ON LONG TIME ASYMPTOTICS OF THE VLASOV-POISSON-BOLTZMANN SYSTEM

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We deal with the long time asymptotics of the Vlasov-Poisson-Boltzmann system. We present some results on the stationary states with fixed mass and temperature, and prove the convergence of the solutions of the evolution problem towards such states. Then, using the conservation laws, we are able to prove the existence and the uniqueness of the asymptotic state, and to identify its parameters in terms of the conserved quantities of the evolution problem.

1. Introduction

The purpose of this paper is to present some results on the Vlasov-Poisson-Boltzmann system, and especially on its long time asymptotic states. We will establish that these states are stationary and we will identify their parameters as functions of the macroscopic conserved quantities of the problem of evolution.

Most of the results which are presented here were obtained in a common work with L. Desvillettes (see [De,Do]).

The Vlasov-Poisson-Boltzmann system is used to describe plasma. We will consider the case of only one species of particles (the jellium approximation), evolving in a bounded domain Ω. Since we are not interested in relativistic phenomena, we assume that the velocities belong to $\mathbb{R}^3$.

We will also neglect the magnetic effects, and consider only the electrostatic field. We assume that it derives from a potential obeying to Poisson's law. The density f satisfies the Vlasov's equation, with a Boltzmann collision term $Q(f,f)$. The Vlasov-Poisson-Boltzmann system (VPB) is therefore

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = Q(f,f) \\ -\Delta_x \phi = \rho(t,x) = \displaystyle\int_{\mathbb{R}^3} f(t,x,v)dv \end{cases}$$

for $t \geq 0$, $x \in \Omega$, $v \in \mathbb{R}^3$. The Boltzmann collision term $Q(f,f)$ has the usual form

$$Q(f,f) = \int\int_{\mathbb{R}^3 \times S^{N-1}} B(v - v_*, \omega)(f' f'_* - f f_*)\, dv_* d\omega$$

$$f = f(t,x,v) \qquad f_* = f(t,x,v_*) \qquad f' = f(t,x,v') \qquad f'_* = f(t,x,v'_*)$$

$$v' = v - (v - v_*).\omega\ \omega$$
$$v'_* = v_* + (v - v_*).\omega\ \omega$$

where v_* belongs to $\mathbb{R}^3$ and ω denotes a unit vector of $\mathbb{R}^3$: $\omega \in S^2$. $B(w,\omega)$ is the cross section. For instance, in the case of the hard spheres, $B(w,\omega) = |w \cdot \omega|$.

In the following, we shall assume that the cross section satisfies the conditions

$(H0)$ $B(w,\omega) = b(|w|, |(w,\omega)|)$
$(H1)$ $B \in L^\infty_{loc}(S^2 \times \mathbb{R}^3)$
$(H2)$ $B > 0$ a.e.
$(H3)$ $\int_{\omega \in S^2} B(v - v_1, \omega)\, d\omega \leq K(1 + |v|^\alpha + |v_1|^\alpha)$

Conditions on the boundary :

$$f(t,x,v) = f(t,x,Rv)$$

$$Rv = v - 2(v \cdot n(x))\, n(x)$$

for all t in $\mathbb{R}^+$, $(x,v) \in \partial\Omega \times \mathbb{R}^3$ such that $v \cdot n(x) \leq 0$, $n(x)$ beeing the outward normal to $\partial\Omega$ at point x. We assume Dirichlet conditions for ϕ. To simplify, we take ϕ constant on the boundary (condition of perfect conductor), and since ϕ is defined up to an additive constant, we assume

$$\phi = 0 \text{ on } \partial\Omega$$

Initial conditions :

$$f(t=0,x,v) = f_0(x,v),$$
$$\phi(t=0,x) = \phi_0(x),$$

satisfying the compatibility assumption :

$$-\Delta\phi_0 = \int_{\mathbb{R}^3} f_0(x,v)\, dv$$

2. Stationary solutions with fixed mass and temperature

In this part, we recall a result of Gogny and Lions (see [G,L]) which gives some motivations to our study. The stationary solutions (f, ϕ) are such that f is a maxwellian function

$$f(x,v) = \frac{1}{(2\pi T)^{3/2}} \cdot \rho(x) \cdot e^{-\frac{|v-u(x)|^2}{2T}}$$

and ϕ satisfies a semilinear elliptic equation

$$-\Delta\phi = \rho$$

Here, the temperature T is a strictly positive constant and the mean velocity $u(x)$ is of the form

$$u(x) = \omega \wedge x$$

where ω is a constant vector of $I\!R^3$. The specular reflection condition ensures that

$$n(x) \cdot u(x) = 0 \quad \forall\, x \in \partial\Omega$$

Then, if
(h) Ω is not a surface of revolution
we get $\omega = 0$. We will also assume that Ω is bounded and regular (of class C^2). If the mass and the temperature are fixed, (VPB) is reduced to

$$\begin{cases} \dfrac{\nabla\rho}{\rho} = -\dfrac{\nabla\phi}{T} \\ -\Delta\phi = \rho \end{cases}$$

with the mass normalization

$$M = \int_\Omega \rho(x)\,dx$$

Finally

$$\rho(x) = M \cdot \frac{e^{-\frac{\phi}{T}}}{\int_\Omega e^{-\frac{\phi}{T}}\,dx}$$

and ϕ is solution in $H_0^1(\Omega$ of

$$-\Delta\phi = M \cdot \frac{e^{-\frac{\phi}{T}}}{\int_\Omega e^{-\frac{\phi}{T}}\,dx}$$

Proposition 1. *The above equation has a unique solution in $H_0^1(\Omega)$. This solution belongs to $C^\infty(\Omega)$.*

Proof. The proof relies on classical minimization arguments for the following functional (the parameters M and T are unessential)

$$J(\phi) = \frac{1}{2}\int_\Omega |\nabla\phi|^2\, dx + \log\left(\int_\Omega e^{-\phi^+}\, dx\right)$$

□

3. Properties of the Vlasov-Poisson-Boltzmann system

3.1 A priori estimates

Using the changes of variables

$$(v, v_*) \mapsto (v_*, v), \quad (v, v_*) \mapsto (v', v_*'), \quad (v, v_*) \mapsto (v', v_*'),$$

we easily get that for all regular function $\psi(v)$

$$\int_{\mathbb{R}^3} Q(f,f)\cdot\psi(v)\, dv$$

$$= \int\int\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^{N-1}} B(v-v_*,\omega)(f'f_*' - ff_*)\cdot\psi(v)\, dvdv_*d\omega$$

$$= -\frac{1}{4}\int\int\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^{N-1}} B(v-v_*,\omega)(f'f_*' - ff_*)\cdot(\psi+\psi_*-\psi'-\psi_*')\, dvdv_*d\omega$$

$\psi(v) = 1$ and $\psi(v) = |v|^2$ are collision invariants (*i.e.* $\psi+\psi_* = \psi'+\psi_*'$), and it is easy to prove (at least formally)

- the conservation of mass

$$\frac{\partial}{\partial t}\int\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\, dxdv = 0$$

$$\int\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\, dxdv = M_0$$

- the conservation of energy

$$\frac{\partial}{\partial t}\left(\int\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\cdot|v|^2\, dxdv + \int_\Omega |\nabla\phi(t,x)|^2\, dx\right) = 0$$

$$\int\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\cdot|v|^2\, dxdv + \int_\Omega |\nabla\phi(t,x)|^2\, dx = E_0$$

One can also establish an H-theorem, using $\psi(v) = \log f(.,.,v)$

$$\frac{\partial}{\partial t}\int\int_{\Omega\times\mathbb{R}^3} (f\log f - f)(t,x,v)\, dxdv$$

$$= -\frac{1}{4}\int\int\int_{\mathbb{R}^3\times\mathbb{R}^3\times S^{N-1}} B(v-v_*,\omega)(f'f_*' - ff_*)\cdot\log\left(\frac{f'f_*'}{ff_*}\right) dvdv_*d\omega \le 0$$

Remark 2. $(f \log f)(.,x,v)$ belongs to $L^1(\Omega \times \mathbb{R}^3)$ as soon as the mass and the energy are conserved and $(f_0 \log f_0)(x,v)$ belongs to $L^1(\Omega \times \mathbb{R}^3)$. Indeed, following an argument of DiPerna and Lions (see [DP,L 1-2] or [Ge]),

$$\begin{aligned} 0 &\le \int\int_{\Omega\times\mathbb{R}^3} |(f\log f)(t,x,v)|\,dxdv \\ &\le \int\int_{\Omega\times\mathbb{R}^3} (f\log f)(t,x,v)\,dxdv \\ &\qquad + Const + 2\int\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\cdot(1+|v|^2)\,dxdv \\ &\le \int\int_{\Omega\times\mathbb{R}^3} (f_0\log f_0)(x,v)\,dxdv \\ &\qquad + Const + 2\int\int_{\Omega\times\mathbb{R}^3} f_0(x,v)\cdot(1+|v|^2)\,dxdv \end{aligned}$$

and, as a consequence

$$\int_0^{+\infty} dt \int\int\int\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times S^{N-1}} B(v-v_*,\omega)(f'f'_* - ff_*) \cdot \log\left(\frac{f'f'_*}{ff_*}\right) dxdvdv_*d\omega < +\infty$$

3.2 Some results on related problems

Before going further, let us give a (non extensive) list of results concerning problems related to the Vlasov-Poisson-Boltzmann system :

- on the Vlasov-Poisson system :
  - existence of weak solutions in $\mathbb{R}^3$ (see [DP,L 3-4])
  - existence of strong solutions in $\mathbb{R}^3$ (see [Pf], [Ho],[L,P])
  - existence of stationary solutions in a bounded domain (see [Po])
- on the spatially inhomogeneous Boltzmann equation :
  - existence of renormalized solutions in$\mathbb{R}^3$ (see [DP,L 1-2], [Ge])
  - existence of renormalized solutions in a bounded domain (see [Ha])
  - large time asymptotics (see [A], [De])
- on the Vlasov-Poisson-Boltzmann system :
  - existence of stationary solutions in a bounded domain (see [G,L])
  - existence of stationary solutions in $\mathbb{R}^3$, with a confining potential (see [Dr], [Do])

4. Long time asymptotics of the Vlasov-Poisson-Boltzmann system

The first result concerns the convergence of a solution of (VPB) towards a long time asymptotic state. Since up to now no existence result is known, we will assume to simplify quite strong conditions, whose essential features are to authorize the use of Ascoli's theorem (if we had to deal with solutions as weak as the renormalized solutions of DiPerna and Lions, we could still adapt the proof, using averaging lemmas and weak-L^1 compactness properties). Let us assume that (f, ϕ) is a solution of (VPB) such that

(i) f is nonnegative, bounded and uniformly continuous on $\mathbb{R}^+ \times \Omega \times \mathbb{R}^3$ and $\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv$ belongs to $L^\infty([0,T] \times \Omega)$

(ii) ϕ belongs to $C^2(\mathbb{R}^+ \times \overline{\Omega})$ and its derivatives up to second order are bounded and uniformly continuous

(iii) $f_0 \not\equiv 0$, $E_0 = \int\int_{\Omega\times\mathbb{R}^3} f_0(x,v)\cdot |v|^2\, dxdv + \int_\Omega |\nabla\phi_0(x)|^2\, dx < +\infty$ and $\int\int_{\Omega\times\mathbb{R}^3} |f_0(x,v)\log f_0(x,v)|\, dxdv < +\infty$

Let $(t_n)_{n\in\mathbb{N}}$ be a sequence of real numbers going to infinity, and T be a strictly positive real number.
We define $f^n(t,x,v) = f(t+t_n,x,v)$ and $\phi^n(t,x,v) = \phi(t+t_n,x,v)$.

Theorem 3. *There exist a subsequence $(t_{n_k})_{k\in\mathbb{N}}$, a function $\psi(x)$ in $C^2(\overline{\Omega})$ and two strictly positive constant numbers ρ and θ such that $(f^{n_k})_{k\in\mathbb{N}}$ converges uniformly on every compact set of $[0,T] \times \Omega \times \mathbb{R}^3$ to*

$$g(x,v) = \frac{\rho}{(2\pi\theta)^{3/2}} \exp -\left(\frac{\psi(x)}{\theta} - \frac{v^2}{2\theta}\right)$$

and $(\phi^{n_k})_{k\in\mathbb{N}}$ converges in $C^2([0,T] \times \overline{\Omega})$ to $\psi(x)$. Moreover, ψ satisfies the following equation:

$$-\Delta\psi = \rho e^{-\psi(x)/\theta}$$

Proof. First, we can notice that the solutions of (VPB) satisfying assumptions (i)-(iii) satisfy also the *a priori* estimates. Ascoli's theorem ensures the existence of a subsequence $(t_{n_k})_{k\in\mathbb{N}}$, a function $g(t,x,v)$ and a function $\psi(t,x)$ such that $(f^{n_k})_{k\in\mathbb{N}}$ converges to g and $(\phi^{n_k})_{k\in\mathbb{N}}$ converges to ψ. Using $(H3)$, one can prove that $(Q(f^{n_k}, f^{n_k}))_{k\in\mathbb{N}}$ converges to $Q(g,g)$ in the sense of distributions. As a consequence of remark 2,

$$\int\int\int\int_{\Omega\times\mathbb{R}^3\times\mathbb{R}^3\times S^{N-1}} B\,(g'g'_* - gg_*)\cdot\log\left(\frac{g'g'_*}{gg_*}\right) dxdvdv_*d\omega = 0$$

g is therefore a maxwellian solution of the Vlasov-Poisson system, and according to [De], g and ψ are like stated in theorem 3 (they do not depend on t any more). □

As quoted before, if the mass and the temperature are given, then ρ is fixed and the equation for ψ has a unique solution. But the asymptotic temperature is not known. In the following, we will prove that it is natural to identify the asymptotic states in terms of the initial mass and energy.

5. Conservation of the macroscopic quantities and consequences for the asymptotic states

The total mass is conserved in the large time limit :

$$\int\int_{\Omega\times\mathbb{R}^3} g(x,v)\, dxdv = M_0$$

but we have only an inequality for the energy :

$$E_\infty = \int\int_{\Omega\times\mathbb{R}^3} g(x,v)\cdot|v|^2\, dxdv + \int_\Omega |\nabla\psi(x)|^2\, dx \leq E_0$$

To get the conservation of energy, we can for instance assume that there exists an $\epsilon > 0$ such that
$(H4)$ $\sup_{t\in\mathbb{R}^+}\int\int_{\Omega\times\mathbb{R}^3} f\cdot|v|^{2+\epsilon}dvdx + \sup_{t\in\mathbb{R}^+}\int_\Omega|\nabla\phi|^{2+\epsilon}dx < +\infty$
This assumption is not natural at all, and has no other justification than the following fact : under this condition, we get

$$E_\infty = E_0$$

Then, making the change of unknown function $\theta U = \psi$, we get

$$M_0 = \rho\int_\Omega e^{-U}\, dx$$

$$E_0 = \int\int_{\Omega\times\mathbb{R}^3} g(x,v)\cdot|v|^2\, dxdv + \theta^2\int_\Omega|\nabla U|^2\, dx$$
$$= \frac{3}{2}\theta\cdot\rho\int_\Omega e^{-U}\, dx + \theta^2\int_\Omega|\nabla U|^2\, dx$$

and U is solution of

$$-\frac{3}{2}\chi\frac{\Delta U}{\left(1+\sqrt{1+\chi\|\nabla_x U\|^2_{L^2(\Omega)}}\right)} = M\frac{e^{-U}}{\int_\Omega e^{-U}\, dx} \quad (x\in\Omega) \qquad (*)$$

with $\chi = \frac{4E}{9M^2}$.

Proposition 4. *The above equation has a unique solution in $H^1_0(\Omega)$. This solution belongs to $C^\infty(\Omega)$.*

Under the previous assumptions,

Corollary 5. *For all $E_0 > 0$, for all $M_0 > 0$, there exist a unique function U solution of $(*)$, a unique temperature θ given by*

$$\theta = \frac{3}{2} M_0 \frac{\chi}{\left(1 + \sqrt{1 + \chi \|\nabla_x U\|^2_{L^2(\Omega)}}\right)}$$

such that, when $\tau \to +\infty$,

$$f^\tau(t,x,v) = f(t+\tau,x,v) \to g(x,v) = \frac{M}{(2\pi\theta)^{\frac{3}{2}} \int_\Omega e^{-U}\, dx} \exp - \left(U(x) + \frac{v^2}{2\theta}\right)$$

uniformly on every compact set of $[0,T] \times \Omega \times \mathbb{R}^3$, and

$$\phi^\tau(t,x) = \phi(t+\tau,x) \to \theta U(x)$$

in $C^2([0,T] \times \overline{\Omega})$.

Proof. Like in section 2, let us introduce on $H_0^1(\Omega)$ the C^1 convex functional

$$J(U) = j\left(\|\nabla_x U\|_{L^2(\Omega)}\right) + \log\left(\int_\Omega e^{-U^+}\, dx\right)$$

where

$$j(t) = \frac{3}{2}\left(\sqrt{1+\chi t^2} - \log(1 + \sqrt{1+\chi t^2})\right) \quad \forall\, t \in \mathbb{R}^+$$

According to the

Lemma 6. *For all U in $H_0^1(\Omega)$, we have*

(i) $\log\left(\int_\Omega e^{-U}\, dx\right) \geq \log|\Omega| - \frac{C(\Omega)}{\sqrt{|\Omega|}} \|\nabla U\|_{L^2(\Omega)}$, *where $C(\Omega)$ is Poincaré's constant.*

(ii) $\log\left(\int_\Omega e^{-U}\, dx\right) \geq -2\log\left(\|\nabla U\|_{L^2(\Omega)}\right)$ *when $\|\nabla U\|_{L^2(\Omega)} \to +\infty$.*

U is a solution of (*) if and only if U is a minimizer for J. This minimizer exists and is unique. The proof of part (i) of lemma 6 is a straightforwrd consequence of Jensen's inequality and Poincaré's inequality. The part (ii) relies on Hardy's inequality. □

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