}\left[\delta\, t^{2\frac{1-m}{m-m1}} + \frac{1+A}{2}\frac{\eta}{t}\left(\left(v-\frac{\eta}{t}x\right)^2 + A\left(\frac{\eta}{t}x\right)^2\right) \text{/. ResA}\eta\right];$$

$$\left\{\text{FullSimplify}\left[d\,(1+A)\,\frac{\eta}{t}\,(1-m)\,\text{P0} - \text{D[P0, v]}^2 - v\,\text{D[P0, x]} - \text{D[P0, t] /. ResA}\eta\right],\right.$$

$$\left.\text{Simplify}\left[-d\,\frac{1+A}{1-A}\,(m-1) - 2\,\frac{1-m}{m-m1} \text{ /. ResA}\eta\right]\right\}$$

Out[5]= {0, 0}

Figure 1

In[6]:=
$$\text{Rm} = \sqrt{2^{\frac{1-m}{m}}-1};\ \text{Am} = \frac{2-m}{2+m};$$

$$\text{R[t\_]} := ((1-\text{Am})\,t)^{\frac{1}{1-\text{Am}}}$$

```
Pos[t_] := (t + Abs[t]) / 2
```

In[9]:=
$$\text{h[m\_]} := \text{Show}\left[\text{Plot}\left[\left(\text{Pos}\left[1+\frac{1-m}{\text{Abs}[1-m]}r^2\right]\right)^{\frac{1}{m-1}},\ \{r, 0, 1.5\},\ \text{PlotStyle} \to \text{Thick}\right],\right.$$

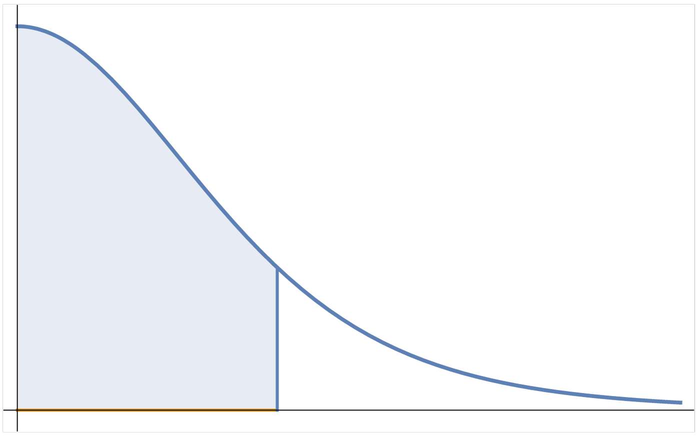
$$\text{Plot}\left[\left\{\left(\text{Pos}\left[1+\frac{1-m}{\text{Abs}[1-m]}r^2\right]\right)^{\frac{1}{m-1}},\ 0\right\},\ \left\{r, 0, \sqrt{\text{Abs}\left[2^{\frac{1-m}{m}}-1\right]}\right\},\right.$$

$$\left.\text{Filling} \to \text{Bottom},\ \text{FillingStyle} \to \text{Opacity}[0.15]\right],\ \text{ListLinePlot}\Big[$$

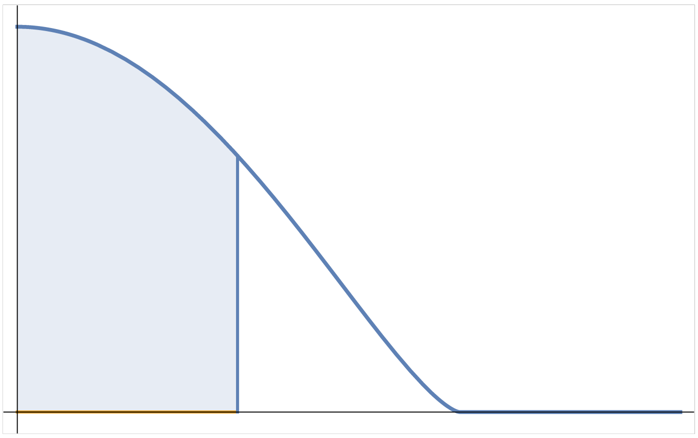
$$\left.\left\{\{r, 0\},\ \left\{r,\ \left(1+\frac{1-m}{\text{Abs}[1-m]}r^2\right)^{\frac{1}{m-1}}\right\}\right\} \text{ /. } r \to \sqrt{\text{Abs}\left[2^{\frac{1-m}{m}}-1\right]}\ \Big],\ \text{Ticks} \to \text{None}\right]$$

In[10]:=
```
h[0.7]
h[1.7]
```

Out[10]=



Out[11]=



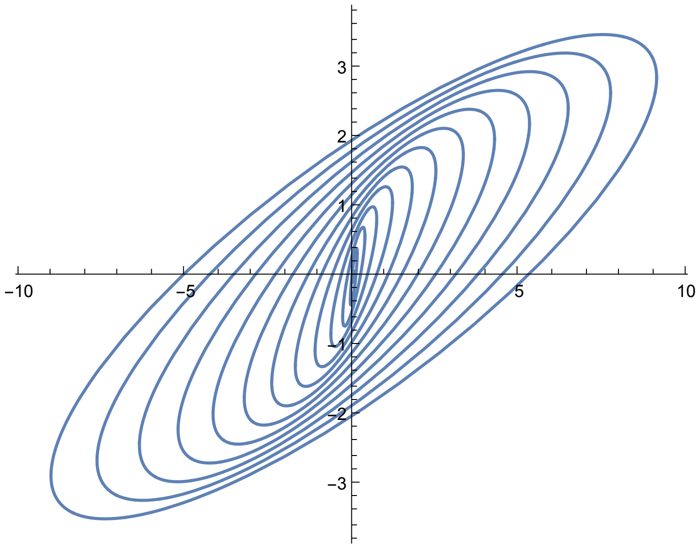
In[12]:=

$$\text{Simplify}\left[\left\{\frac{\text{Rm}\,\text{Cos}[\theta]}{\sqrt{\text{Am}\,(1+\text{Am})/2}}\,\text{R}[t],\ \left(\frac{\text{Rm}\,\text{Sin}[\theta]}{\sqrt{(1+\text{Am})/2}}+\frac{\text{Rm}\,\text{Cos}[\theta]}{\sqrt{\text{Am}\,(1+\text{Am})/2}}\right)\text{R}[t]^{\text{Am}}\right\}\right];$$

$$\text{H}[m\_,\ t\_] := \text{ParametricPlot}\left[\sqrt{\text{Abs}\left[2^{\frac{1}{m}}-2\right]}\ \left\{\frac{2^{-\frac{1}{2}+\frac{1}{m}}\left(\frac{m\,t}{2+m}\right)^{\frac{1}{2}+\frac{1}{m}}\text{Cos}[\theta]}{\frac{\sqrt{2-m}}{2+m}},\ \frac{\left(2^{\frac{1}{2}+\frac{1}{m}}\left(\frac{m\,t}{2+m}\right)^{\frac{1}{2}+\frac{1}{m}}\right)^{\frac{2-m}{2+m}}\left(\frac{1}{\sqrt{2+m}}\text{Cos}[\theta]+\frac{\sqrt{2-m}}{2+m}\text{Sin}[\theta]\right)}{2\ \frac{\sqrt{2-m}}{(2+m)^{3/2}}}\right\},\ \{\theta,\ 0,\ 2\,\pi\}\right]$$

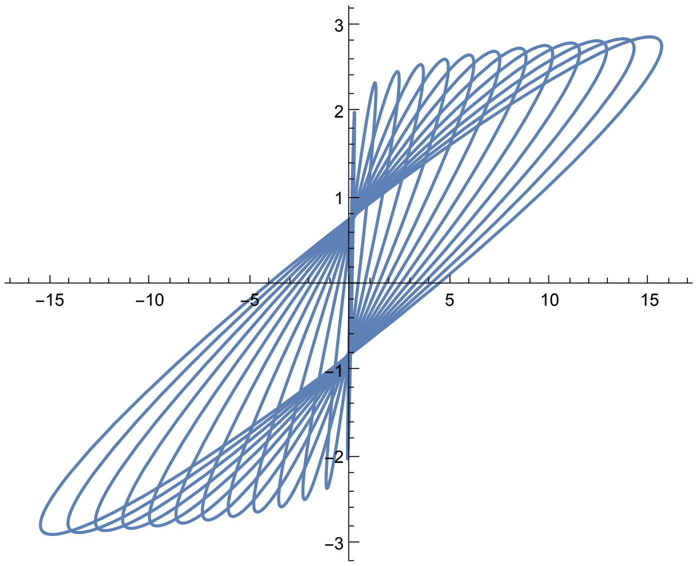
In[14]:= `Show[Table[H[0.7, t], {t, 0.1, 6.2, 0.5}], PlotRange → All, AspectRatio → 0.8]`

Out[14]=



In[15]:= `Show[Table[H[1.7, t], {t, 0.1, 6.1, 0.5}], PlotRange → All, AspectRatio → 0.8]`

Out[15]=



• Section 4

Fast diffusion case: computation of γ_* (case $d = 1$)

In[16]:=
$$\frac{2\pi}{\sqrt{A}} \text{Integrate}\left[\left(\frac{1-m}{m}\left(\gamma + \frac{1+A}{2} r^2\right)\right)^{\frac{1}{m-1}} r,\ \{r, 0, \infty\}, \text{Assumptions} \to 0 < m < 1 \,\&\&\, \gamma > 0 \,\&\&\, A > 0\right]$$
$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%^{\frac{1-m}{m}}\right]\right]$$
$$\% \text{ /. } \gamma \to 1$$
$$\text{Res}\gamma 1 = \text{FullSimplify}\left[\text{PowerExpand}\left[\text{Simplify}\left[\% \text{ /. } A \to \frac{2-m}{2+m}\right] \text{ /. } \left\{\text{Gamma}\left[\frac{1}{1-m}\right] \to X,\ \text{Gamma}\left[-\frac{m}{-1+m}\right] \to \frac{1}{1-m} X\right\}\right]\right]$$

Out[16]=
$$\frac{2\pi\gamma\left(\left(-1+\frac{1}{m}\right)\gamma\right)^{\frac{1}{-1+m}} \text{Gamma}\left[-\frac{m}{-1+m}\right]}{\sqrt{A}\,(1+A)\,\text{Gamma}\left[\frac{1}{1-m}\right]}$$

Out[17]=
$$\frac{A^{\frac{-1+m}{2m}}(1+A)^{\frac{-1+m}{m}}\left(-1+\frac{1}{m}\right)^{-1/m}(2\pi)^{-1+\frac{1}{m}}\,\text{Gamma}\left[\frac{1}{1-m}\right]^{\frac{-1+m}{m}}\,\text{Gamma}\left[-\frac{m}{-1+m}\right]^{-1+\frac{1}{m}}}{\gamma}$$

Out[18]=
$$A^{\frac{-1+m}{2m}}(1+A)^{\frac{-1+m}{m}}\left(-1+\frac{1}{m}\right)^{-1/m}(2\pi)^{-1+\frac{1}{m}}\,\text{Gamma}\left[\frac{1}{1-m}\right]^{\frac{-1+m}{m}}\,\text{Gamma}\left[-\frac{m}{-1+m}\right]^{-1+\frac{1}{m}}$$

Out[19]=
$$\left(-1+\frac{1}{m}\right)^{-1/m}(1-m)^{1-\frac{1}{m}}(2-m)^{\frac{-1+m}{2m}}(2+m)^{\frac{3}{2}\left(-1+\frac{1}{m}\right)}\left(\frac{2}{\pi}\right)^{\frac{-1+m}{m}}$$

Porous medium case: computation of γ_* (case $d = 1$)

In[20]:=
$$\frac{2\pi}{\sqrt{A}} \text{Integrate}\left[\left(\frac{m-1}{m}\left(\gamma - \frac{1+A}{2} r^2\right)\right)^{\frac{1}{m-1}} r, \left\{r, 0, \sqrt{\frac{2\gamma}{1+A}}\right\}, \text{Assumptions} \to m > 1 \,\&\&\, \gamma > 0 \,\&\&\, A > 0\right]$$
$$\text{FullSimplify}\left[\text{PowerExpand}\left[\%^{\frac{1-m}{m}}\right]\right]$$
$$\% \,/.\, \gamma \to 1$$
$$\text{Res}\gamma 2 = \text{FullSimplify}\left[\text{PowerExpand}\left[\text{Simplify}\left[\% \,/.\, A \to \frac{2-m}{2+m}\right] /. \left\{\text{Gamma}\left[\frac{1}{1-m}\right] \to X, \text{Gamma}\left[-\frac{m}{-1+m}\right] \to \frac{1}{1-m} X\right\}\right]\right]$$

Out[20]=
$$\frac{2\pi\left(\frac{(-1+m)\gamma}{m}\right)^{\frac{m}{-1+m}}}{\sqrt{A}\,(1+A)}$$

Out[21]=
$$\frac{A^{\frac{-1+m}{2m}}(1+A)^{\frac{-1+m}{m}}\, m\,(2\pi)^{-1+\frac{1}{m}}}{(-1+m)\,\gamma}$$

Out[22]=
$$\frac{A^{\frac{-1+m}{2m}}(1+A)^{\frac{-1+m}{m}}\, m\,(2\pi)^{-1+\frac{1}{m}}}{-1+m}$$

Out[23]=
$$\frac{2^{1-\frac{1}{m}}(2-m)^{\frac{1}{2}-\frac{1}{2m}}\, m\,(2+m)^{-\frac{3}{2}+\frac{3}{2m}}\,\pi^{-1+\frac{1}{m}}}{-1+m}$$

Figure 2: Spectrum (m=0.8, Rx=18, Rv=28, d=1, Rectangle [-Rx,Rx] x [-Rv,Rv])

In[24]:=
```
Off[NDEigenvalues::femcscd]
Off[Eigensystem::maxit2]
Off[Eigensystem::chnpdef]
```

In[27]:=
```
Rx = 18; Rv = 28;
```
$$N\left[\left\{\sqrt{A}, \frac{Rx}{Rv}\right\} /. m \to 0.8\right]$$

Out[28]= $\left\{\sqrt{A}, 0.642857\right\}$

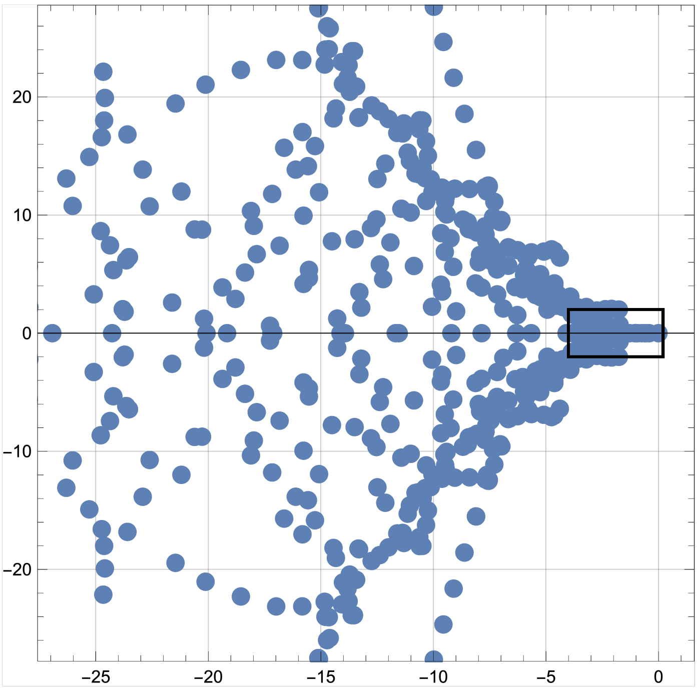
In[29]:=
$$P[x\_, v\_] := \frac{1-m}{m}\left(\gamma + \frac{1+A}{2}\left(v^2 + A x^2\right)\right) /. \left\{A \to \frac{2-m}{2+m}, \gamma \to \left(-1+\frac{1}{m}\right)^{-1/m}(1-m)^{1-\frac{1}{m}}(2-m)^{\frac{-1+m}{2m}}(2+m)^{\frac{3}{2}\left(-1+\frac{1}{m}\right)}\left(\frac{2}{\pi}\right)^{\frac{-1+m}{m}}\right\}$$

In[30]:=
```
H[x_, v_, m_] :=
  Evaluate[P[x, v]^((m-2)/(2 (m-1))) (m D[P[x, v]^(m/(2 (m-1))) f[x, v], {v, 2}] + (1 + A) P[x, v]^((2-m)/(2 (m-1))) f[x, v] +
      ((1 + A) v + A x) D[P[x, v]^((2-m)/(2 (m-1))) f[x, v], v] -
      v D[P[x, v]^((2-m)/(2 (m-1))) f[x, v], x]) /. A → (2 - m)/(2 + m)]
```

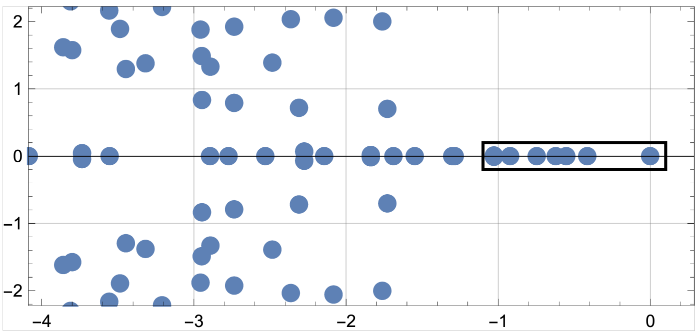
In[31]:=
```
{ℒ, ℬ} = {H[x, v, 0.8], DirichletCondition[f[x, v] == 0, True]};
```

In[32]:=
```
NDEigenvalues[{ℒ, ℬ}, f[x, v], {x, -Rx, Rx}, {v, -Rv, Rv}, 500];
Pl = ComplexListPlot[%, PlotRange → All,
    AspectRatio → 1, PlotMarkers → {Automatic, Medium},
    Frame → True, Axes → True, GridLines → Automatic];
Pc = ComplexListPlot[{0 + 0.0 ⅈ, -1 + 0.0 ⅈ, -A + 0.0 ⅈ, A - 1 + 0.0 ⅈ},
    PlotMarkers → "OpenMarkers", PlotStyle → Red];
Show[Pl, ListLinePlot[{{-4, -2}, {0.2, -2}, {0.2, 2}, {-4, 2}, {-4, -2}},
   PlotStyle → Black], PlotRange → {{-27, 1}, {-25, 25}}]
Show[Pl, ListLinePlot[{{-1.1, -0.2}, {0.1, -0.2}, {0.1, 0.2},
    {-1.1, 0.2}, {-1.1, -0.2}}, PlotStyle → Black],
 PlotRange → {{-4, 0.2}, {-2, 2}}, AspectRatio → 0.45]
Show[Pl, Pc, PlotRange → {{-1.1, 0.1}, {-0.2, 0.2}}, AspectRatio → 0.2]
```

Out[35]=



Out[36]=



Out[37]=

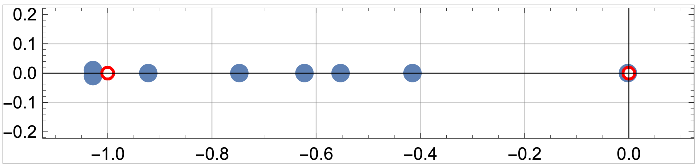
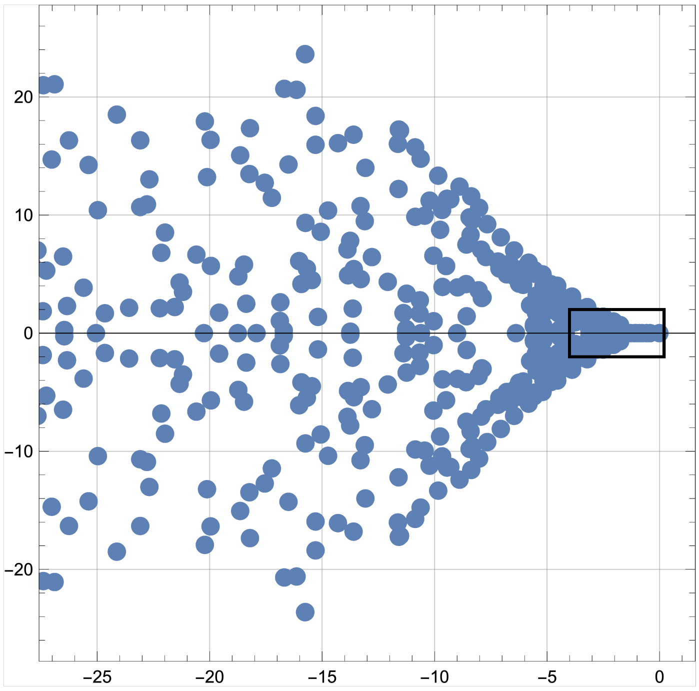


Figure 2: Spectrum, m=0.8, d=1, Ellipse with semi-axes (Rx,Rv) = $\frac{2}{\sqrt{\pi}}$(18,28)

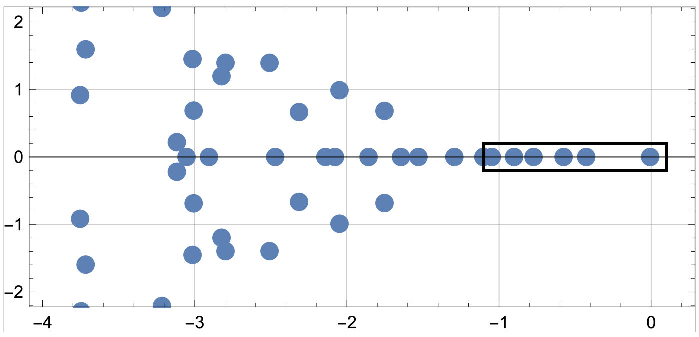
In[38]:=
```
NDEigenvalues[{ℒ, ℬ}, f[x, v], {x, v} ∈ Disk[{0, 0}, 2/√π {Rx, Rv}], 500];

Pl = ComplexListPlot[%, PlotRange → All,
   AspectRatio → 1, PlotMarkers → {Automatic, Medium},
   Frame → True, Axes → True, GridLines → Automatic];
Pc = ComplexListPlot[{0 + 0.0 ⅈ, -1 + 0.0 ⅈ, -A + 0.0 ⅈ, A - 1 + 0.0 ⅈ},
   PlotMarkers → "OpenMarkers", PlotStyle → Red];
Show[Pl, ListLinePlot[{{-4, -2}, {0.2, -2}, {0.2, 2}, {-4, 2}, {-4, -2}},
  PlotStyle → Black], PlotRange → {{-27, 1}, {-25, 25}}]
Show[Pl, ListLinePlot[{{-1.1, -0.2}, {0.1, -0.2}, {0.1, 0.2},
   {-1.1, 0.2}, {-1.1, -0.2}}, PlotStyle → Black],
 PlotRange → {{-4, 0.2}, {-2, 2}}, AspectRatio → 0.45]
Show[Pl, Pc, PlotRange → {{-1.1, 0.1}, {-0.2, 0.2}}, AspectRatio → 0.2]
```

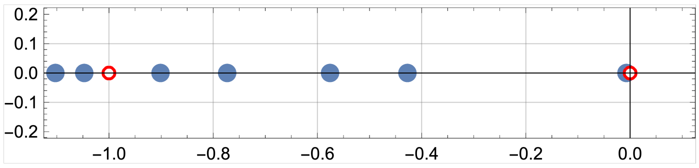
Out[41]=



Out[42]=



Out[43]=

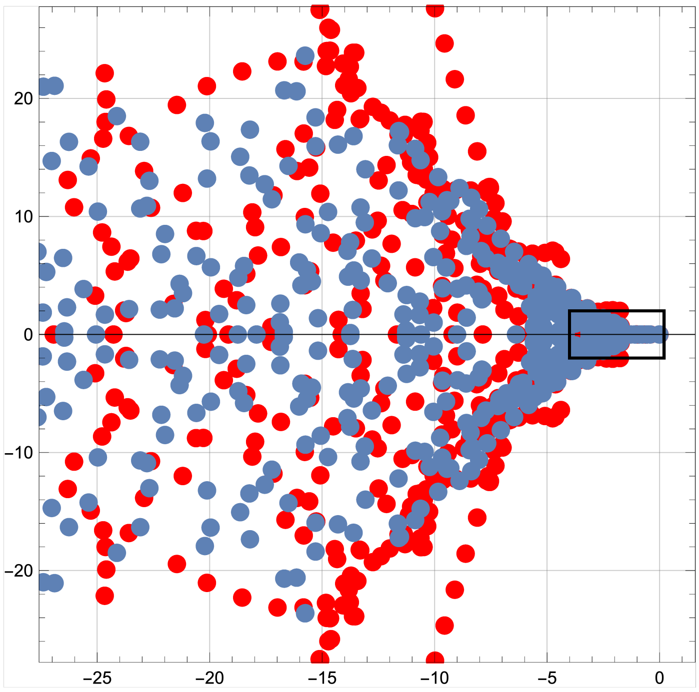


How much do these spectra differ ?

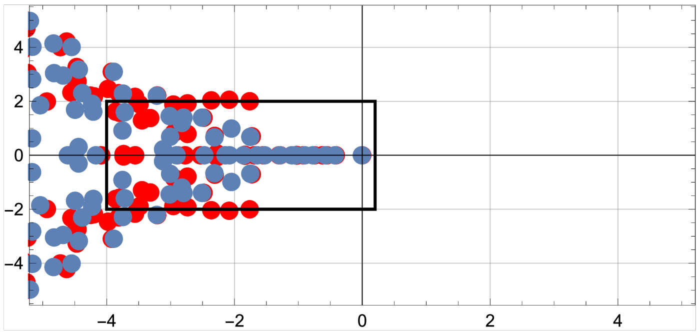
In[44]:=
```
{ℒ, ℬ} = {H[x, v, 0.8], DirichletCondition[f[x, v] == 0, True]};
Pc = ComplexListPlot[{0 + 0.0 i, -1 + 0.0 i, -A + 0.0 i, A - 1 + 0.0 i},
   PlotMarkers → "OpenMarkers", PlotStyle → Red];
P1 = Show[ComplexListPlot[NDEigenvalues[{ℒ, ℬ}, f[x, v],
      {x, -Rx, Rx}, {v, -Rv, Rv}, 500], PlotRange → All,
    AspectRatio → 1, PlotMarkers → {Automatic, Medium}, Frame → True,
    Axes → True, GridLines → Automatic, PlotStyle → Red],
   ListLinePlot[{{-4, -2}, {0.2, -2}, {0.2, 2}, {-4, 2}, {-4, -2}},
    PlotStyle → Black], PlotRange → {{-27, 1}, {-25, 25}}];
P2 = Show[ComplexListPlot[NDEigenvalues[{ℒ, ℬ}, f[x, v],
      {x, v} ∈ Disk[{0, 0}, 2/Sqrt[π] {Rx, Rv}], 500], PlotRange → All,
    AspectRatio → 1, PlotMarkers → {Automatic, Medium},
    Frame → True, Axes → True, GridLines → Automatic],
   ListLinePlot[{{-4, -2}, {0.2, -2}, {0.2, 2}, {-4, 2}, {-4, -2}},
    PlotStyle → Black], PlotRange → {{-27, 1}, {-25, 25}}];
```

In[47]:=
```
Show[P1, P2]
Show[%, PlotRange → {{-5, 5}, {-5, 5}}, AspectRatio → 0.45]
Show[%, PlotRange → {{-4, 0.2}, {-2, 2}}, AspectRatio → 0.45]
```

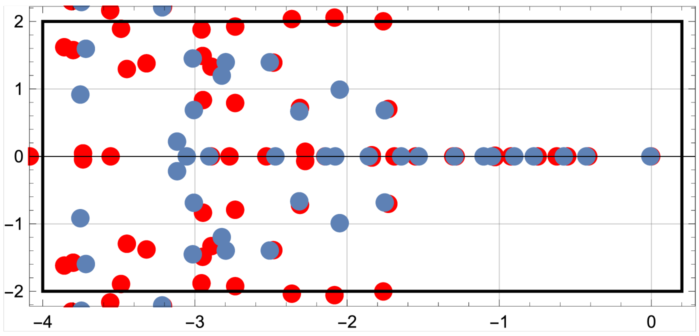
Out[47]=



Out[48]=



Out[49]=



```
In[52]:= Take[Re[Chop[NDEigenvalues[{ℒ, ℬ}, f[x, v], {x, -Rx, Rx}, {v, -Rv, Rv}, 500]]], 10]
         Take[Re[Chop[
           NDEigenvalues[{ℒ, ℬ}, f[x, v], {x, v} ∈ Disk[{0, 0}, 2/Sqrt[π] {Rx, Rv}], 500]]], 10]

Out[52]= {-0.0017549, -0.415162, -0.553207, -0.622438,
          -0.74739, -0.922198, -1.0284, -1.0284, -1.28614, -1.30249}

Out[53]= {-0.0069042, -0.427246, -0.575762, -0.773169,
          -0.90112, -1.0475, -1.10284, -1.29357, -1.52965, -1.64433}
```