

The fundamental solution of a nonlinear kinetic Fokker-Planck equation

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This paper is devoted to a fundamental solution of a nonlinear kinetic equation involving a porous medium or fast diffusion operator acting on velocities. Such a nonlinearity has interesting scaling properties, which result in a self-similar behaviour of the fundamental solution. Here fundamental solution means a Dirac distribution initial datum which moreover governs the large time asymptotics of a large class of solutions. Using a self-similar change of variables, the equation becomes a nonlinear kinetic Fokker-Planck equation with harmonic confinement and the intermediate asymptotics regime is transformed into a stability property of a special stationary solution, which attracts the solutions for large times. In the homogeneous case (pure nonlinear diffusion), the problem is reduced to a classical nonlinear diffusion equation with Barenblatt-Pattle self-similar profiles. Unexpectedly, this beautiful structure is preserved at kinetic level, with remarkable consequences for relative entropy estimates, detailed intermediate asymptotics and nonlinear diffusion limits in adapted functional spaces.

Fundamental solution | Nonlinear kinetic Fokker-Planck equation | Self-similar solutions | Fast diffusion | Porous media | Maximum Principle | Intermediate asymptotics | Entropy methods | Diffusion limit

1. Introduction and main results

The computation of the Green function for the linear Vlasov-Fokker-Planck equation by A. Kolmogorov in (1) was the starting point of the hypo-ellipticity theory developed later by L. Hörmander in (2). Barenblatt-Pattle self-similar solutions of (3, 4) play the role of *fundamental solutions* for the porous medium and the fast diffusion equations (see for instance (5)), in the sense that they have Dirac distributions as initial data and govern the large time behaviour of finite mass nonnegative solutions in the appropriate range of parameters. Over the years, these solutions became an essential tool in the theory of nonlinear diffusion equations, see (6, 7). Our purpose is to extend this notion of fundamental solution to a class of nonlinear kinetic equations and draw some consequences.

Let us consider the nonlinear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f^m, \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d. \quad [1]$$

We generalise to the case $m \neq 1$ a few results known for $m = 1$. Inspired by the Green function obtained by A. Kolmogorov in (1) in the linear case $m = 1$, and the Barenblatt-Pattle functions in the homogeneous case (*i.e.*, when f does not depend on x and solves a nonlinear diffusion equation), we look for self-similar solutions of Eq. (1) with finite mass and Dirac distribution δ as initial data. Let us introduce the *pressure* variable

$$P := \frac{m}{1-m} f^{m-1}$$

and, on the support of f , rewrite Eq. (1) as

$$\partial_t P = (1-m) P \Delta_v P - |\nabla_v P|^2 - v \cdot \nabla_x P. \quad [2]$$

The differential operator in the right-hand side preserves second order polynomials in (x, v) with t -dependent coefficients. An elementary computation shows that

$$P_\star(t, x, v) = \beta(t) + \frac{(1+A)}{2(1-A)t} \left(\left| v - \frac{x}{(1-A)t} \right|^2 + A \left| \frac{x}{(1-A)t} \right|^2 \right) \quad \text{with} \quad A = \frac{1+d-dm}{3-d+dm}$$

is a solution of Eq. (2). If

$$m \in (m_1, 1) \cup (1, m_2) \quad \text{with} \quad m_1 := 1 - \frac{1}{d}, \quad m_2 := 1 + \frac{1}{d}, \quad [3]$$

the corresponding *fundamental solution* of Eq. (1) is given by

$$f_\star(t, x, v) = \left(\frac{1-m}{m} P_\star(t, x, v) \right)_+^{\frac{1}{m-1}}. \quad [4]$$

Significance Statement

The sixth Hilbert problem addresses the mathematical foundations of the laws of physics and specifically the question of the diffusion limits in kinetic theory. Kolmogorov exhibited a fundamental solution of the Vlasov-Fokker-Planck equation which plays a crucial role in the hypoellipticity theory and the study of large time asymptotics (hypocoercivity) in linear evolution equations. In the spatially homogeneous case, the self-similar function found by Barenblatt and Pattle extends the notion of fundamental solution to nonlinear diffusions. We obtain such a fundamental solution for a nonlinear kinetic equation and draw some important consequences for large time asymptotics. Using pressure variables, the expressions are unexpectedly simple, thus paving the road to diffusion limits, hypocoercivity methods and sharp estimates in nonlinear kinetic theory.

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Here $\beta(t) = ((1-A)t)^{2(1-m)/(m-m_1)} \gamma_\star$ is an explicit normalization constant (see Eq. (11)). We have not found such a solution in the current literature, but it is straightforward to check that one recovers Kolmogorov's solution (1) in the limit as $m \rightarrow 1$.

Our main result is that the *fundamental solution* f_\star is an interesting special solution which governs the large time behaviour of all solutions of Eq. (1) with initial datum f_0 such that,

$$\begin{aligned} f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d), \quad \|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1, \\ g_{\gamma_1}(x, v-x) \leq f_0(x, v) \leq g_{\gamma_2}(x, v-x) \end{aligned} \quad [5]$$

for any $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $(1-m)\gamma_2 > (1-m)\gamma_1 > 0$ and

$$g_\gamma(x, v) := \left(\frac{1-m}{m} \left(\gamma + \frac{1+A}{2} |v|^2 + \frac{1+A}{2} A |x|^2 \right) \right)_+^{\frac{1}{m-1}}. \quad [6]$$

In simpler words, we assume that f_0 is nonnegative, of mass 1, and trapped between two self-similar profiles of masses $M_i = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_{\gamma_i}(x, v) dx dv$ with $M_1 < 1 < M_2$.

Theorem 1. *Assume that $d \geq 1$ and m satisfies (3), with the additional condition that $1/2 < m < 3/2$ when $d = 1$. Then*

$$\lim_{t \rightarrow +\infty} \|f(t, \cdot, \cdot) - f_\star(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 0$$

if f solves Eq. (1) with an initial datum f_0 satisfying Eq. (5).

Since Eq. (1) conserves mass for any $t \geq 0$ but $f_\star(t, \cdot, \cdot) \rightarrow 0$ as $t \rightarrow +\infty$ in $L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$, the result of Theorem 1 is an *intermediate asymptotics* estimate. The level lines of the *fundamental solution* f_\star given by Eq. (4) are ellipses, which rotate and expand in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$ as t increases. See Fig. 1 for plots of the ellipse $\mathfrak{E}_m(t)$ which encloses half of the mass of f_\star .

As in the study of porous media and fast diffusions equations, it is convenient to replace *intermediate asymptotics* estimates by estimates of convergence to a stationary solution after a *time-dependent rescaling*. The expression of f_\star suggests to introduce *self-similar variables* and consider g such that

$$f(t, x, v) = R(t)^{-d(1+A)} g\left(\log R(t), \frac{x}{R(t)}, \frac{v}{R(t)^A} - \frac{x}{R(t)}\right) \quad [7]$$

with

$$R(t) = \left(R_0^{1-A} + (1-A)t \right)^{\frac{1}{1-A}} \quad \text{and} \quad A = \frac{1+d-dm}{3-d+dm} \quad [8]$$

where $R_0 = R(0) \geq 0$, and $A = A(m)$ is the same as in the definition of P_\star . With this change of variables, if f is a solution of Eq. (1), then g solves

$$\partial_t g + v \cdot \nabla_x g - A x \cdot \nabla_v g = \Delta_v g^m + (1+A) \nabla_v \cdot (v g). \quad [9]$$

If $R_0 = 1$, the initial datum is

$$g(0, x, v) = g_0(x, v) := f_0(x, v+x) \quad [10]$$

if f solves Eq. (1) with initial datum f_0 . It is also elementary to check that Eq. (9) admits a nonnegative stationary solution g_γ for some $\gamma \in \mathbb{R}$ such that $(1-m)\gamma > 0$. Notice that the integrability of g_γ induces a restriction on m .

Lemma 2. *Let $d \geq 1$. Then g_γ is a function in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ if and only if m satisfies (3). In that range, Eq. (1) admits a fundamental solution f_\star given by Eq. (4), i.e., by (6) with $\gamma = \gamma_\star$ and $g_\star := g_{\gamma_\star}$ using Eq. (7) and Eq. (8) with $R_0 = 0$.*

The Herrero-Pierre exponent corresponding to the threshold case of integrability of Barenblatt profiles in \mathbb{R}^d is $m_c := (d-2)/d$ according to (8). The exponent $m_1 = (d-1)/d = (2d-2)/(2d)$ in Lemma 2 is the Herrero-Pierre exponent in $\mathbb{R}^d \times \mathbb{R}^d$, while the limitation $m < m_2$ arises from $A(m) > 0$. We also refer to (9) for more detailed references and to (10) for an extended range of m . From here on, we shall assume that (3) holds and take $\gamma = \gamma_\star$ in (6). Our goal is to prove Theorem 1 and establish large time asymptotic properties of the solutions of Eq. (1) using nonlinear parabolic methods in Section 2 and relative entropy methods applied to the solutions of Eq. (9) in Section 3. At formal level, we shall also sketch consequences on the diffusion limit and the linearization of Eq. (9) around g_\star in Section 4.

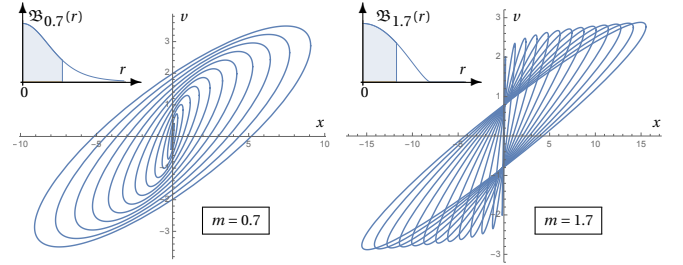


Fig. 1. In dimension $d = 1$, the ellipse $\mathfrak{E}_m(t)$ is represented in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$ at times $t = 0.1, 0.6, 1.1, \dots, 6.1$ for $m = 0.7$ (left) and $m = 1.7$ (right). In the upper left corners, the Barenblatt profiles $\mathfrak{B}_m(r) := (1 \pm r^2)_+^{1/(m-1)}$ are shown, with shaded areas corresponding to half of the mass. Here \pm denotes the sign of $(1-m)$ and r is such that $r^2 = A|x|^2 + |v|^2$: the function $f_\star(t, \cdot, \cdot)$ has compact support if $m > 1$.

2. Mass, contraction and comparison

A. The mass parameter. Let f be an arbitrary nonnegative function in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$, $M > 0$ a real number and

$$f_M(t, x, v) := M f\left(M^{2\zeta} t, M^\zeta x, M^{-\zeta} v\right) \quad \text{with} \quad \zeta := -\frac{1}{4}(1-m).$$

Lemma 3. *With the above notation, f_M solves Eq. (1) if and only if f solves Eq. (1).*

Considering solutions of Eq. (1) or Eq. (9) with initial data f_0 such that $\|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ is therefore not a restriction. Let \pm denote the sign of $(1-m)$. The condition $\|g_\star\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ determines $\gamma = \gamma_\star$ in Eq. (9) such that

$$\frac{2^d}{(\sqrt{A}(1-A))^d} \left| \frac{1-m}{m} \right|^{\frac{1}{m-1}} \gamma_\star^d \int_{\mathbb{R}^d \times \mathbb{R}^d} (1 \pm |z|^2)_+^{\frac{1}{m-1}} dz = 1. \quad [11]$$

B. The Comparison, L^1 -contraction and consequences.

Proposition 4. *Under the assumptions of Theorem 1, Eqs. (9)-(10) has at most one solution in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ with*

$$g_{\gamma_1}(x, v) \leq g(t, x, v) \leq g_{\gamma_2}(x, v) \quad \text{and} \quad \|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1.$$

(t, x, v) $\in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ a.e. and $\|f(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$.

Any solution $f \in \mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ of Eq. (1) can be rewritten by solving the characteristics of the free transport operator, i.e., by solving

$$\partial_t h = \left(\Delta_v - 2t \nabla_x \cdot \nabla_v + t^2 \Delta_x \right) h^m \quad [12]$$

where $h(t, x, v) = f(t, x + tv, v)$. Even if Eq. (12) is degenerate, classical methods for porous media ($m > 1$) and fast diffusion ($m < 1$) equations apply. We refer to (6–8, 10) for further results on nonlinear diffusion equations in $L^1_{\text{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$. Because fundamental solutions are positive with finite mass, we shall only consider solutions of Eq. (1) with finite $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ norm and nonnegative initial data. In order to prove Proposition 4, the first step is the L^1 -contraction property.

Lemma 5. *Let $d \geq 1$ and $m \in (m_1, 1) \cup (1, m_2)$. If f_1 and f_2 solve Eq. (1) in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$ in the sense of distributions, then $t \mapsto \iint_{\mathbb{R}^d \times \mathbb{R}^d} (f_2(t, x, v) - f_1(t, x, v))_+ dx dv$ is nonincreasing.*

Here $(\cdot)_+$ denotes the positive part. The result follows by standard methods of approximation as in (8). A straightforward consequence of Lemma 5 is the following *Maximum Principle*.

Corollary 6. *Let $d \geq 1$ and $m \in (m_1, 1) \cup (1, m_2)$. If f_1 and f_2 solve Eq. (1) in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$, in the sense of distributions, respectively with initial data $f_{1,0}$ and $f_{2,0}$ such that $f_{1,0} \leq f_{2,0}$ a.e., then $f_1(t, \cdot, \cdot) \leq f_2(t, \cdot, \cdot)$ a.e. for all $t \in \mathbb{R}^+$.*

With the results of Corollary 6, the proof of Proposition 4 is now classical (see for instance (10)).

C. Existence of a solution.

Proposition 7. *Under the assumptions of Theorem 1, Eqs. (9)–(10) has a solution g in $\mathcal{C}(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$.*

Proof. We rewrite Eq. (9) for $g(t, x, v) = G(\sqrt{At}, A^{1/4}x, A^{-1/4}v)$ as

$$\partial_t G + v \cdot \nabla_x G - x \cdot \nabla_v G = \frac{1}{A} \Delta_v G^m + \frac{1+A}{\sqrt{A}} \nabla_v \cdot (vG) \quad [13]$$

so that the stationary solution g_γ for Eq. (9) is transformed into the stationary solution for Eq. (13) given by

$$G_\gamma(x, v) := \left(\frac{1-m}{m} \left(\gamma + \frac{1}{2} (1+A) \sqrt{A} (|v|^2 + |x|^2) \right) \right)_+^{\frac{1}{m-1}}.$$

We consider a function f_0 satisfying the assumptions of Theorem 1 and consider the following *splitting scheme*:

- for any $t \in [0, 1/n]$, $n \in \mathbb{N} \setminus \{0\}$, we solve

$$\begin{aligned} \frac{\partial G}{\partial t} &= 2 \left(\frac{1}{A} \Delta_v G^m + \frac{1+A}{\sqrt{A}} \nabla_v \cdot (vG) \right) & \text{if } 0 \leq t \leq \frac{1}{2n}, \\ \frac{\partial G}{\partial t} + 2(v \cdot \nabla_x G - x \cdot \nabla_v G) &= 0 & \text{if } \frac{1}{2n} \leq t \leq \frac{1}{n}, \end{aligned}$$

with initial datum $G_0(x, v) = g_0(A^{-1/4}x, A^{1/4}v)$. We denote the solution by $S_n(t)G_0$, $t \in [0, 1/n]$.

- Let $n \in \mathbb{N} \setminus \{0\}$ and define the function G_n on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \ni (t, x, v)$ by

$$G_n \left(t + \frac{k}{n}, x, v \right) = S_n(t) G_n \left(\frac{k}{n}, x, v \right) \quad \forall (k, t) \in \mathbb{N} \times [0, 1/n].$$

The function G_n inherits of all good properties of the nonlinear diffusion and free transport equations: mass conservation, maximum principle, L^1 -contraction, uniqueness. A remarkable feature is that $G_\gamma(x, v) = g_\gamma(A^{-1/4}x, A^{1/4}v)$ is invariant under both steps of the splitting scheme: $S_n(t)G_\gamma = G_\gamma$ for any $t \in [0, 1/n]$. This is true in particular for $G_\star := G_{\gamma_\star}$. Moreover, the initial bound Eq. (5) is transformed into $G_{\gamma_1} \leq G_n \leq G_{\gamma_2}$. As a consequence, the sequences $(|x|^2 + |v|^2)(G_n - G_\star)_{n \in \mathbb{N}}$ and $((1 + |x|^2 + |v|^2)^{-1} G_n^m)_{n \in \mathbb{N}}$ are bounded in $L^\infty(\mathbb{R}^+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$.

Moreover, there is a positive constant λ and a modulus of continuity ω , such that $\lim_{\delta \rightarrow 0} \omega(\delta) = 0$, depending only on f_0 , such that for every $\delta > 0$, one has

$$\sup_{|y|^2 + |w|^2 \leq \delta^2} \|G_n(t, y + \cdot, w + \cdot) - G_n(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} \leq \omega(e^{\lambda t} \delta)$$

for every $t \geq 0$ and $n \in \mathbb{N} \setminus \{0\}$. By the Riesz-Fréchet-Kolmogorov theorem, we learn that for each $T > 0$, $[0, T] \ni t \mapsto G_n(t, \cdot, \cdot)$ remains in a compact subset of $L^1(\mathbb{R}^d \times \mathbb{R}^d)$. Moreover $\partial_t G_n$ is uniformly bounded in $L^\infty(\mathbb{R}^+; X^*)$ where X is the Banach space of $\mathcal{C}^{1,1}(\mathbb{R}^d \times \mathbb{R}^d)$ -test functions φ for which

$$\begin{aligned} \|\varphi\|_X &:= \|\varphi\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} + \|(|x| + |v|) \nabla \varphi\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \\ &\quad + \|(1 + |x|^2 + |v|^2) D^2 \varphi\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \end{aligned}$$

is finite and X^* denotes its topological dual so that $L^1(\mathbb{R}^d \times \mathbb{R}^d)$ is continuously embedded in X^* . The Arzelà-Ascoli Theorem then enables us to find a subsequence of G_n which converges both in $\mathcal{C}([0, T]; X^*)$ and in $L^1((0, T) \times \mathbb{R}^d \times \mathbb{R}^d)$ to some limit $G \in \mathcal{C}([0, T]; L^1(\mathbb{R}^d \times \mathbb{R}^d))$. The fact that the limit G solves Eq. (13) then easily follows. \square

D. Scalings and the four-step method. A possible strategy for proving Theorem 1 is to adapt the *four-step method* introduced by S. Kamin and J.L. Vázquez in (11) and summarized for instance in (7, Section 18.2), with the following key issues. Scaling properties of the equation have already been taken into account in the self-similar change of variables Eq. (7). With A as in Eq. (8), notice in particular that

$$f_\star(t, x, v) = t^{-d} \frac{1+A}{1-A} f_\star \left(1, t^{-\frac{1}{1-A}} x, t^{-\frac{A}{1-A}} v \right) \quad [14]$$

for all $t > 0$. Within our framework, regularity properties can probably be derived from Eq. (12) using standard parabolic methods for nonlinear diffusions and hypoelliptic techniques but, to our knowledge, such a theory is still to be done. Compactness properties would then follow. The limit as $t \rightarrow +\infty$ of the solution has to be one of the self-similar solutions for Eq. (1) with unit mass. After the change of variables Eq. (8), we shall prove that the unique possible limit turns out to be g_\star . In order to prove Theorem 1, we ultimately rely on entropy methods, which are well adapted to kinetic equations.

Incidentally, we can observe that Eq. (1) admits a scale invariance $f \mapsto f_\lambda$ for any $\lambda > 0$ that preserves mass, with

$$f_\lambda(t, x, v) := \lambda^4 f \left(\lambda^{2(m-m_1)} t, \lambda^{m-m_3} x, \lambda^{m_2-m} v \right),$$

$m_1 = 1 - 1/d$ and $m_2 = 1 + 1/d$ as in (3) and $m_3 := 1 - 3/d$. We recover Eq. (14) with the ansatz $\lambda^{2(m-m_1)} t = 1$. Combined with the mass invariance of Lemma 3, this allows us to relax Eq. (5) to a more general condition on the behaviour of $f_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ as $|x|^2 + |v|^2 \rightarrow \infty$ than (5).

3. Entropy methods in self-similar variables

A. Moments. According to Proposition 4, g_\star has finite mass for any $m > m_1$. However, g_\star admits a moment $(|x|^2 + |v|^2)$ only if

$$m > \tilde{m}_1 = \frac{d}{d+1} > m_1. \quad [15]$$

The missing range is covered by considering the difference with g_\star .

Lemma 8. *Under the assumptions of Theorem 1, the solution g of Eqs. (9)-(10) is such that*

$$\sup_{t \geq 0} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (|x|^2 + |v|^2) |g(t, x, v) - g_\star(x, v)| dx dv < \infty.$$

Proof. This is a consequence of Proposition 4 and

$$\begin{aligned} |g(t, x, v) - g_\star(x, v)| &\leq g_{\gamma_2}(x, v) - g_{\gamma_1}(x, v) \\ &= O\left(\frac{|x|^2 + |v|^2}{m-1}\right) \end{aligned}$$

as $(|x|^2 + |v|^2) \rightarrow \infty$. Against $(|x|^2 + |v|^2)$, the right-hand side is indeed integrable if and only if $m > m_1$. \square

Let us define the *relative entropy* functional by

$$\mathcal{E}[g] := \mathcal{H}[g] - \mathcal{H}[g_\star]$$

with $\mathcal{H}[g] := \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{g^m}{m-1} + \frac{1+A}{2} (|v|^2 + A|x|^2) g \right) dx dv$ for any m satisfying (3), and with the help of Lemma 8 if $m \leq \tilde{m}_1$. This definition generalizes the entropy introduced in (12, 13) for nonlinear diffusions and was used for instance in (14) for a kinetic relaxation model with same local equilibria. We shall write $\mathcal{E}(t) = \mathcal{E}[g(t, \cdot, \cdot)]$ if g solves Eq. (9). If $m < 1$, notice that

$$\mathcal{E}[g] = \frac{1}{m-1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(g^m - g_\star^m - m g_\star^{m-1} (g - g_\star) \right) dx dv.$$

In the range $m_1 < m < \tilde{m}_1$, the relative entropy can be understood as the functional acting on $w = g/g_\star$ given by

$$w \mapsto \frac{1}{m-1} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (w^m - 1 - m(w-1)) g_\star^m dx dv.$$

Cf. (10) for details.

B. Relative entropy estimates. Let us consider a nonnegative solution g of Eq. (9) with the pressure variables pressure

$$Q := \frac{m}{1-m} g^{m-1} \quad \text{and} \quad Q_\star := \gamma_\star + \frac{1+A}{2} (|v|^2 + A|x|^2)$$

where Q_\star is the pressure associated to g_\star .

Lemma 9. *Under the assumptions of Theorem 1, if g solves Eqs. (9)-(10), then*

$$\frac{d}{dt} \mathcal{E}(t) = - \iint_{\mathbb{R}^d \times \mathbb{R}^d} g |\nabla_v Q - \nabla_v Q_\star|^2 dx dv \quad \forall t > 0.$$

Proof. By Lemma 8, $g(t, \cdot, \cdot) - g_\star$ is bounded uniformly in t in $L^1(\mathbb{R}^d \times \mathbb{R}^d, (|x|^2 + |v|^2) dx dv)$. A direct computation using

$$\Delta_v g^m + (1+A) \nabla_v \cdot (v g) = \nabla_v \cdot (g (\nabla_v Q_\star - \nabla_v Q))$$

shows that

$$\begin{aligned} \frac{1}{m-1} \frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} (g^m - m g_\star^{m-1} g) dx dv \\ = \iint_{\mathbb{R}^d \times \mathbb{R}^d} (Q_\star - Q) \nabla_v \cdot (g (\nabla_v Q_\star - \nabla_v Q)) dx dv. \end{aligned}$$

An integration by parts completes the proof as in (10). \square

C. Convergence in L^1 and intermediate asymptotics. The proof of Theorem 1 has to be completed by identifying the limit as $t \rightarrow +\infty$ of the solution of Eq. (9): we have to prove that

$$\lim_{t \rightarrow +\infty} \|g(t, \cdot, \cdot) - g_\star\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$$

if g solves Eqs. (9)-(10) under Assumption (5).

Proof of Theorem 1. By Proposition 4, g is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$ with bounded moments for any $t \geq 0$. Lemma 9 yields that $\mathcal{E}(t)$ decreases to a limit and

$$\lim_{t \rightarrow +\infty} \int_t^{+\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} g |\nabla_v Q - \nabla_v Q_\star|^2 dx dv ds = 0. \quad [16]$$

Let us define the *spatial density* of g by

$$\rho_g(t, x) := \int_{\mathbb{R}^d} g(t, x, v) dv$$

and the corresponding *local equilibrium* by

$$\tilde{g}(t, x, v) := \left(\mu(t, x) + \frac{1-m}{2m} (1+A) |v|^2 \right)_+^{\frac{1}{m-1}} \quad [17]$$

where μ is found by inverting

$$\rho_g(t, x) = \int_{\mathbb{R}^d} \left(\mu(t, x) + \frac{1-m}{2m} (1+A) |v|^2 \right)_+^{\frac{1}{m-1}} dv,$$

i.e., $\mu(\tau, x) = \mu_1 (\rho_g(\tau, x))^k$ for some $\mu_1 > 0$ with k such that

$$\frac{1}{k-1} = \frac{d}{2} + \frac{1}{m-1}. \quad [18]$$

Notice that ρ_g is uniformly bounded on $\mathbb{R}^+ \times \mathbb{R}^d$ by Proposition 4. If $m < 1$, using the Csiszár-Kullback inequality of (9, Section 2.4.6) and the entropy-entropy production inequality (9, Lemma 1.12), which amounts to a Gagliardo-Nirenberg inequality in the v variable only, according to (15), we obtain

$$\int_{\mathbb{R}^d} g |\nabla_v Q - \nabla_v Q_\star|^2 dv \geq \mathcal{C} \frac{\|g(t, x, \cdot) - \tilde{g}(t, x, \cdot)\|_{L^1(\mathbb{R}^d)}^2}{(\rho_g(t, x))^{2-k}}$$

(t, x) a.e. for some $\mathcal{C} > 0$ depending only on d and m . With $|g - \tilde{g}| = \rho_g^{1-k/2} \cdot |g - \tilde{g}| / \rho_g^{1-k/2}$ and a Cauchy-Schwarz inequality, we obtain

$$\lim_{t \rightarrow \infty} \int_t^{t+1} \|g(s, \cdot, \cdot) - \tilde{g}(s, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}^2 ds = 0. \quad [19]$$

Take any sequence $(t_n)_{n \in \mathbb{N}}$ such that $0 < t_n \rightarrow \infty$, set $g_n(t, x, v) := g(t_n + t, x, v)$, $\tilde{g}_n(t, x, v) := \tilde{g}(t_n + t, x, v)$ and $\rho_n := \rho_{g_n}$. We already know that $g_n - \tilde{g}_n \rightarrow 0$ as $n \rightarrow +\infty$ in $L^1((0, 1) \times \mathbb{R}^d \times \mathbb{R}^d)$ and $g_n (\nabla_v Q_n - \nabla_v Q_\star) \rightarrow 0$ in $L^2((0, 1) \times \mathbb{R}^d \times \mathbb{R}^d)$ by Eq. (16), where $Q_n := \frac{m}{1-m} g_n^{m-1}$. Via averaging Lemmas (16, Theorem 5) applied to Eq. (9), $\rho_n^R(t, x) := \int_{\mathbb{R}^d} g_n(t, x, v) \psi_R(v) dv$ is bounded in $H^{1/4}((0, 1) \times \mathbb{R}^d)$ where $\psi_R \in C_c^\infty(\mathbb{R}^d)$ is a smooth cut-off function such that $0 \leq \psi_R \leq 1$, $\psi_R = 1$ in B_R and $\psi_R = 0$ in $\mathbb{R}^d \setminus B_{R+1}$. By a diagonal argument as $n \rightarrow \infty$ and $R \rightarrow \infty$, we establish the a.e. and $L^1((0, 1) \times \mathbb{R}^d)$ convergence of $(\rho_n)_{n \in \mathbb{N}}$ to some limit ρ_∞ , up to the extraction of a subsequence. By continuity of the mapping $\rho_n(t, \cdot) \mapsto \mu(t + t_n, \cdot) =: \mu_n(t, \cdot)$, the sequence $(\mu_n)_{n \in \mathbb{N}}$ converges a.e. to a limit μ_∞ , therefore $(\tilde{g}_n)_{n \in \mathbb{N}}$ converges to some g_∞ a.e. and in $L^1((0, 1) \times \mathbb{R}^d \times \mathbb{R}^d)$. As a consequence of Eq. (19), so does $(g_n)_{n \in \mathbb{N}}$, with same limit g_∞ . Since $g_\infty = \tilde{g}_\infty$ is given by Eq. (17) with $\mu = \mu_\infty$, we notice that

$$\partial_t g_\infty + v \cdot \nabla_x g_\infty - A x \cdot \nabla_v g_\infty = 0$$

by passing to the limit in Eq. (9). We learn from Eq. (17) that g_∞ is radially symmetric in v for every fixed (t, x) . By integrating the equation in v , we readily get $\partial_t \rho_\infty = 0$, so that μ_∞ only depends on x and $\partial_t g_\infty = 0 = v \cdot \nabla_x g_\infty - A x \cdot \nabla_v g_\infty$, yielding $m \nabla \mu_\infty(x) = (1-m)(1+A)Ax$. Since g_∞ and g_\star have same mass, we conclude that $g_\infty = g_\star$. This limit is unique, which proves that

$$\lim_{n \rightarrow +\infty} \|g_n(t, \cdot, \cdot) - g_\star\|_{L^1((0,1) \times \mathbb{R}^d \times \mathbb{R}^d)} = 0.$$

along any sub-sequence of $(t_n)_{n \in \mathbb{N}}$ and also along any sequence $(t_n)_{n \in \mathbb{N}}$. By dominated convergence and monotonicity of $t \mapsto \mathcal{E}(t)$, we deduce that $\lim_{t \rightarrow +\infty} \mathcal{E}(t) = 0$, yielding $g(t, \cdot, \cdot) \rightarrow g_\star$, a.e. and in $L^1(\mathbb{R}^d \times \mathbb{R}^d)$, as $t \rightarrow +\infty$.

If $1 < m < \min\{m_2, 3/2\}$, one has to replace $\|g - \tilde{g}\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)}$ in Eq. (19) by $\iint_{\mathbb{R}^d \times \mathbb{R}^d} |g(s, \cdot, \cdot) - \tilde{g}(s, \cdot, \cdot)| \tilde{g}^{m-1}(s, \cdot, \cdot) dx dv$ in order to use the generalized Csiszár-Kullback inequality of (15, Proposition 15, (ii)), which is sufficient to prove the strong convergence on the support of g_\star . The conclusion holds since g and g_\star are nonnegative functions with same mass. \square

By Hölder interpolation and the change of variables Eq. (7), we can obtain more general *intermediate asymptotics* than in Theorem 1.

Corollary 10. *Under the assumptions of Theorem 1, if f solves Eq. (1) with initial datum f_0 and $p \in [1, \infty)$, then*

$$\lim_{t \rightarrow +\infty} t^d \frac{p-1}{p} \frac{1+A}{1-A} \|f(t, x, v) - f_\star(t, x, v)\|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} = 0.$$

D. Decay rates of the spatial density. Here we consider the *spatial density* $\rho_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv$. For the sake of simplicity, we argue in rescaled variables given by Eq. (7). To ensure that moments in v are finite, we also impose $m > \tilde{m}_1$ as in Eq. (15).

Let us recall a classical interpolation inequality in kinetic theory (see for instance (17, Corollary 2)).

Lemma 11. *If $g \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d, |v|^2 dx dv) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d, dx dv)$, then we have*

$$\|\rho_g\|_{L^{1+\frac{2}{d}}(\mathbb{R}^d)} \leq \mathcal{C}_d \|g\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}^{\frac{2}{d+2}} \left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 g(x, v) dx dv \right)^{\frac{d}{d+2}}$$

with $\mathcal{C}_d := 2^{d/(d+2)} \frac{d+2}{2d} |\mathbb{S}^{d-1}|^{2/(d+2)}$.

Proof. The proof is elementary. If $|v| \leq R$, we estimate ρ_g with $\|g\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)}$, and $g|v|^2/R^2$ if $|v| > R$, so that

$$\rho_g(x) \leq |\mathbb{S}^{d-1}| \|g\|_{L^\infty(\mathbb{R}^d \times \mathbb{R}^d)} \frac{R^d}{d} + \frac{1}{R^2} \int_{\mathbb{R}^d} |v|^2 g(x, v) dv.$$

We minimize the right-hand side on $R > 0$ and conclude by integrating with respect to $x \in \mathbb{R}^d$ after taking the power $(d+2)/d$ of both sides of the estimate. \square

We take $\gamma = \gamma_\star$ in (6) so that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} g_\star dx dv = 1$. If $m < 1$, we can write $\mathcal{E}[g] = Z_m \iint_{\mathbb{R}^d \times \mathbb{R}^d} \phi(g/g_\star) g_\star^m dx dv$ with

$$\phi(s) := \frac{Z_m^{-1}}{m-1} (s^m - 1 - m(s-1)), \quad Z_m = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_\star^m dx dv.$$

The function $\phi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is invertible if it is restricted to $[1, +\infty)$. Let us denote by ψ this inverse and notice that $\phi(s) \leq t$ implies $s \leq \psi(t)$ for any $s \in (0, +\infty)$. The following estimate is a straightforward consequence of Jensen's inequality.

Lemma 12. *If $m \in (\tilde{m}_1, 1)$, then we have*

$$(1+A) \frac{1-m}{2m} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 g dx dv \leq Z_m \Psi \left(Z_m^{-1} \mathcal{E}[g] \right) \quad [20]$$

for any $g \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d, (1+|v|^2) dx dv) \cap L^m(\mathbb{R}^d \times \mathbb{R}^d, dx dv)$.

In the case $m > 1$, we shall notice that

$$\frac{1}{2} (1+A) \iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^2 g dx dv \leq \mathcal{E}[g] + \mathcal{H}[g_\star] \quad [21]$$

If g solves Eq. (9), we learn from Lemma 9 and Lemma 11 combined with Eq. (20) and Eq. (21) that $\|\rho_g(t, \cdot)\|_{L^{1+2/d}(\mathbb{R}^d)}$ is uniformly bounded in terms of g_0 . Taking Eq. (7) and Eq. (8) into account, this proves the following result.

Corollary 13. *Under the assumptions of Theorem 1, if f is a solution of Eq. (1) with $f_0 \in L^1_+(\mathbb{R}^d \times \mathbb{R}^d, (|x|^2 + |v|^2) dx dv) \cap L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, then for any $t > 0$, we have*

$$\|\rho_f(t, \cdot)\|_{L^{1+\frac{2}{d}}(\mathbb{R}^d)} \leq \mathcal{C}[f_0] (1 + (1-A)t)^{-\frac{3-d+dm}{(d+2)(m-1)}}$$

for some explicit constant $\mathcal{C}[f_0]$ and with A as in Eq. (8).

4. Further results

In this section, we collect various observations, mostly at formal level, which explains why the fundamental solution defined by Eq. (4) is of interest as a model case for nonlinear kinetic equations.

A. Formal diffusion limit. In order to investigate the diffusion limit, or overdamped regime, let us consider the parabolic scaling applied to a solution f of Eq. (1) such that

$$f(t, x, v) = \left(\frac{\varepsilon^2 \eta}{R(\varepsilon t)} \right)^d h_\varepsilon \left(\varepsilon^2 \tau(\varepsilon t), \varepsilon^{\eta+1} x, \frac{\varepsilon^{\eta-1} v}{R(\varepsilon t)} \right),$$

as $\varepsilon \rightarrow 0$, with $2(d-m-d+1)\eta = 3$, and assume that

$$R(s) = (1+s/\alpha)^\alpha \quad \forall s \geq 0 \quad \text{with } 1/\alpha = d-m-d+2 = d(m-m_c)$$

so that $(\tau, x, v) \mapsto h_\varepsilon(\tau, x, v)$ solves

$$\varepsilon \frac{d\tau}{ds}(s) \partial_\tau h_\varepsilon + R(s) v \cdot \nabla_x h_\varepsilon = \frac{\sigma(s)}{\varepsilon} (\Delta_v h_\varepsilon^m + \nabla_v \cdot (v h_\varepsilon)) \quad [22]$$

where $\sigma(s) := (1+s/\alpha)^{-1}$, $\tau = \tau(s)$ has to be chosen appropriately, and $s = \varepsilon t$. Notice that the rescaling differs from Eq. (8).

At formal level, see (14, 18), a standard Hilbert expansion shows that

$$h_\varepsilon(\tau, x, v) \approx H(\tau, x, v) := \left(\mu(\tau, x) + \frac{1-m}{2m} |v|^2 \right)_+^{\frac{1}{m-1}},$$

where $\mu(\tau, x) = \mu_1(\rho(\tau, x))^{k-1}$ is obtained by solving

$$\rho(\tau, x) = \int_{\mathbb{R}^d} \left(\mu(\tau, x) + \frac{1-m}{2m} |v|^2 \right)_+^{\frac{1}{m-1}} dv = \left(\frac{\mu(\tau, x)}{\mu_1} \right)^{\frac{d}{2} + \frac{1}{m-1}}$$

with $k = 1 + 2\alpha(m-1)$ as in Eq. (18) and an explicit numerical constant $\mu_1 > 0$. The spatial density $\rho_\varepsilon(\tau, x) := \int_{\mathbb{R}^d} h_\varepsilon(\tau, x, v) dv$ solves the *local mass conservation* equation

$$\frac{1}{R} \frac{d\tau}{ds} \partial_\tau \rho_\varepsilon + \nabla_x \cdot j_\varepsilon = 0$$

obtained by formally integrating Eq. (22) with respect to v , while the flux $j_\varepsilon(\tau, x) := \frac{1}{\varepsilon} \int_{\mathbb{R}^d} v h_\varepsilon(\tau, x, v) dv$ obeys to

$$\frac{\varepsilon}{R} \frac{d\tau}{ds} \partial_\tau j_\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v h_\varepsilon(\tau, x, v) dv = -\frac{\sigma}{R} j_\varepsilon.$$

This equation is obtained by multiplying Eq. (12) by v and formally integrating with respect to v . Heuristically, we may drop the $O(\varepsilon)$ term and write, at leading order in ε ,

$$\begin{aligned} -\frac{\sigma(s)}{R(s)} j_\varepsilon(\tau, x) &\approx \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v H(\tau, x, v) dv \\ &= \frac{1}{d} \nabla_x \int_{\mathbb{R}^d} |v|^2 \left(\mu(\tau, x) + \frac{1-m}{2m} |v|^2 \right)_+^{\frac{1}{m-1}} dv = \nu_1 \nabla_x \left(\rho(\tau, x)^k \right) \end{aligned}$$

for some explicit numerical constant $\nu_1 > 0$. Collecting our estimates, we obtain that $\rho_\varepsilon \approx \rho$ solves the nonlinear diffusion equation

$$\partial_\tau \rho = \Delta_x \rho^k \quad \forall (\tau, x) \in \mathbb{R}^+ \times \mathbb{R}^d \quad [23]$$

in the limit as $\varepsilon \rightarrow 0$ if we choose the time scale $s \mapsto \tau(s)$ such that

$$\frac{d\tau}{ds} = \nu_1 \frac{R(s)^2}{\sigma(s)}$$

with the condition $\tau(0) = 0$, i.e.,

$$\tau(t) = \frac{\alpha \nu_1}{2(1+\alpha)} \left((1+t/\alpha)^{2(1+\alpha)} - 1 \right) \quad \forall t \geq 0.$$

With this choice, the initial datum for Eq. (23) is given in terms of f_0 by $\rho(\tau = 0, x) = \varepsilon^{-d(\eta+1)} \int_{\mathbb{R}^d} f_0(\varepsilon^{-(\eta+1)} x, v) dv$ where f_0 stands for the initial datum for Eq. (1). The self-similar Barenblatt-Pattle solution ρ of Eq. (23) with initial datum $\rho(0, \cdot) = \delta$ takes the form $\rho(\tau, x) = (\tau/\beta)^{-d\beta} \rho_\star((\tau/\beta)^{-\beta} x)$ where $1/\beta = d(k-1)+2$ and, for some integration constant $c_\star > 0$ such that $\|\rho_\star\|_{L^1(\mathbb{R}^d)} = 1$,

$$\rho_\star(x) = \left(c_\star + \frac{1-k}{2k} |x|^2 \right)_+^{\frac{1}{k-1}} \quad \forall x \in \mathbb{R}^d.$$

Assuming that α is positive, that is, $m > m_c$, we notice that $k-1$ and $m-1$ have the same sign because $k-1 = 2\alpha(m-1)$. However, this formal computation requires at least a second moment, which means $m > \tilde{m}_1$, or equivalently, $k > d/(d+2)$. This last condition ensures that $\int_{\mathbb{R}^d} |x|^2 \rho_\star(x) dx$ is finite.

B. Translation and Galilean invariance. Although Eq. (1) is nonlinear, it makes sense to consider solutions with measure valued initial data corresponding to non-centred Dirac distributions. Since Eq. (1) is invariant under translations in position, that is, $(x, v) \mapsto (x - x_0, v)$, and under Galilean transformations, that is, $(x, v) \mapsto (x - t v_0, v - v_0)$, the solutions are simple.

Proposition 14. *The function $t \mapsto f_\star(t, x - x_0 - t v_0, v - v_0)$ solves Eq. (1) with measure initial datum $\delta(x - x_0, v - v_0)$ for any $(x_0, v_0) \in \mathbb{R}^d \times \mathbb{R}^d$.*

C. Linearization, eigenfunctions and asymptotic rates. If g solves Eq. (9), let us consider a perturbation h such that $g(t, x, v) = g_\star(x, v) + \varepsilon h(t, x, v)$ and consider the limit as $\varepsilon \rightarrow 0$. At order ε , we obtain the linear evolution equation

$$\partial_t h = \mathcal{L} h \quad [24]$$

where the operator \mathcal{L} is defined by

$$\mathcal{L} h := m \Delta_v (g_\star^{m-1} h) + (1+A) \nabla_v \cdot (v h) - v \cdot \nabla_x h + A x \cdot \nabla_v h.$$

We assume that $\|g_\star\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$ and recall that by (6),

$$m (g_\star(x, v))^{m-1} := \left((1-m) \left(\gamma_\star + \frac{1+A}{2} |v|^2 + \frac{1+A}{2} A |x|^2 \right) \right)_+.$$

By taking the limit as $\varepsilon \rightarrow 0$ in Lemma 9 and collecting terms at order ε^2 , we obtain

$$\begin{aligned} \frac{d}{dt} \iint_{\mathbb{R}^d \times \mathbb{R}^d} |h|^2 g_\star^{m-2} dx dv \\ = -2m \iint_{\mathbb{R}^d \times \mathbb{R}^d} g_\star |\nabla_v (g_\star^{m-2} h)|^2 dx dv, \end{aligned} \quad [25]$$

which can also be recovered directly from Eq. (24). The case $m > 1$ raises various difficulties due to the compactness of the support of g_\star and is omitted here.

Lemma 15. *Let $d \geq 1$ and $m \in (m_1, 1)$. The operator \mathcal{L} defined on $L^2(\mathbb{R}^d \times \mathbb{R}^d, g_\star^{m-2} dx dv)$ is such that*

$$\text{Spec}(\mathcal{L}) \subset \{ \lambda \in \mathbb{C} : \text{Re}(\lambda) \leq 0 \}$$

and the kernel of \mathcal{L} is generated by $h_0 := g_\star^{2-m}$.

Proof. A derivative with respect to γ of both sides of Eq. (9) written for the stationary solution g_\star provides us with an eigenfunction with eigenvalue 0, that can be written as $\partial_\gamma g_\star$ at $\gamma = \gamma_\star$, with g_\star as in (6). Up to a multiplicative constant, this function is h_0 . Any function h such that $\mathcal{L} h = 0$ satisfies $\nabla_v (g_\star^{m-2} h) = 0$ a.e. on the support of g_\star by Eq. (25). Hence, we have to solve $\mathcal{L} h = 0$ with

$$h(x, v) = \rho(x) h_0(x, v) \quad \forall (x, v) \in \mathbb{R}^d \times \mathbb{R}^d$$

for some function ρ . We deduce from $0 = \mathcal{L} h = -h_0 v \cdot \nabla_x \rho$ that ρ is constant on the support of g_\star . Testing the equation $\mathcal{L} h = 0$ with $(|v|^2 + A|x|^2) h$ shows that $h \equiv 0$ on $\mathbb{R}^d \times \mathbb{R}^d \setminus \text{supp}(g_\star)$. This proves that the kernel of \mathcal{L} is generated by h_0 and $\lambda_0 = 0$ is an eigenvalue of \mathcal{L} . Finally, from Eq. (25), we read that \mathcal{L} has no eigenvalue $\lambda \in \mathbb{C}$ such that $\text{Re}(\lambda) > 0$, which completes the proof of Lemma 15. \square

From the invariances of Eq. (1), we can identify several real eigenvalues $\lambda \in \mathbb{R}$, of $(-\mathcal{L})$ such that $\lambda > 0$. Indeed, by constructing special solutions g of Eq. (9) such that

$$g(t, x, v) - g_\star(x, v) = e^{-\lambda t} h(x, v) + o(e^{-\lambda t}) \quad \text{as } t \rightarrow +\infty,$$

and passing to the limit in the equation at main order, we find explicit solutions of $\mathcal{L} h + \lambda h = 0$. In practice, we are able to identify three eigenvalues λ_1, λ_2 and λ_3 whose values eventually depend on m . The indices i of λ_i do not reflect any order in their values.

• *A mode based on scaling.* With R as in Eq. (8) such that $R(t_0) = \tilde{R}_0 \neq 1$, the function $f(t, x, v) = f_\star(t + t_0, x, v)$ solves Eq. (1) with initial datum

$$f_0(x, v) = f_\star(t_0, x, v) = \tilde{R}_0^{-d(1+A)} g_\star(\tilde{R}_0^{-1} x, \tilde{R}_0^{-A} v - \tilde{R}_0^{-1} x).$$

Using the rescaling of Eq. (7), the corresponding solution g of Eqs. (9)-(10) is such that

$$g(\tau, x, v) = G_{R(\tau)}(x, v), \quad G_R(x, v) := R^{-d(1+A)} g_\star\left(\frac{x}{R}, \frac{v}{R} - \frac{x}{R}\right)$$

with $R(\tau) := (1 + (\tilde{R}_0^{1-A} - 1) e^{-(1-A)\tau})^{1/(1-A)}$. Using the fact that $\lim_{\tau \rightarrow +\infty} R(\tau) = 1$, we can identify $\partial_R G_R$ at $R = 1$ as an eigenfunction, that is, $d(1+A) g_\star + x \cdot (\nabla_x g_\star - \nabla_v g_\star) + A v \cdot \nabla_v g_\star = 2d(1-m) h_1$ with

$$\frac{h_1(x, v)}{h_0(x, v)} = \frac{B}{m} \left(\gamma_\star + (B+AC) |v|^2 + C x \cdot v + A(B-C) |x|^2 \right),$$

$B = (1 + A)/2$ and $1/C = d(1 - m)$. Since $R(\tau) - 1 \sim e^{-(1-A)\tau}$ as $\tau \rightarrow +\infty$, by writing that

$$g(\tau, x, v) = g_\star(x, v) + 2d(1 - m)h_1(x, v)(R - 1) + O((R - 1)^2),$$

we also identify $\lambda_1 = 1 - A$ as an eigenvalue of $-\mathcal{L}$.

• *A mode based on translations with respect to v in the phase space.* The solution g of Eq. (9) with initial datum $f_0(x, v) = g_\star(x, v - v_0)$ for any $v_0 \neq 0$ is given up to $o(e^{-A\tau})$ terms by

$$g(\tau, x, v) \approx g_\star\left(x - \frac{1}{1-A}e^{-A\tau}v_0, v + \frac{A}{1-A}e^{-A\tau}v_0\right)$$

and we identify $\lambda_2 = A$ as an eigenvalue of $-\mathcal{L}$ corresponding to any of the eigenfunctions $A\partial_{v_i}g_\star - \partial_{x_i}g_\star$, that is, of

$$h_{2,i}(x, v) := (v_i - x_i)h_0(x, v), \quad i = 1, 2, \dots, d.$$

• *A mode based on translations with respect to x in the phase space.* The solution g of Eq. (9) with initial datum $f_0(x, v) = g_\star(x - x_0, v)$ is given by $g(\tau, x, v) = g_\star(x - e^{-\tau}x_0, v + e^{-\tau}x_0)$ for any $x_0 \neq 0$ so that we can identify $\lambda_3 = 1$ as an eigenvalue of $-\mathcal{L}$ corresponding to any of the eigenfunctions $\partial_{v_i}g_\star - \partial_{x_i}g_\star$ which can also be written as

$$h_{3,i}(x, v) := (v_i - Ax_i)h_0(x, v), \quad i = 1, 2, \dots, d.$$

A numerical computation of the spectrum of \mathcal{L} is shown in Fig. 2, which suggests that, at least for some values of m , the operator \mathcal{L} is sectorial and

$$\text{Spec}(\mathcal{L}) \setminus \{0\} \subset \{\lambda \in \mathbb{C} : \text{Re}(\lambda) \leq -\min\{A, 1 - A\}\}.$$

A simpler question would be to prove that

$$\sup\{\text{Re}(\lambda) : \lambda \in \text{Spec}(\mathcal{L}) \setminus \{0\}\} < 0.$$

Coming back to the nonlinear evolution problem Eq. (9), it is an open question to decide whether there is some $\lambda_\star > 0$ such that

$$\limsup_{t \rightarrow +\infty} e^{\lambda_\star t} \mathcal{E}(t) < +\infty \quad [26]$$

for any function g satisfying the assumptions of Theorem 1, or not. One can also wonder if $\lambda_\star = \min\{A, 1 - A\}$ and ask if Eq. (26) is also true under more general conditions on the initial datum.

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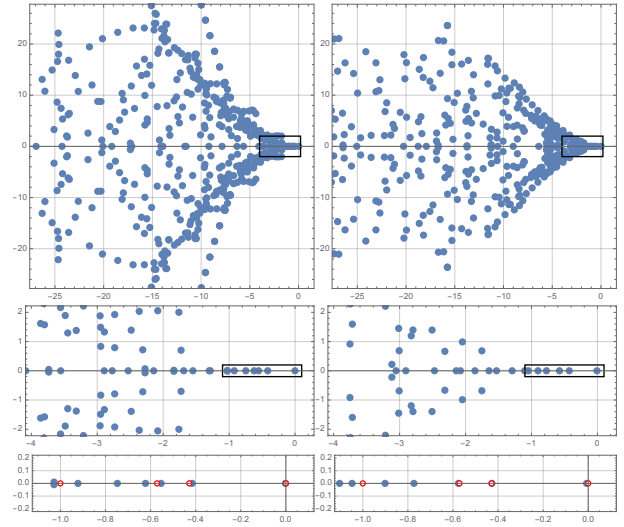


Fig. 2. Numerical computation of the spectrum of the operator L defined by $Lf = g_\star^{(m-2)/2} \mathcal{L}(g_\star^{(2-m)/2} f)$ for a smooth function f supported in the rectangle $[-18, 18] \times [-28, 28] \ni (x, v)$ (left) and in the ellipse (right) with semi-axes $2 * 18/\sqrt{\pi}$ for x and $2 * 28/\sqrt{\pi}$ for v , with zero Dirichlet boundary conditions and $m = 0.8$. Here L is defined on $L^2(\Omega, dx dv)$ where Ω is either the rectangle or the ellipse (both have same area). The spectrum of L is seen as an approximation of the spectrum of \mathcal{L} on $L^2(\mathbb{R}^d \times \mathbb{R}^d, g_\star^{m-2} dx dv)$, with dimension $d = 1$, for the eigenvalues which are the closest to 0. Local enlargements close to $\lambda = 0$ are shown below the general figure of the spectrum. On the last row, the eigenvalues $-\lambda_i$ with $i = 0, 1, 2, 3$ are marked by red circles. The highest non-zero eigenvalue of L is of the order of -0.4152 (left, rectangle) and -0.4272 (right, ellipse), to be compared with $-\lambda_1 = -A \approx -0.4286$.

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