Entropies relatives pour le système de Vlasov-Poisson dans des domaines bornés

Naoufel BEN ABDALLAH [a], Jean DOLBEAULT [b]

[a] M.I.P. (U.M.R. C.N.R.S. no. 5640), Université Paul Sabatier - Toulouse III, 118, route de Narbonne, 31062 Toulouse Cédex 3, France. Courriel : naoufel@mip.ups-tlse.fr

[b] Ceremade (UMR CNRS no. 7534), Université Paris IX-Dauphine, Place de Lattre de Tassigny, 75775 Paris Cédex 16, France. Courriel : dolbeaul@ceremade.dauphine.fr

Résumé. A l'aide d'entropies relatives, nous étudions l'irréversibilité pour le système de Vlasov-Poisson avec conditions d'injection au bord en présence comme en l'absence de collisions. Nous en déduisons que s'il y a convergence en temps grand, la solution limite est à trace donnée au bord. Nous montrons qu'en l'absence de terme de collision cela n'est possible que pour l'unique solution stationnaire, sous réserve d'hypothèses de régularité très fortes et en dimension un. © Académie des Sciences/Elsevier, Paris

Relative entropies for the Vlasov-Poisson system in bounded domains

Abstract. *Using relative entropies we study irreversibility for the Vlasov-Poisson system with injection conditions on the boundary with or without collisions. If the solution converges for large times, this allows to deduce that the limit has a given trace on the boundary. In the one dimensional collisionless case and under strong regularity assumptions, this is possible only for the unique stationary solution.* © Académie des Sciences/Elsevier, Paris

---

Version Française abrégée

Considérons le système de Vlasov-Poisson (avec, éventuellement, des collisions)

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - (\nabla_x\phi + \nabla_x\phi_0)\cdot\nabla_v f = Q(f)\ , \quad -\Delta\phi = \displaystyle\int f(x,v,t)\ dv \\ f_{|t=0} = f_0\ , \quad \phi_{|\partial\omega} = 0\ , \quad f_{|\Gamma^-\times I\!\!R^+} = \gamma(\frac{1}{2}|v|^2 + \phi_0(x)) \end{cases} \tag{1}$$

où $(x,v,t) \in \overline{\omega}\times I\!\!R^d \times I\!\!R^+$, ω est un domaine borné régulier de $I\!\!R^d$, $\Omega = \omega\times I\!\!R^d$, $\Gamma = \partial\Omega$, Γ^- (resp. Γ^+) est la partie de Γ correspondant aux vitesses entrantes (resp. sortantes) et $d\sigma(x,v)$ est la mesure induite par la mesure de Lebesgue sur Γ avec poids $|\nu(x)\cdot v|$, où $\nu(x)$ est le vecteur unitaire normal sortant en $x\in\partial\omega$. On dira que γ possède la propriété (P) si et seulement si γ est définie sur $(\min_{x\in\omega}\phi_0(x),+\infty)$, bornée, régulière, strictement décroissante, à valeurs positives, et rapidement décroissante vers 0 à l'infini. On notera $\gamma^{(-1)}$ l'inverse de γ étendue par une fonction arbitraire fixée strictement décroissante à $I\!\!R$ et $M(x,v) = \gamma(\frac{1}{2}|v|^2 + U[M](x) + \phi_0(x))$ où $U[M]$ est l'unique point critique dans $H^1_0(\omega)$ de $U \mapsto \frac{1}{2}\int_\omega |\nabla U|^2\ dx - 2^{d/2-1}|S^{d-1}|\cdot\int_\omega\ dx\cdot[\int_0^{U+\phi_0(x)}\ dw(\int_0^{+\infty} s^{d/2-1}\gamma(s+$

$w)\ ds)]$. De manière plus générale, on notera $U[f]$ la solution $\phi \in H^1_0(\omega)$ de $-\Delta\phi = \int f\ dv$. On supposera que le noyau de collision vérifie

$$\int_{I\!R^d} Q(g)[\frac{1}{2}|v|^2 - \gamma^{(-1)}(g)]\ dv \leq 0\ , \tag{H1}$$

et les quatre principaux exemples de cette note correspondent aux noyaux de collision suivants:

1. Système de Vlasov-Poisson pur: $Q \equiv 0$.
2. Opérateur de Fokker-Planck linéaire: $Q_{FP}(f) = \beta \mathrm{div}_v(vf + \theta\nabla_v f)$, $\gamma(u) = \exp(-\frac{1}{\theta}(u-\mu_0))$.
3. Approximation BGK de l'opérateur de collision de Boltzmann pour des fermions ($\alpha \geq 0$): $Q_\alpha(f) = \int \sigma(v,v')[M_0(v)f(v')(1-\alpha f(v)) - M_0(v')f(v)(1-\alpha f(v'))]\ dv'$, où $M_0(v) = (2\pi\theta_0)^{-d/2}\exp[-\frac{|v|^2}{2\theta_0}]$, et $\gamma(u) = (\alpha + \exp((u-\mu_0)/\theta_0))^{-1}$.
4. Opérateur linéaire pour les collisions élastiques: $Q_E(f) = \int_{I\!R^d} \chi(v,v')(f(v') - f(v))\delta(|v'|^2 - |v|^2)dv'$.

L'outil principal de cette étude est l'entropie relative:
$\Sigma_\gamma[g|h] = \int_\Omega (\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h))\ dxdv + \frac{1}{2}\int_\omega |\nabla U[g-h]|^2\ dx$ où $\beta_\gamma(g) = -\int_0^g \gamma^{(-1)}(z)\ dz$.

Théorème 1 – *Soit $f_0 \in L^1(\Omega)$ une fonction positive telle que $\Sigma_\gamma[f_0|M]$ soit finie. Sous les hypothèses (P) et (H1), une solution de (1) vérifie $\Sigma_\gamma[f_0|M] \geq \Sigma_\gamma[f|M] \geq 0$, $\Sigma_\gamma[f|M] = 0$ si et seulement si $f = M$ p.p., et $\frac{d}{dt}\Sigma_\gamma[f(t)|M] = \int_\Omega Q(f)[\frac{1}{2}|v|^2 - \gamma^{(-1)}(f)]\ dxdv - \int_{\Gamma+}(\beta_\gamma(g) - \beta_\gamma(M)) - (g-M)\beta'_\gamma(M)\ d\sigma \leq 0$.*

A l'aide de cet outil, nous avons un contrôle uniforme en temps sur $f(t)$ (par exemple si $\gamma(u) = \exp(-u/\theta_0)$, on obtient une borne uniforme sur $||f\log f(t)||_{L^1_{x,v}}$), ainsi qu'un contrôle sur le terme de production de l'entropie (et sur $f(t)_{|\Gamma+}$). Dans l'étude de la convergence en temps grand vers la solution stationaire, la deuxième étape consiste à montrer que la suite $(f(t+t_n), \phi(t+t_n))$, où t_n est une suite tendant vers l'infini, converge, à extraction près, vers une solution (f_∞, ϕ_∞) de Vlasov-Poisson, telle que $Q(f) = 0$ et $f_{|\partial\Omega} = M$. Cette étape dépasse le cadre de la présente note, mais nous montrons que s'il y a convergence, alors (f_∞, ϕ_∞) est solution d'un problème à trace au bord fixée. Il s'agit ensuite de montrer que (f_∞, ϕ_∞) est stationnaire. C'est le cas des exemples 2, 3 et 4, le noyau de Q étant composé de fonctions ne dépendant que de $|v|^2$. Il suffit alors d'appliquer le

Lemme 1 – *Soit f une solution de l'équation de Vlasov. Si f est paire (ou impaire) par rapport à la variable v, elle est stationnaire.*

En l'absence de collisions, la stationnarité de la solution est un problème ouvert. Néanmoins, sous réserve d'une hypothèse de régularité très forte et difficile à vérifier, et en dimension $d = 1$ uniquement, une telle solution est stationnaire.

Théorème 2 – *Considérons une solution (f,ϕ) du système de Vlasov-Poisson pour $\omega = (0,1)$, telle que $f(x,v,t) = \gamma(\frac{1}{2}|v|^2 + \phi_0(x))$ pour tous $t \in I\!R^+$, $(x,v) \in \Gamma$, et supposons que ϕ est analytique en x avec des coefficients C^∞ en temps. Si $-\Delta\phi_0 \geq 0$, (f,ϕ) est l'unique solution stationnaire donnée par $f = M$, $\phi = U[M]$.*

1. Preliminaries and examples

Consider the Vlasov-Poisson system for charged particles transport (with zero or non zero collision term $Q(f)$)

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - (\nabla_x\phi + \nabla_x\phi_0)\cdot\nabla_v f = Q(f)\ , \quad -\Delta\phi = \displaystyle\int f(x,v,t)\ dv \\ f_{|t=0} = f_0\ , \quad \phi_{|\partial\omega} = 0\ , \quad f_{|\Gamma^-\times I\!R^+} = \gamma(\frac{1}{2}|v|^2 + \phi_0(x)) \end{cases} \tag{1}$$

where $(x,v,t)\in\overline{\omega}\times I\!R^d\times I\!R^+$, ω is a smooth bounded domain in $I\!R^d$, $\Omega=\omega\times I\!R^d$, $\Gamma=\partial\Omega$, and if $\nu(x)$ denotes the outgoing normal unit vector at $x\in\partial\omega$, $\Sigma^\pm(x)=\{v\in I\!R^d\ :\ \pm v\cdot\nu(x)>0\}$ and $\Gamma^\pm=\{(x,v)\in\Gamma\ :\ v\in\Sigma^\pm(x)\}$. We will denote by $d\sigma(x,v)$ the measure $|\nu(x)\cdot v|\ d_\Gamma(x,v)$ where $d_\Gamma(x,v)$ is the measure induced by Lebesgue's measure on Γ.

$\nabla_x\phi_0$ is an external electrostatic field, whereas $\nabla_x\phi$ accounts for self-consistent effects. We say that γ has Property (P) if and only if it is defined on $(\min_{x\in\omega}\phi_0(x),+\infty)$, bounded, smooth and strictly decreasing with values in $I\!R^+_*$, and rapidly decreasing at infinity: $\sup_{x\in\omega}\int_0^{+\infty}s^{d/2-1}\gamma(s+\phi_0(x))\ ds<+\infty$. We denote by $\gamma^{(-1)}$ the inverse function of γ extended by an arbitrary, fixed, strictly decreasing function to $I\!R$. For any function $g\in L^1(\Omega)$, $U[g]=u$ denotes the unique solution in $W_0^{1,d/(d-1)}(\omega)$ of $-\Delta u=\int g(x,v)\ dv$. The operator U is linear and satisfies

$$\int_\Omega g\ U[f]\ dxdv = \int_\Omega f\ U[g]\ dxdv = \int_\omega\left(\left|\nabla_x U\big[\frac{f+g}{2}\big]\right|^2 - \left|\nabla_x U\big[\frac{f-g}{2}\big]\right|^2\right)\ dx\ .$$

Q is a collision term preserving the mass: $\int Q(f)\ dv=0$. When $Q\equiv 0$, we define the following stationary solution

$$M(x,v)=\gamma(\frac{1}{2}|v|^2+U[M](x)+\phi_0(x))\quad \forall\ (x,v)\in\Omega\ . \tag{2}$$

Note that M is uniquely defined since $U[M]$ is the unique critical point in $H_0^1(\omega)$ of the strictly convex coercive functional

$$U\mapsto\frac{1}{2}\int_\omega|\nabla U|^2\ dx - 2^{d/2-1}|S^{d-1}|\cdot\int_\omega dx\cdot\left[\int_0^{U+\phi_0(x)}dw\left(\int_0^{+\infty}s^{d/2-1}\gamma(s+w)\ ds\right)\right]\ .$$

When $Q\not\equiv 0$, we still use the same definition for M and require the following property (compatibility of the collisions with the boundary conditions)

$$\int_{I\!R^d}Q(g)[\frac{1}{2}|v|^2-\gamma^{(-1)}(g)]\ dv\le 0\ . \tag{H1}$$

In this note, we study the irreversibility due to the boundary conditions and compatible collisions, characterize the possible limits as $t\to+\infty$ as solutions with no flux at the boundary and prove in two cases that such solutions are stationary solutions. The main examples we have in mind are

Example 1. Pure Vlasov-Poisson system: $Q\equiv 0$. The choice of γ is arbitrary. See [1] for a scattering result for large times asymptotics.

Example 2. The Vlasov-Poisson-Fokker-Planck: $Q_{FP}(f)=\mathrm{div}_v(vf+\theta\nabla_v f)$ where $\theta>0$ is a temperature, $\gamma(u)=\exp(-\frac{1}{\theta}(u-\mu_0))$, $\gamma^{(-1)}(f)=\theta\log(f)-\mu_0$. See [2,3] for related results.

Example 3. BGK approximation of the Boltzmann collision operator for fermions

$$Q_\alpha(f) = \int \sigma(v,v')[M_0(v)f(v')(1-\alpha f(v)) - M_0(v')f(v)(1-\alpha f(v'))]\, dv' \; .$$

σ is a nonnegative symmetric cross-section and $M_0(v) = (2\pi\theta_0)^{-d/2}\exp[-\frac{|v|^2}{2\theta_0}]$ is a fixed Maxwellian function with a given temperature $\theta_0 > 0$. Here α is a nonnegative parameter (the case $\alpha < 0$ also makes sense in the context of bosonic particles), $\gamma(u) = (\alpha + \exp((u-\mu_0)/\theta_0))^{-1}$ and $\gamma^{(-1)}(f) = \mu_0 - \theta_0 \log(\frac{f}{1-\alpha f})$.

Example 4. Linear elastic collision operator

$$Q_E(f) = \int_{\mathbb{R}^d} \chi(v,v')\; (f(v') - f(v))\; \delta(|v'|^2 - |v|^2)\; dv' \; .$$

χ is a symmetric positive cross-section, and similarly to Example 1, the choice of γ is left arbitrary. Indeed, for any given function f in $L^1(\mathbb{R}^d)$, any bounded continuous function ψ and any nondecreasing function H,

$$\int Q_E(f)\cdot\psi(|v|^2)\; dv = 0 \quad \text{and} \quad \int Q_E(f)\cdot H(f)\; dv \le 0 \; .$$

Consequently, (H1) is satisfied for any function γ having Property (P) and the kernel of Q is spanned by the functions depending only on $|v|^2$.

Further examples will be provided in a forthcoming paper [4]. The main ingredient we use here is the construction of a *relative entropy* depending on the boundary data. This denomination generalizes the usual notion of entropy for the Boltzmann equation and tends to be used for other kinetic models or even for parabolic equations (see [5,6]).

2. Relative entropy and irreversibility

Let us define the relative entropy of two functions g, h of the (x,v) variables by

$$\Sigma_\gamma[g|h] = \int_\Omega (\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h))\; dxdv + \frac{1}{2}\int_\omega |\nabla U[g-h]|^2\; dx \; ,$$

where β_γ is the real function defined by $\beta_\gamma(g) = -\int_0^g \gamma^{(-1)}(z)\; dz$. Since γ is strictly decreasing, it is readily seen that $\Sigma_\gamma[g|h]$ is always nonnegative and vanishes if and only if $g = h$ a.e.

Theorem 1 – *Let f be a solution of (1) and assume that γ and Q satisfy the properties (P) and (H1) respectively. Then the relative entropy $\Sigma_\gamma[f(t)|M]$ where M is defined by (2) satisfies*

$$\frac{d}{dt}\Sigma_\gamma[f(t)|M] = -\Sigma^+_\gamma[f|M] + \int_\Omega Q(f)[\frac{1}{2}|v|^2 - \gamma^{(-1)}(f)]\; dxdv \le 0 \; . \tag{3}$$

Here Σ^+_γ is the boundary entropy flux defined by $\Sigma^+_\gamma[g|h] = \int_{\Gamma^+}(\beta_\gamma(g) - \beta_\gamma(h) - (g-h)\beta'_\gamma(h))\; d\sigma$.

To prove the theorem, we first recall the following identities satisfied by renormalized solutions of the Vlasov equation [7]

$$\frac{d}{dt}\int_\Omega \beta(f)\; dxdv = \int_{\Gamma^-}\beta(f)\; d\sigma - \int_{\Gamma^+}\beta(f)\; d\sigma + \int_\Omega \beta'(f)Q(f)\; dxdv \; , \tag{4}$$

$$\frac{d}{dt}\int_\Omega f(\frac{|v|^2}{2}+\frac{1}{2}U[f]+\phi_0)\;dxdv=\int_{\Gamma^-} f(\frac{|v|^2}{2}+\phi_0)\;d\sigma-\int_{\Gamma^+} f(\frac{|v|^2}{2}+\phi_0)\;d\sigma+\int_\Omega \frac{|v|^2}{2}Q(f)\;dxdv\;. \tag{5}$$

Identity (3) is obtained, after some simple calculations, by summing (4) and (5) (with $\beta=\beta_\gamma$) and taking advantage of (2), and of the identity $f=M$ on Γ^-.

Since $\Sigma_\gamma[g|h]$ is always nonnegative, the above theorem provides a uniform in time control on $f(t)$. Like in whole space problems [5,6,8], we obtain a Lyapunov functional for the study of the long time behaviour. The main difference with whole space problems and with previous studies of boundary value problems [2,3] is that the total mass is not conserved.

3. The characterization of the limit Problem

The rigorous treatment of how to pass to the limit in time is beyond the scope of this note. We shall only give the general strategy. We assume that

$$\int_{I\!R^d} Q(g)[\frac{1}{2}|v|^2]-\gamma^{(-1)}(g)]\;dv=0\Longleftrightarrow Q(g)=0\;. \tag{H2}$$

This is satisfied in each of the examples of Section 1. When $Q\equiv 0$ or $Q=Q_E$, the solution $f(t)$ is uniformly bounded in L^∞, and we may pass to the limit.

Corollary 1 – *Let γ and Q satisfy Property (P) and (H1)-(H2) respectively, and consider a solution f of (1) such that the initial datum is bounded in $L^1\cap L^\infty(\Omega)$. Consider an unbounded increasing sequence $(t_n)_{n\in I\!N}$ and (f^n,ϕ^n) defined by $(f^n(x,v,t),\phi^n(x,t))=(f(x,v,t+t_n),\phi(x,t+t_n))$. If (f^n,ϕ^n) converges to some (f^∞,ϕ^∞) weakly in $L^\infty_{\rm loc}(dt,L^1(\Omega))\times L^\infty_{\rm loc}(dt,H^1_0(\omega))$ and if $Q(f^n)\overset{\mathcal{D}'}{\to}Q(f^\infty)$, then $\partial_t f^\infty+v\cdot\nabla_x f^\infty-\nabla_x(\phi^\infty+\phi_0)\cdot\nabla_v f^\infty=0$ in $I\!R^+\times\omega\times I\!R^d$, $f^\infty(x,.,t)$ belongs to the kernel of Q for any $(t,x)\in I\!R^+\times\omega$ and $f_{|\Gamma^+}(x,v,t)=\gamma(|v|^2/2+\phi_0(x))$ for any $t\in I\!R^+$, $(x,v)\in\Gamma^+$.*

When there is no L^∞ uniform bound on f and for γ sufficiently decreasing, in the sense that $\lim_{g\to+\infty}(\beta_\gamma(g)/g)=+\infty$, it is still possible to pass to the limit in the equation in some cases (Vlasov-Poisson-Fokker-Planck system or BGK equation, for instance) using renormalized solutions. To identify the limit as $t\to+\infty$, one has to prove that $Q(f^n)\to Q(f^\infty)$, which is not straightforward when Q is nonlinear (example 3 with $\alpha\neq 0$). This difficulty is not specific to bounded domains.

4. Solutions with a given trace at the boundary

In this section, we prove that the solution of the limit problem is stationary. If $f^\infty\in\mathrm{Ker}\,\mathrm{Q}$ depends only on $|v|^2$ (examples 2, 3 and 4), we may apply the

Lemma 1 – *Let f be a solution of the Vlasov equation. If f is even (or odd) with respect to the v variable, then it does not depend on t.*

For the pure Vlasov-Poisson system ($Q\equiv 0$) proving that f^∞ is stationary is an interesting open problem. It is true when $d=1$ if the potential is analytic.

Theorem 2 – *Assume that γ satisfies Property (P) and consider a solution (f,ϕ) of the Vlasov-Poisson system on the interval $\omega=(0,1)$ such that $f(x,v,t)=\gamma(\frac{1}{2}|v|^2+\phi_0(x))$ for any $t\in I\!R^+$, $(x,v)\in\Gamma$ and assume that ϕ is analytic in x with C^∞ (in time) coefficients. If $-\Delta\phi_0\geq 0$ on ω, then (f,ϕ) is the unique stationary solution corresponding to $f=M$, $\phi=U[M]$.*

Analyticity results are available only for whole space or periodic evolution problems [9] (note that analyticity is the standard framework for the study of the Landau damping [1,10,11]). We are not aware of any analyticity result for boundary value problems. In the above theorem, the assumption on ϕ_0 is made only to avoid closed characteristics [12]. We shall refer to [4] for a complete proof and will only provide a sketch of the computations. Let us write $\phi(x,t) = \sum_{n=0}^{+\infty} a_n(t)x^n$ with $a_n \in C^\infty$. The first term $a_0(t)$ vanishes identically because of the boundary conditions and $\inf_{t>0} a_1(t) > 0$ by Hopf's lemma.

Next, we shall consider particles injected in the domain with a small velocity at $x = 0$: their exit time is small too ($\phi + \phi_0$ is strictly concave). As long as v does not change sign, we may parametrize the characteristics with x by writing $\frac{dv}{dt} = \frac{\partial\phi}{\partial x}(x,t)$ and $\frac{dv^2}{dx} = -2\frac{\partial\phi}{\partial x}(x,t)$ and using $dt = dx/v$. Letting $e^\pm(x) = \frac{|v|^2}{2}(x)$ for particles going forward ($+$) and backward ($-$) respectively, we have

$$\frac{de^\pm}{dx} = -\frac{\partial\phi}{\partial x}(x, t_0 \mp \frac{1}{\sqrt{2}}\int_{x_0}^{x} \frac{dy}{\sqrt{e^\pm(y)}}) , \quad e^\pm(x_0) = 0 ,$$

where x_0 is the point where the velocity changes sign (at time $t = t_0$). The monotonicity of γ and the control of f on the whole boundary Γ implies that $e^+(0) = e^-(0)$ for any $x_0 > 0$, small, and an expansion in $\epsilon = \sqrt{x_0}$ shows that $\frac{\partial}{\partial t}\frac{\partial^n\phi}{\partial x^n}(0,t) = 0$ for any $t \in I\!R^+$. □

As a concluding remark, we notice that if the stationary Vlasov-Poisson problem has a unique solution (this is true for example in dimension $d = 1$, when $-\Delta\phi_0 \geq 0$), this solution is the long time limit of $f(t)$. The results contained in this note can be extended to relativistic or periodic in momentum models simply by replacing $|v|^2/2$ by an energy $\varepsilon(k)$ depending on the wave vector (or momentum) k, v by $\nabla_k\varepsilon(k)$ and dv by dk, provided $k \mapsto \varepsilon(k)$ is even, smooth and non degenerate.

Acknowledgments. The authors acknowledge support from the GdR SPARCH of the Centre National de la Recherche Scientifique (France) and from the T.M.R. project "Asymptotic Methods in Applied kinetic Theory" # ERB FMRX CT97 0157, run by the European Community. They thank the referee for his very relevant and detailed comments.

Références bibliographiques

[1] Caglioti, E.; Maffei, C. Time asymptotics for solutions of Vlasov-Poisson equation in a circle. J. Statist. Phys. 92 no. 1-2 (1998) 301–323.

[2] J.A. Carrillo, Global weak solutions for the initial-boundary value problems to the Vlasov-Poisson-Fokker-Planck system, Math. Meth. Appl. Sci. 21 (1998) 907–938.

[3] L.L. Bonilla, J.A. Carrillo, J. Soler, Asymptotic behaviour of the initial boundary value problem for the three-dimensional Vlasov-Poisson-Fokker-Planck system, SIAM J. Appl. Math. 57 (1997) 1343–1372.

[4] N. Ben Abdallah, J. Dolbeault, in preparation.

[5] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, On logarithmic Sobolev inequalities, Csiszár-Kullback inequalities, and the rate of convergence to equilibrium for Fokker-Planck type equations, preprint T.M.R. "Asymptotic Methods in Kinetic Theory" (no. 1, 1998).

[6] J. Dolbeault, Free energy and solutions of the Vlasov-Poisson-Fokker-Planck system: external potential and confinement (large time behavior and steady states), J. Math. Pures et Appliquées 78 (1999) 121–157.

[7] S. Mischler, On the trace problem for solutions of the Vlasov equation, to appear in Comm. P. D. E.

[8] L. Desvillettes, C. Villani, On the trend to golbal equilibrium in spatially inhomogeneous entropy-dissipating systems, Part I: the linear Fokker-Planck equation, preprint TMR "Asymptotic Methods in Kinetic Theory" (no. 79, 1999).

[9] S. Benachour, Analyticité des solutions des équations de Vlassov-Poisson. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 16 no. 1 (1989) 83–104.

[10] P. Degond, Spectral theory of the linearized Vlasov-Poisson equation. Trans. Amer. Math. Soc. 294 no. 2 (1986) 435–453.

[11] R. Glassey, J. Schaeffer, On time decay rates in Landau damping. Comm. Partial Differential Equations 20 no. 3-4 (1995) 647–676.

[12] F. Poupaud, Boundary value problems for the stationary Vlasov-Maxwell system, Forum Math. 4 (1992) 499–527.