INEQUALITIES INVOLVING AHARONOV-BOHM MAGNETIC POTENTIALS IN DIMENSIONS 2 AND 3

DENIS BONHEURE

Département de mathématique, Faculté des sciences, Université Libre de Bruxelles, Campus Plaine CP 213, Bld du Triomphe, B-1050 Brussels, Belgium

E-mail: dbonheur@ulb.ac.be

JEAN DOLBEAULT

CEREMADE (CNRS UMR n° 7534), PSL university, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

E-mail: dolbeaul@ceremade.dauphine.fr

MARIA J. ESTEBAN

CEREMADE (CNRS UMR n° 7534), PSL university, Université Paris-Dauphine, Place de Lattre de Tassigny, 75775 Paris 16, France

E-mail: esteban@ceremade.dauphine.fr

ARI LAPTEV

Department of Mathematics, Imperial College London, Huxley Building, 180 Queen's Gate, London SW7 2AZ, UK

E-mail: a.laptev@imperial.ac.uk

MICHAEL LOSS

School of Mathematics, Skiles Building, Georgia Institute of Technology, Atlanta GA 30332-0160, USA

E-mail: loss@math.gatech.edu

This paper is devoted to interpolation inequalities of Gagliardo-Nirenberg type associated with Schrödinger operators involving Aharonov-Bohm magnetic potentials and related magnetic Hardy inequalities in dimensions 2 and 3. The focus is on symmetry properties of the optimal functions, with explicit ranges of symmetry and symmetry breaking in terms of the intensity of the magnetic potential.

Keywords: Aharonov-Bohm magnetic potential; radial symmetry; cylindrical symmetry; symmetry breaking; magnetic Hardy inequality; magnetic interpolation inequality; optimal constants; magnetic Schrödinger operator; magnetic Keller-Lieb-Thirring inequality; magnetic rings.

Mathematics Subject Classification 2010: Primary: 81V10, 81Q10, 35Q60; Secondary: 35Q40, 49K30, 35P30, 35J10, 35Q55, 46N50, 35J20.

1. Introduction

A quantum charged particle described by a complex valued wave function interacts with an electromagnetic potential even in regions in which both magnetic and electric fields are vanishing, *i.e.*, in regions in which a classical particle would not be affected by any force. From a mathematical point of view, the wave function is a nonlocal object which detects the fields even if they are supported in a zero measure set and equations have to be written in the sense of distributions. In 1959 Y. Aharonov and D. Bohm proposed experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects* (see [1]). They suggested to use a long, thin solenoid to produce a magnetic field such that the region in which the magnetic field is non-zero, approximated by a line, and the region in which the particle evolves essentially do not overlap. However the magnetic potential is everywhere non-zero and produces a phase shift of the wave function which can be detected experimentally by looking for interferences inducing variations of the particle density. This experiment reveals the role of the phase in Quantum Mechanics. It is one of the few experiments that have been realized so far to question the very foundations of Quantum Mechanics and its relevance for the description of matter. It is therefore of importance to clarify the mathematical framework, study the optimal solutions for the underlying functional inequalities and gain as much qualitative insight as possible. Although the problems studied in this paper are non-linear, we give quantitative estimates, which are in some cases remarkably accurate (see [5]) or even sharp.

In dimensions $d=2$ and $d=3$, the magnetic field can be considered as a singular measure supported in the set $x_1=x_2=0$, where $(x_i)_{i=1}^d$ is a system of cartesian coordinates.

On the Euclidean space $\mathbb{R}^d$, the *magnetic Laplacian* is defined via a *magnetic potential* $\mathbf{A}$ by

$$-\Delta_{\mathbf{A}}\,\psi=-\Delta\,\psi-\,2\,i\,\mathbf{A}\cdot\nabla\psi+|\mathbf{A}|^2\psi-\,i\,(\operatorname{div}\mathbf{A})\,\psi\,.$$

The *magnetic field* is $\mathbf{B}=\operatorname{curl}\mathbf{A}$. The quadratic form associated with $-\Delta_{\mathbf{A}}$ is given by $\int_{\mathbb{R}^3}|\nabla_{\mathbf{A}}\,\psi|^2\,dx$ and well defined for all functions in the space

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d):=\left\{\psi\in\mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\,\psi\in\mathrm{L}^2(\mathbb{R}^d)\right\}$$

where the *magnetic gradient* takes the form

$$\nabla_{\mathbf{A}}\,:=\nabla+\,i\,\mathbf{A}\,.$$

Let us introduce *polar coordinates* (r,θ) with

$$r=|x|=\sqrt{x_1^2+x_2^2}\quad\text{and}\quad r\,e^{i\theta}=x_1+i\,x_2$$

in dimension $d=2$ and cylindrical coordinates (ρ,θ,z) with

$$\rho=\sqrt{x_1^2+x_2^2}\,,\quad\rho\,e^{i\theta}=x_1+i\,x_2\quad\text{and}\quad z=x_3$$

in dimension $d = 3$. In this paper we shall consider *Aharonov-Bohm magnetic fields* defined by a magnetic potential such that

$$\mathbf{A} = \frac{a}{r^2}\,(x_2, -x_1) = \frac{a}{r^2}\,\mathbf{e}_\theta$$

in dimension $d = 2$, where a is a real constant and $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ with $\mathbf{e}_r = \frac{x}{r}$ denotes the orthogonal basis associated with our polar coordinates. With similar notations, we shall also consider the magnetic potential

$$\mathbf{A} = \frac{a}{\rho^2}\,(x_2, -x_1, 0)$$

in dimension $d = 3$. In both cases, $\mathbf{A}$ is singular at $x_1 = x_2 = 0$ and the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ is a measure supported in the set $x_1 = x_2 = 0$. The magnetic gradient and the magnetic Laplacian are explicitly given in our systems of coordinates by

$$\nabla_{\mathbf{A}} = \left(\frac{\partial}{\partial r}, \frac{1}{r}\left(\frac{\partial}{\partial \theta} - i\,a\right)\right), \quad -\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\left(\frac{\partial}{\partial \theta} - i\,a\right)^2$$

in dimension $d = 2$, and

$$\nabla_{\mathbf{A}} = \left(\frac{\partial}{\partial \rho}, \frac{1}{\rho}\left(\frac{\partial}{\partial \theta} - i\,a\right), \frac{\partial}{\partial z}\right), \quad -\Delta_{\mathbf{A}} = -\frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2}\left(\frac{\partial}{\partial \theta} - i\,a\right)^2 - \frac{\partial^2}{\partial z^2}$$

in dimension $d = 3$. Adapted definitions will be given later in the case of the circle, the sphere and the torus.

The primary goal of this paper is to prove new interpolation inequalities of Gagliardo-Nirenberg and Caffarelli-Kohn-Nirenberg or Hardy-Sobolev type in presence of an Aharonov-Bohm magnetic potential. One of the key features is that the corresponding magnetic Laplacian has the same scaling properties as the non-magnetic Laplacian and that the spectrum is explicit. On the other hand, our inequalities involve L^p norms in *superquadratic* (case $p > 2$) and *subquadratic* (case $p < 2$) regimes. The dual counterpart of these estimates are estimates of Keller-Lieb-Thirring type, which allow us to give a lower bound of the ground state energy of Schrödinger operators involving an Aharonov-Bohm magnetic potential and a potential with an appropriate L^q regularity. Such spectral estimates differ from semi-classical estimates. As a special case, we are also interested in various Hardy inequalities corresponding to a singularity at $x = 0$ that goes like $|x|^{-2}$ but may have some anisotropy.

In the absence of a magnetic potential, a typical *Gagliardo-Nirenberg inequality* asserts that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathcal{X})} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathcal{X})} \ge \mathcal{C}_{\mathrm{GN}}\,\|u\|^2_{\mathrm{L}^p(\mathcal{X})}\,,$$

where $\mathcal{X}$ denotes either the Euclidean space $\mathbb{R}^d$ or a manifold, and $\mathcal{C}_{\mathrm{GN}}$ is a positive, finite constant. By default, we shall always consider the optimal constant. When adding a magnetic potential, similar inequalities hold true as a consequence of the *diamagnetic inequality.* We shall speak of *Hardy-Sobolev* inequalities when a term $\int_{\mathcal{X}} |x|^{-2}\,|u|^2\,dx$ is subtracted from the kinetic energy and of *Caffarelli-Kohn-Nirenberg* inequalities when various pure power weights are taken into account. Proving the inequalities is a rather straightforward task but we are interested in more detailed issues: how do the constants depend on the parameters? Can we characterize the optimal constants and eventually compute them? Are there optimal functions and can we compute them?

A central issue is the question of *symmetry* and *symmetry breaking* of the optimal functions: are the optimal functions radially symmetric when $d = 2$ or axially symmetric when $d = 3$ for low magnetic fields? Can we estimate the range of the fields for which there is symmetry? This is a difficult question, but a linear instability analysis shows that symmetry breaking occurs for large magnetic fields in $\mathbb{R}^2$ as was recently proved in [5]. A first result of symmetry with explicit (and actually optimal range) has been established in [11] in the case $d = 1$ and our main goal is to characterize various cases in higher dimensions in which we are able to give a quantitative answer.

Our results are mostly devoted to the dimensions $d = 1$ on circles, $d = 2$ (Euclidean space, two-dimensional torus and two-dimensional sphere) and $d = 3$ in the axisymmetric case compatible with the Aharonov-Bohm magnetic potential. It is a remarkable fact that, in presence of a magnetic potential, a Hardy inequality can be established in the two-dimensional case (see [11,17,19,21]). Here we try to systematically derive the Hardy inequality from our Keller-Lieb-Thirring estimates. From a more mathematical viewpoint, the overall question is to determine the functional spaces which are adapted to magnetic Schrödinger operators involving an Aharonov-Bohm magnetic potential. In that sense, this is the continuation of [10] in the case of the whole space, for general and constant magnetic fields.

This paper is organized as follows. Section 2 is devoted to some preliminary results and also collects some previous results that we need later. Section 3 is devoted to subquadratic magnetic interpolation inequalities on the circle and on the torus, with some applications to Hardy inequalities in dimensions $d = 2$ and $d = 3$. In Section 4 we consider interpolation inequalities in $\mathbb{R}^2$ in the presence of an Aharonov-Bohm magnetic field. We conclude in Section 5 by further considerations on Hardy inequalities on $\mathbb{R}^3$ in the axisymmetric case. For more details, we refer to the *table of contents* at the end of the paper.

2. General set-up and preliminary results

2.1. *Non-magnetic interpolation inequalities on* $\mathbb{S}^d$

On the sphere $\mathbb{S}^d$, we consider the uniform probability measure $d\sigma$, which is the measure induced by the Lebesgue measure in $\mathbb{R}^{d+1}$, duly normalized and denote by $\|\cdot\|_{\mathrm{L}^q(\mathbb{S}^d)}$ the corresponding L^q norm.

2.1.1. *Interpolation inequalities without weights*

The interpolation inequalities

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \tag{2.1}$$

hold for any $p \in [1,2) \cup (2,+\infty)$ if $d=1$ and $d=2$, and for any $p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, where $2^* := 2\,d/(d-2)$ is the Sobolev critical exponent. See [3,4] for $p>2$ and [2] if $d=1$ or $d \geq 2$ and $p \leq \big(2\,d^2+1\big)/(d-1)^2$.

If $p>2$, we know from [9] that there exists a concave monotone increasing function $\lambda \mapsto \mu_{0,p}(\lambda)$ on $(0,+\infty)$ such that $\mu_{0,p}(\lambda)$ is the optimal constant in the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu_{0,p}(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^d) \tag{2.2}$$

and that $\mu_{0,p}(\lambda) = \lambda$ if and only if $\lambda \leq d/(p-2)$. In this range, equality is achieved if and only if u is a constant function: this is a *symmetry* range. On the opposite, if $\lambda > d/(p-2)$, the optimal function is not constant and we shall say that there is *symmetry breaking.*

The case $1 \leq p < 2$ is similar: there exists a concave monotone increasing function $\mu \mapsto \lambda_{0,p}(\mu)$ on $(0,+\infty)$ such that $\lambda_{0,p}(\mu)$ is the optimal constant in the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \geq \lambda_{0,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^d) \tag{2.3}$$

and that $\lambda_{0,p}(\mu) = \mu$ if and only if $\mu \leq d/(2-p)$. In this symmetry range, constants are the optimal functions, while there is symmetry breaking if $\mu > d/(2-p)$: optimal functions are non-constant.

In the symmetry range, positive constants are actually the only *positive* solutions of the Euler-Lagrange equation

$$-\,\varepsilon\,\Delta u + \lambda\,u = u^{p-1}$$

where $\varepsilon = \pm 1$ is the sign of $(p-2)$, while there are multiple solutions in the symmetry breaking range. The limit case $p=2$ can be obtained by taking the limit as $p \to 2$ and the corresponding inequality is the logarithmic Sobolev inequality. Much more is known and we refer to [9] for further details.

2.1.2. *A weighted Poincaré inequality for the ultra-spherical operator*

Using cylindrical coordinates $(z,\omega) \in [-1,1] \times \mathbb{S}^{d-1}$, we can rewrite the Laplace-Beltrami operator on $\mathbb{S}^d$ as

$$\Delta = \mathcal{L}_d + \frac{1}{1-z^2}\,\Delta_\omega \quad \text{with} \quad \mathcal{L}_d\,u := \big(1-z^2\big)\,u'' - d\,z\,u'$$

where Δ_ω denotes the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ and $\mathcal{L}_d$ is the *ultra-spherical operator*. In other words, $\mathcal{L}_d$ is the Laplace-Beltrami operator on $\mathbb{S}^d$ restricted to functions which depend only on z. The operator $\mathcal{L}_d$ has a basis of eigenfunctions $G_{\ell,d}$, the Gegenbauer polynomials, associated with the eigenvalues $\ell\,(\ell+d-1)$ for any $\ell\in\mathbb N$ (see [23]). Here d is not necessarily an integer.

Let us consider the eigenvalue problem

$$-\,\mathcal L_2\,f+\frac{4\,\mathsf a^2}{1-z^2}\,f=\lambda\,f\,. \tag{2.4}$$

By changing the unknown function according to $f(z)=\big(1-z^2\big)^{\mathsf a}\,g(z)$, we obtain that g solves

$$-\,\mathcal L_{2\,(2\,\mathsf a+1)}\,g+2\,\mathsf a\,(1+2\,\mathsf a)\,g=\lambda\,g$$

which determines the eigenvalues $\lambda=\lambda_{\ell,\mathsf a}$ given by

$$\lambda_{\ell,\mathsf a}=\ell\,\big(\ell+2\,(2\,\mathsf a+1)-1\big)+2\,\mathsf a\,(1+2\,\mathsf a)=(\ell+2\,\mathsf a)\,(\ell+2\,\mathsf a+1)\,,\quad \ell\in\mathbb N\,. \tag{2.5}$$

We shall denote by $g_{\ell,\mathsf a}(z)=G_{\ell,2\,(2\,\mathsf a+1)}(z)$ the associated eigenfunctions and define $f_{\ell,\mathsf a}(z):=\big(1-z^2\big)^{\mathsf a}\,g_{\ell,\mathsf a}(z)$. By considering the lowest positive eigenvalue, we obtain a *weighted Poincaré inequality.*

Lemma 2.1. *For any* $\mathsf a\in\mathbb R$ *and any function* $f\in\mathrm H^1_0[-1,1]$*, we have*

$$\int_{-1}^1\left(\big(1-z^2\big)\,\big|f'(z)\big|^2+\frac{4\,\mathsf a^2}{1-z^2}\,|f(z)|^2\right)dz\ge\lambda_{1,\mathsf a}\int_{-1}^1\big|f(z)-\bar f(z)\big|^2\,dz\,,$$

where

$$\bar f(z)=\big(1-z^2\big)^{\mathsf a}\,\frac{\int_{-1}^1 f(z)\,\big(1-z^2\big)^{\mathsf a}\,dz}{\int_{-1}^1\big(1-z^2\big)^{2\,\mathsf a}\,dz}\,.$$

Equality is achieved by a function f *if and only if* f *is proportional to* $f_{1,\mathsf a}(z)=z\,\big(1-z^2\big)^{\mathsf a}$.

Notice for consistency that, if $f(z)=\big(1-z^2\big)^{\mathsf a}\,g(z)$, then

$$\int_{-1}^1\left(\big(1-z^2\big)\,\big|f'(z)\big|^2+\frac{4\,\mathsf a^2}{1-z^2}\,|f(z)|^2\right)dz$$
$$=\int_{-1}^1\left(\big(1-z^2\big)\,\big|g'(z)\big|^2+2\,\mathsf a\,(1+2\,\mathsf a)\,|g(z)|^2\right)\big(1-z^2\big)^{2\,\mathsf a}\,dz\,,$$

where the right-hand side is the Dirichlet form associated with the operator

$$-\,\mathcal L_{2\,(2\,\mathsf a+1)}+2\,\mathsf a\,(1+2\,\mathsf a)\,.$$

2.2. *Magnetic rings: superquadratic inequalities on* $\mathbb S^1$

In this section, we review a series of results which have been obtained in [11] in the superquadratic case $p>2$, in preparation for an extension to the subquadratic case $p\in[1,2)$ that will be studied in Section 3.

2.2.1. *Magnetic interpolation inequalities and consequences*

Let us consider the superquadratic case $p > 2$ in dimension $d = 1$. We recall that $d\sigma = (2\pi)^{-1}\,d\theta$ where $\theta \in [0, 2\pi) \approx \mathbb{S}^1$. As in [11] we consider the space $\mathrm{H}^1(\mathbb{S}^1)$ of the 2π-periodic functions $u \in C^{0,1/2}(\mathbb{S}^1)$, such that $u' \in \mathrm{L}^2(\mathbb{S}^1)$. Inequality (2.2) can be rewritten as

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \lambda\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^1) \tag{2.6}$$

for any $\lambda \in (0, 1/(p-2)]$. We also have the inequality

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} \ge \frac{1}{4}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^1)\,, \tag{2.7}$$

according to [16], with the convention that $\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} = 0$ if u^{-2} is not integrable and, as a special case, if u changes sign. Notice that inequality (2.7) is formally the case $p = 2\,d/(d-2)$ and $\lambda = d/(p-2)$ of (2.2) when $d = 1$ (see [11, Appendix A]).

In [11], it was shown that the inequality (for complex valued functions)

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\lambda)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{S}^1, \mathbb{C}) \tag{2.8}$$

is equivalent, after eliminating the phase, to the inequality

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^1)\,.$$

The equivalence is relatively easy to prove if ψ does not vanish, but some care is required otherwise: see [11] for details. Here we denote by $\mu_{a,p}(\lambda)$ the optimal constant in (2.8). Using (2.8) and then (2.6), we obtain that

$$\begin{aligned}
&\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \\
&\quad = (1 - 4\,a^2)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + 4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \tfrac14\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right) \\
&\qquad\qquad \ge (1 - 4\,a^2)\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \tfrac{a^2+\lambda}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right) \ge (a^2 + \lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}
\end{aligned}$$

under the condition $(a^2 + \lambda)/(1 - 4\,a^2) \le 1/(p-2)$, which provides an estimate of $\mu_{a,p}(\lambda)$. This estimate turns out to be optimal.

Proposition 2.1 ([11]). *Let $p > 2$, $a \in [0, 1/2]$, and $\lambda > -\,a^2$.*

(i) If $a^2\,(p+2) + \lambda\,(p-2) \le 1$, then $\mu_{a,p}(\lambda) = a^2 + \lambda$ and equality in (2.8) is achieved only by the constants.

(ii) If $a^2\,(p+2) + \lambda\,(p-2) > 1$, then $\mu_{a,p}(\lambda) < a^2 + \lambda$ and equality in (2.8) is not achieved by the constants.

The condition $a \in [0, 1/2]$ is not a restriction. First, replacing ψ by $e^{iks}\,\psi(s)$ for any $k \in \mathbb{Z}$ shows that $\mu_{a+k,p}(\mu) = \mu_{a,p}(\mu)$ so that we can assume that $a \in [0, 1]$. Then by considering $\chi(s) = e^{-is}\,\overline{\psi}(s)$, we find that

$$|\psi' - i\,a\,\psi|^2 = |\chi' + i\,(1 + a)\,\chi|^2\,,$$

hence $\mu_{a,p}(\mu) = \mu_{1+a,p}(\mu)$.

2.2.2. *Magnetic Hardy inequalities on* $\mathbb{S}^1$ *and* $\mathbb{R}^2$

As in [11], we can draw an easy consequence of Proposition 2.1 on a Hardy-type inequality. By Hölder's inequality applied with $q = p/(p-2)$, we have

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu^{-1}\int_{\mathbb{S}^1}\phi\,|\psi|^2\,d\theta \ge \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu^{-1}\,\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\,.$$

Using (2.8) with $\lambda = 0$ and μ such that $\mu^{-1}\,\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)} = \mu_{a,p}(0)$, we know that the right-hand side is nonnegative. See [11] for more details. Altogether we obtain the following *magnetic Hardy inequality on* $\mathbb{S}^1$*: for any* $a \in \mathbb{R}$*, any* $p > 2$ *and* $q = p/(p-2)$*, if* ϕ *is a non-trivial potential in* $\mathrm{L}^q(\mathbb{S}^1)$*, then*

$$\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}}\int_{\mathbb{S}^1}\phi\,|\psi|^2\,d\sigma \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^1)\,. \tag{2.9}$$

This is a special case of the more general interpolation inequality

$$\begin{aligned}\|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \int_{\mathbb{S}^1}\phi\,|\psi|^2\,d\theta &\ge \|\psi' - i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\\ &\ge -\,\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\end{aligned} \tag{2.10}$$

with $\mu = \|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}$, where we denote by $\lambda_{a,p}(\mu)$ the inverse function of $\lambda \mapsto \mu_{a,p}(\lambda)$, as defined in Proposition 2.1. See [10] for details.

The standard non-magnetic Hardy inequality on $\mathbb{R}^d$, *i.e.*,

$$\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx \ge \frac{1}{4}\,(d-2)^2\int_{\mathbb{R}^d}\frac{|\psi|^2}{|x|^2}\,dx \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)\,,$$

degenerates if $d = 2$, but this degeneracy is lifted in the presence of a Aharonov-Bohm magnetic field. According to [21], we have

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \min_{k\in\mathbb{Z}}\,(a-k)^2\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)\,.$$

It is natural to ask whether an improvement can be obtained if the singularity $|x|^{-2}$ is replaced by a weight which has an angular dependence. Using polar coordinates $x \approx (r,\theta)$ and interpolation inequalities of [9], the inequality

$$\int_{\mathbb{R}^d}|\nabla\psi|^2\,dx \ge \frac{(d-2)^2}{4\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^{d-1})}}\int_{\mathbb{R}^d}\frac{\varphi(\theta)}{|x|^2}\,|\psi|^2\,dx \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)$$

was proved in [19], under the condition that $q \ge 1 + \frac{1}{2}\,(d-2)^2/(d-1)$, again with normalized measure on $\mathbb{S}^{d-1}$. Magnetic and non-radial improvements have been combined in [11]. Let us give a statement in preparation for similar extensions to the case of dimension $d = 3$.

Corollary 2.1 ([11]). *Let* $p > 2$*,* $a \in [0, 1/2]$*,* $q = p/(p-2)$ *and assume that* φ *is a non-negative function in* $\mathrm{L}^q(\mathbb{S}^1)$*. With the above notations, the inequality*

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \tau\int_{\mathbb{R}^2}\frac{\varphi(\theta)}{|x|^2}\,|\psi|^2\,dx \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)$$

holds with a constant $\tau > 0$ which is the unique solution of the equation

$$\lambda_{a,p}\big(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\big) = 0\,.$$

Moreover, $\tau = a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $a^2 \le 1/(p+2)$.

2.3. *Magnetic interpolation inequalities on* $\mathbb{S}^2$

2.3.1. *A magnetic ground state estimate*

Let us consider the magnetic Laplacian on $\mathbb{S}^2$ and the associated Dirichlet form $\int_{\mathbb{S}^1} |\nabla_{\mathbf{A}}\, u|^2\, d\theta$ where $d\sigma$ is the uniform probability measure on $\mathbb{S}^2$. Using cylindrical coordinates $(\theta, z) \in [0, 2\pi) \times [-1, 1]$, we can write that $d\sigma = \frac{1}{4\,\pi}\, dz\, d\theta$ and assume that the magnetic gradient takes the form

$$\nabla_{\mathbf{A}}\, u = \begin{pmatrix} \sqrt{1-z^2}\,\frac{\partial u}{\partial z} \\ \frac{1}{\sqrt{1-z^2}}\,\big(\frac{\partial u}{\partial \theta} - i\,a\,u\big) \end{pmatrix}$$

where $a > 0$ is a magnetic flux, so that

$$|\nabla_{\mathbf{A}}\, u|^2 = \big(1-z^2\big)\left|\frac{\partial u}{\partial z}\right|^2 + \frac{1}{1-z^2}\left|\frac{\partial u}{\partial \theta} - i\,a\,u\right|^2 .$$

Lemma 2.2. *Assume that $a \in \mathbb{R}$. With the above notations, we have*

$$\int_{\mathbb{S}^2} |\nabla_{\mathbf{A}}\, u|^2\, d\sigma \ge \Lambda_a \int_{\mathbb{S}^2} |u|^2\, d\sigma \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^2)$$

with optimal constant

$$\Lambda_a = \min_{k\in\mathbb{Z}} |k-a|\,\big(|k-a|+1\big)\,. \tag{2.11}$$

Notice that $\Lambda_a \le \Lambda_{1/2} = 3/4$.

Proof. We can write u using a Fourier decomposition

$$u(z,\theta) = \sum_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{Z}} u_{k,\ell}(z)\, e^{i\,k\,\theta}$$

and observe that

$$|\nabla_{\mathbf{A}}\, u|^2 = \sum_{\ell\in\mathbb{N}}\sum_{k\in\mathbb{Z}} \left(\big(1-z^2\big)\,\big|u_{k,\ell}'(z)\big|^2 + \frac{(k-a)^2}{1-z^2}\,\big|u_{k,\ell}(z)\big|^2\right)$$

where

$$u_{k,\ell}(z) = f_{\ell,|k-a|/2}(z)\int_{\mathbb{S}^2} u(z,\theta)\,\frac{2\,f_{\ell,|k-a|/2}(z)\,e^{-i\,k\,\theta}}{\int_{-1}^1 \big(1-z^2\big)^{|k-a|}\,dz}\,d\sigma$$

and $f_{\ell,|k-a|/2}$ is an eigenfunction of (2.4) with eigenvalue $\lambda = \lambda_{\ell,\mathsf{a}}$ such that $2\,\mathsf{a} = |k-a|$. Using (2.5), we conclude that the spectrum of $-\Delta_{\mathbf{A}}$ is given by

$$\big(\ell + |k-a|\big)\,\big(\ell + |k-a| + 1\big)\,, \quad k \in \mathbb{Z}\,,\ \ell \in \mathbb{N}\,.$$

□

2.3.2. *Superquadratic interpolation inequalities and consequences*

Proposition 2.2. *Let $a \in \mathbb{R}$ and $p > 2$. There exists a concave monotone increasing function $\lambda \mapsto \mu_{a,p}(\lambda)$ on $(-\Lambda_a, +\infty)$ such that $\mu_{a,p}(\lambda)$ is the optimal constant in the inequality*

$$\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \lambda\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \ge \mu_{a,p}(\lambda)\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^2)} \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^2).$$

Furthermore, $\mu_{a,p}(\lambda) \ge 2\,(\lambda + \Lambda_a)/\big(2 + (p-2)\,\Lambda_a\big)$ and $\lim_{\lambda \to -\Lambda_a} \mu_{a,p}(\lambda) = 0$, with Λ_a given by (2.11).

Proof. The proof is adapted from [10, Proposition 3.1]. For an arbitrary $t \in (0,1)$, we can write that

$$\begin{aligned} \|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \lambda\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} &\ge t \left(\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} - \Lambda_a\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \right) \\ &\quad + (1-t) \left(\|\nabla |u|\|^2_{\mathrm{L}^2(\mathbb{S}^2)} + \frac{\lambda + t\,\Lambda_a}{1-t}\, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \right) \\ &\ge (1-t)\, \mu_{0,p} \left(\frac{\lambda + t\,\Lambda_a}{1-t} \right) \|u\|^2_{\mathrm{L}^p(\mathbb{S}^2)}\,, \end{aligned}$$

as a consequence of Lemma 2.2 and of the *diamagnetic inequality* (see e.g. [22, Theorem 7.21])

$$\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \ge \|\nabla |u|\|^2_{\mathrm{L}^2(\mathbb{S}^2)}\,.$$

If $\lambda < 2/(p-2)$, the estimate is obtained by choosing t such that

$$\frac{\lambda + t\,\Lambda_a}{1-t} = \frac{2}{p-2}$$

and recalling that $\mu_{0,p}\,(2/(p-2)) = 2/(p-2)$. The limit as $\lambda \to -\Lambda_a$ is obtained by taking the ground state of $-\Delta_{\mathbf{A}}$ on $\mathrm{H}^1(\mathbb{S}^2)$ as test function. □

With the same method as for the proof of (2.9), we can deduce a Hardy-type inequality.

Corollary 2.2. *Let $a \in \mathbb{R}$, $p > 2$ and $q = p/(p-2)$. If ϕ is a non-trivial potential in $\mathrm{L}^q(\mathbb{S}^2)$, then*

$$\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{S}^2)} \ge \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^2)}} \int_{\mathbb{S}^2} \phi\, |u|^2\, d\sigma \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{S}^2)\,.$$

3. Subquadratic magnetic interpolation inequalities

This section is devoted to results on inequalities involving L^p norms with $1 < p < 2$, which are generically known as *subquadratic inequalities*.

3.1. *Magnetic rings: subquadratic interpolation inequalities on* $\mathbb{S}^1$

We extend to the range $1 < p < 2$ the results of [11] on (2.8) and (2.9) (see summary in Section 2.2.1).

3.1.1. *Statement of the inequality*

As a special case of (2.1) corresponding to $d=1$, we have the non-magnetic interpolation inequality

$$(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\ge\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^1)\tag{3.1}$$

for any $p\in[1,2)$. Our first result is the magnetic counterpart of this inequality.

Lemma 3.1. *Let $a\in\mathbb{R}$ and $p\in[1,2)$. Then there exists a concave monotone increasing function $\mu\mapsto\lambda_{a,p}(\mu)$ on $\mathbb{R}^+$ such that*

$$\|\psi'-i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\ge\lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\quad\forall\,\psi\in\mathrm{H}^1(\mathbb{S}^1,\mathbb{C})\,.\tag{3.2}$$

Here we denote by $\lambda_{a,p}(\mu)$ the optimal constant in (3.1).

Proof. The existence of $\lambda_{a,p}(\mu)$ is a consequence of (3.1) and of the *diamagnetic inequality*: let $\rho=|\psi|$ and ϕ be such that $\psi=\rho(\theta)\,\exp\big(i\,\phi(\theta)\big)$. Since

$$|\psi'-i\,a\,\psi|^2=|\rho'|^2+|\phi'-a|^2\,\rho^2\ge|\rho'|^2\,,$$

we have that $\|\psi'-i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\ge\|\,|\psi|'\,\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$. The concavity of $\mu\mapsto\lambda_{a,p}(\mu)$ is a consequence of the definition of $\lambda_{a,p}(\mu)$ as the optimal constant, *i.e.*, the infimum on $\mathrm{H}^1(\mathbb{S}^1)\ni\psi$ of an affine function of μ. □

3.1.2. *Existence of an optimal function*

Lemma 3.2. *For all $a\in[0,1/2]$, $p\in[1,2)$ and $\mu\ge-a^2$, equality in (3.2) is achieved by at least one function in $\mathrm{H}^1(\mathbb{S}^1)$.*

Proof. We consider a minimizing sequence $\{\psi_n\}$ for

$$\lambda_{a,p}(\mu)=\inf\Big\{\|\psi'-i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\mu\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\,:\,\psi\in\mathrm{H}^1(\mathbb{S}^1)\,,\ \|\psi\|_{\mathrm{L}^2(\mathbb{S}^1)}=1\Big\}\,.$$

By the diamagnetic inequality we know that the sequence $(\psi_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{H}^1(\mathbb{S}^1)$. By the compact Sobolev embeddings, this sequence is relatively compact in $\mathrm{L}^p(\mathbb{S}^1)$ and in $\mathrm{L}^2(\mathbb{S}^1)$. The map $\psi\mapsto\|\psi'-i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is lower semicontinuous by Fatou's lemma, which proves the claim. □

3.1.3. *A non-vanishing property*

Lemma 3.3. *Asssume that $a\in(0,1/2)$, $p\in[1,2)$ and $\mu\ge-a^2$. If $\psi\in\mathrm{H}^1(\mathbb{S}^1)$ is an optimal function for (3.2) with $\|\psi\|_{\mathrm{L}^p(\mathbb{S}^1)}=1$, then $\psi(s)\neq0$ for any $s\in\mathbb{S}^1$.*

Proof. The proof goes as in [11]. Let us decompose $v(s)=\psi(s)\,e^{ias}$ as a real and an imaginary part, respectively v_1 and v_2, which both solve the same Euler-Lagrange equation

$$-\,v_j''-\,\mu\,\big(v_1^2+v_2^2\big)^{\frac{p}{2}-1}\,v_j=\lambda_{a,p}(\mu)\,v_j\,,\quad j=1\,,\,2\,.$$

Notice that $v \in C^{0,1/2}(\mathbb{S}^1)$ and the nonlinear term is continuous, hence v is smooth. The Wronskian $w = (v_1\, v_2' - v_1'\, v_2)$ is constant. If both v_1 and v_2 vanish at the same point, then w vanishes identically, which means that v_1 and v_2 are proportional. With $a \in (0, 1/2)$, ψ is not 2π-periodic, a contradiction. □

3.1.4. *A reduction to a scalar minimization problem*

We refer to Section 2.1.1 if $a = 0$ and assume in the proofs that $a > 0$. The main steps of the reduction are similar to the case $p > 2$ of [11]. We repeat the key points for completeness. Let us define

$$\mathcal{Q}_{a,p,\mu}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\, \|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\,.$$

Notice that if $u \in \mathrm{H}^1(\mathbb{S}^1)$ is such that $u(s_0) = 0$ for some $s_0 \in (-\pi, \pi]$, then

$$|u(s)|^2 = \left(\int_{s_0}^s u'\, ds\right)^2 \le \sqrt{2\pi}\, \|u'\|_{\mathrm{L}^2(\mathbb{S}^1)}\, \sqrt{|s - s_0|}$$

and u^{-2} is not integrable. In this case, as mentioned earlier, we adopt the convention that

$$\mathcal{Q}_{a,p,\mu}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\,. \tag{3.3}$$

Lemma 3.4. *For any $a \in [0, 1/2]$, $p \in [1, 2)$, $\mu > -\,a^2$,*

$$\lambda_{a,p}(\mu) = \min_{u \in \mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\mu}[u]\,.$$

Proof. We consider functions on $\mathbb{S}^1$ as 2π-periodic functions on $\mathbb{R}$. If $\psi \in \mathrm{H}^1(\mathbb{S}^1)$, then $v(s) = \psi(s)\, e^{ias}$ satisfies the condition

$$v(s + 2\pi) = e^{2i\pi a}\, v(s) \quad \forall\, s \in \mathbb{R} \tag{3.4}$$

and

$$\lambda_{a,p}(\mu) = \min \frac{\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|v\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|v\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

where the minimization is taken on the set of the functions $v \in C^{0,1/2}(\mathbb{R})$ such that $v' \in \mathrm{L}^2(-\pi, \pi)$ and (3.4) holds.

With $v = u\, e^{i\phi}$ written in polar form, the boundary condition becomes

$$u(\pi) = u(-\pi)\,, \quad \phi(\pi) = 2\pi\,(a + k) + \phi(-\pi) \tag{3.5}$$

for some $k \in \mathbb{Z}$, and $\|v'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\, \phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ so that

$$\lambda_{a,p}(\mu) = \min \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \|u\, \phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}$$

where the minimization is taken on the set of the functions $(u,\phi)\in C(\mathbb{R})^2$ such that u', $u\,\phi'\in \mathrm{L}^2(\mathbb{S}^1)$ and (3.5) holds.

Up to a multiplication of u by a constant so that $\|u\|_{\mathrm{L}^p(\mathbb{S}^1)}=1$, the Euler-Lagrange equations are

$$-\,u''+|\phi'|^2\,u+\mu\,|u|^{p-2}\,u=\lambda_{a,p}(\mu)\,u\quad\text{and}\quad(\phi'\,u^2)'=0\,.$$

If $a\in(0,1/2)$, by integrating the second equation and using Lemma 3.3, we find a constant L such that $\phi'=L/u^2$. Taking (3.5) into account, we deduce from

$$L\int_{-\pi}^{\pi}\frac{ds}{u^2}=\int_{-\pi}^{\pi}\phi'\,ds=2\pi\,(a+k)$$

that

$$\|u\,\phi'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}=L^2\int_{-\pi}^{\pi}\frac{ds}{u^2}=\frac{(a+k)^2}{\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}\,.$$

This establishes that

$$\lambda_{a,p}(\mu)=\min_{u,\,k}\mathcal{Q}_{a+k,p,\mu}[u]$$

where the minimization is taken on all $k\in\mathbb{Z}$ and on all functions $u\in\mathrm{H}^1(\mathbb{S}^1)$. Because of the restriction $a\in(0,1/2)$, the minimum is achieved by $k=0$.

The case $a=1/2$ is a limit case that can be handled as in [11, Theorem III.7]. In this case the result holds also true, with the minimizer being in $\mathrm{H}^1_0(\mathbb{S}^1)\setminus\{0\}$, and with the convention defined in (3.3) for the expression of $Q_{a,p,\mu}[u]$ when u vanishes in $\mathbb{S}^1$. □

3.1.5. *A rigidity result*

If $a\in(0,1/2)$, as in [11], the study of (3.2) is reduced to the study of the inequality

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)}+\mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\ge\lambda_{a,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^1)\qquad(3.6)$$

where u is now a real valued function. Necessary adaptations to the trivial case $a=0$ and to the limit case $a=1/2$ are straightforward and left to the reader. The lemma below is the equivalent of Proposition 2.1 for the case $1<p<2$.

Theorem 3.1. *Let $p\in(1,2)$, $a\in(0,1/2)$, and $\mu>0$.*

(i) If $\mu\,(2-p)+4\,a^2\le1$, then $\lambda_{a,p}(\mu)=a^2+\mu$ and equality in (3.6) is achieved only by the constants.

(ii) If $\mu\,(2-p)+4\,a^2>1$, then $\lambda_{a,p}(\mu)<a^2+\mu$ and equality in (3.6) is not achieved by the constants.

Proof. In case (i) we can write

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$
$$= (1-4\,a^2)\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{\mu}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}\right)$$
$$+ 4\,a^2\left(\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{4}\,\|u^{-1}\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right),$$

deduce from (3.1) that

$$\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{\mu}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)} \ge \frac{\mu}{1-4\,a^2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

if $\mu/(1-4\,a^2) \le 1/(2-p)$ and conclude using (2.7).

In case (ii), let us consider the test function $u_\varepsilon := 1 + \varepsilon\,w_1$, where w_1 is the eigenfunction corresponding to the first non-zero eigenvalue of $-\,d^2/ds^2$ on $\mathrm{H}^1(\mathbb{S}^1)$, with periodic boundary conditions, namely, $w_1(s) = \cos s$ and $\lambda_1 = 1$. A Taylor expansion shows that

$$\mathcal{Q}_{a,p,\mu}[u_\varepsilon] = \big(1 + a^2 - \,\mu\,(2-p)\big)\,\varepsilon^2 + o(\varepsilon^2)\,,$$

which proves the result. Notice that the Taylor expansion is also valid if $a = 0$, so that $(p-2)$ is the optimal constant in (3.1), and also that a similar Taylor expansion holds in case of (2.7), which formally corresponds to $p = -\,2$. □

3.2. *Aharonov-Bohm magnetic interpolation inequalities on* $\mathbb{T}^2$

Let us consider the flat torus $\mathbb{T}^2 = \mathbb{S}^1\times\mathbb{S}^1 \approx [-\pi,\pi)\times[-\pi,\pi) \ni (x,y)$ with periodic boundary conditions in x and y. We denote by $d\sigma$ the uniform probability measure $d\sigma = dx\,dy/(4\pi^2)$ and consider the magnetic gradient

$$\nabla_{\mathbf{A}}\,\psi := \big(\psi_x, \psi_y - i\,a\,\psi\big)$$

and the magnetic kinetic energy

$$\|\nabla_{\mathbf{A}}\,u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} = \iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma = \iint_{\mathbb{T}^2} \big(|\psi_x|^2 + |\psi_y - \,i\,a\,\psi|^2\big)\,d\sigma\,.$$

3.2.1. *A magnetic ground state estimate*

Lemma 3.5. *Assume that* $a \in [0,1/2]$. *Then*

$$\iint_{\mathbb{T}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,d\sigma \ge a^2 \iint_{\mathbb{T}^2} |\psi|^2\,d\sigma \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2)\,.$$

Proof. We make a Fourier decomposition on the basis $(e^{i\,\ell\,x}\,e^{i\,k\,y})_{k,\ell\in\mathbb{Z}}$. We find that the lowest modes are given by

$$k=0\,,\ \ell=0\ :\ \lambda_{00} = a^2\,,$$
$$k=1\,,\ \ell=0\ :\ \lambda_{10} = (1-a)^2 \ge a^2 \ \text{ since } a\in[0,1/2],$$
$$k=0\,,\ \ell=1\ :\ \lambda_{01} = 1 + a^2\,.$$

Therefore, λ_{00} is the lowest mode. □

3.2.2. *The Bakry-Emery method applied to the 2-dimensional torus*

We consider the flow given by

$$\frac{\partial u}{\partial t} = \Delta u + (p-1)\,\frac{|\nabla u|^2}{u}$$

and observe that

$$\frac{d}{dt}\,\|u(t,\cdot)\|^2_{\mathrm{L}^p(\mathbb{T}^2)} = 0$$

on the one hand, and

$$-\frac{1}{2}\,\frac{d}{dt}\left(\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right)$$
$$= \|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + (p-1)\iint_{\mathbb{T}^2}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma - \lambda\,(2-p)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$$

on the other hand. Integrations by parts show that

$$\|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} = \|\mathrm{Hess}\,u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$$

and

$$\iint_{\mathbb{T}^2}\Delta u\,\frac{|\nabla u|^2}{u}\,d\sigma = -\,2\iint_{\mathbb{T}^2}\mathrm{Hess}\,u\,:\,\frac{\nabla u\otimes\nabla u}{u}\,d\sigma + \iint_{\mathbb{T}^2}\frac{|\nabla u|^4}{u^2}\,d\sigma\,.$$

Hence

$$-\frac{1}{2}\,\frac{d}{dt}\left(\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right)$$
$$= (2-p)\left(\|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right) + (p-1)\,\left\|\mathrm{Hess}\,u - \tfrac{\nabla u\otimes\nabla u}{u}\right\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,.$$

We know from the Poincaré inequality that

$$\|\Delta u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \ge \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\,,$$

with optimal constant 1, so we can conclude in the case $1 \le p < 2$ that $\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$ is monotone nonincreasing if $0 \le \lambda \le 1$. As a consequence, we have the following result.

Proposition 3.1. *For any $p \in [1,2)$, we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\,u \in \mathrm{H}^1(\mathbb{T}^2)\,.$$

3.2.3. *A tensorization result without magnetic potential*

A result better than Proposition 3.1 follows from a tensorization argument that can be found in [8,14].

Proposition 3.2. *For any $p \in [1,2)$, we have*

$$(2-p)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2)\,. \tag{3.7}$$

Moreover the factor $(2-p)$ is the optimal constant.

Proof. By taking on $\mathbb{T}^2$ a function depending only on $x \in \mathbb{S}^1$, it is clear that the constant in (3.7) cannot be improved. The proof of (3.7) can be done with the Bakry-Emery method applied to $\mathbb{S}^1$ and goes as follows.

Let us consider the flow given by

$$\frac{\partial u}{\partial t} = u'' + (p-1)\,\frac{|u'|^2}{u}$$

and observe that $\frac{d}{dt}\,\|u(t,\cdot)\|^2_{\mathrm{L}^p(\mathbb{S}^1)} = 0$ on the one hand, and

$$\begin{aligned} -\,\frac{1}{2}\,\frac{d}{dt}&\left(\|u'(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right) \\ &= \|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + (p-1)\int_{\mathbb{S}^1} u''\,\frac{|u'|^2}{u}\,d\sigma - \lambda\,(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \\ &= \|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \frac{1}{3}\,(p-1)\int_{\mathbb{S}^1} \frac{|u'|^4}{u^2}\,d\sigma - \lambda\,(2-p)\,\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \end{aligned}$$

on the other hand. Hence

$$-\,\frac{1}{2}\,\frac{d}{dt}\left(\|u'(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)} - \lambda\,\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^1)}\right) \le 0$$

if $\lambda\,(2-p) \le 1$, because of the Poincaré inequality $\|u''\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$. Up to a sign change of λ, this computation also holds if $p > 2$ or if $p = -2$, as noticed in [11], and it is straightforward to extend it to the limit case $p = 2$ corresponding to the logarithmic Sobolev inequality.

According to [8, Proposition 3.1] or [14, Theorem 2.1] and up to a straightforward adaptation to the periodic setting, the optimal constant for the inequality on $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ is the same as for the inequality on $\mathbb{S}^1$, provided $1 \le p < 2$. □

As a consequence of Proposition 3.2, we have the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\,\|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \ge \Lambda_{0,p}(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{T}^2)\,, \tag{3.8}$$

where $\mu \mapsto \Lambda_{0,p}(\mu)$ is a concave monotone increasing function on $(0,+\infty)$ such that $\Lambda_{0,p}(\mu) = \mu$ for any $\mu \in \big(0, 1/(2-p)\big)$.

3.2.4. *A magnetic interpolation inequality in the flat torus*

Now let us consider the generalization of (3.8) to the case $a \neq 0$.

Lemma 3.6. *Assume that $p \in [1,2)$ and $a \in [0,1/2]$. There exists a concave monotone increasing function $\mu \mapsto \Lambda_{a,p}(\mu)$ on $(0,+\infty)$ such that $\lim_{\mu\to 0_+} \Lambda_{a,p}(\mu) = a^2$ where $\Lambda_{a,p}(\mu)$ is the optimal constant in the inequality*

$$\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)} \geq \Lambda_{a,p}(\mu)\, \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \quad \forall\, u \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{T}^2)\,. \tag{3.9}$$

Moreover, we have that

$$\Lambda_{a,p}(\mu) \geq \mu + \big(1 - \mu\,(2-p)\big)\, a^2 \quad \textit{for any} \quad \mu \leq \frac{1}{2-p}\,.$$

Proof. For an arbitrary $t \in (0,1)$, we can write that

$$\begin{aligned}
&\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\, \|u\|^2_{\mathrm{L}^p(\mathbb{T}^2)}\\
&\geq t\left(\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} - a^2\, \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\right)\\
&\quad + (1-t)\left(\|\nabla|u|\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \frac{\mu}{1-t}\, \|u\|^p_{\mathrm{L}^2(\mathbb{T}^2)}\right) + t\, a^2\, \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}\\
&\geq \left[(1-t)\, \Lambda_{0,p}\left(\frac{\mu}{1-t}\right) + t\, a^2\right] \|u\|^2_{\mathrm{L}^2(\mathbb{T}^2)}
\end{aligned}$$

using the diamagnetic inequality $\|\nabla_{\mathbf{A}}\, u\|^2_{\mathrm{L}^2(\mathbb{T}^2)} \geq \|\nabla|u|\,\|^2_{\mathrm{L}^2(\mathbb{T}^2)}$. Inequality (3.8) applies with $\mu = 1/(2-p)$ and $t = 1 - \mu\,(2-p)$. □

3.2.5. *A symmetry result in the subquadratic regime*

As an application of the results on magnetic rings of Theorem 3.1, we can prove a symmetry result for the optimal functions in (3.9) in the case $p < 2$. Let $\Lambda_{a,p}(\mu)$ be the optimal constant in (3.9).

Theorem 3.2. *Assume that $a \in [0,1/2]$ and $p \in [1,2)$. Then*

$$\Lambda_{a,p}(\mu) = \lambda_{a,p}(\mu) \quad \textit{if} \quad \mu \leq \frac{1}{p-2}$$

and any optimal function for (3.2) is then constant w.r.t. x. Moreover, $\Lambda_{a,p}(\mu) = a^2 + \mu$ if and only if $\mu\,(2-p) + 4\,a^2 \leq 1$ and equality in (3.9) is then achieved only by the constants.

Proof. Let us use the notation $⨍ f\, dx := \frac{1}{2\pi}\int_{-\pi}^{\pi} f\, dx$ in order to denote a normalized integration with respect to the single variable x, where y is considered as a parameter. For almost every $x \in \mathbb{S}^1$ we can apply (3.2) to the function $\psi(x,\cdot)$ and get

$$\begin{aligned}
&\|\nabla_{\mathbf{A}}\, \psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\, \|\psi\|^2_{\mathrm{L}^p(\mathbb{T}^2)}\\
&\qquad \geq \|\partial_x \psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \lambda_{a,p}(\mu)\, \|\psi\|^2_{\mathrm{L}^2(\mathbb{T}^2)} + \mu\, \|\psi\|^2_{\mathrm{L}^p(\mathbb{T}^2)} - \mu ⨍ \left(⨍ |\psi|^p\, dy\right)^{\frac{2}{p}} dx
\end{aligned}$$

Let us define $u := |\psi|$, $v(x) := \left(\fint |u(x,y)|^p\,dy\right)^{1/p}$ and observe that

$$|v_x| = v^{1-p} \fint u^{p-1}\,u_x\,dy \le v^{1-p} \left(\fint u^p\,dy\right)^{\frac{p-1}{p}} \left(\fint |u_x|^2\,dy\right)^{\frac12} \left(\fint 1\,dy\right)^{\frac12-\frac{p-1}{p}}$$

by Hölder's inequality, under the condition $p \le 2$, that is,

$$|v_x|^2 \le \fint |u_x|^2\,dy \le \fint |\partial_x\psi|^2\,dy\,.$$

We conclude that if $\mu \le 1/(2-p)$,

$$\int_{\mathbb S^1} |v_x|^2\,d\sigma + \mu \left(\int_{\mathbb S^1} |v|^p\,d\sigma\right)^{2/p} - \mu \int_{\mathbb S^1} |v|^2\,d\sigma + \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm L^2(\mathbb T^2)} \ge \lambda_{a,p}(\mu)\,\|\psi\|^2_{\mathrm L^2(\mathbb T^2)}$$

using (3.1). The equality is achieved by functions v which are constant w.r.t. x and Theorem 3.1 applies. □

3.3. *Some consequences in the subquadratic regime*

In this section, we draw some consequences of our results on magnetic rings of Section 3.1. Here $d\sigma$ denotes the uniform probability measure on $\mathbb S^1$.

3.3.1. *Keller-Lieb-Thirring inequalities on the circle*

As in [10], by duality we obtain a spectral estimate.

Proposition 3.3. *Assume that $a \in [0,1/2]$ and $p \in [1,2)$. If ϕ is a nonnegative potential such that $\phi^{-1} \in \mathrm L^q(\mathbb S^1)$, then the lowest eigenvalue λ_1 of $-(\partial_y - i\,a)^2 + \phi$ is bounded from below according to*

$$\lambda_1 \ge \lambda_{a,p}\left(\|\phi^{-1}\|^{-1}_{\mathrm L^q(\mathbb S^1)}\right)$$

and equality is achieved by a constant potential ϕ if $\|\phi^{-1}\|^{-1}_{\mathrm L^q(\mathbb S^1)}\,(2-p) + 4\,a^2 \le 1$.

Proof. Using Hölder's inequality with exponents $2/(2-p)$ and $2/p$, we get that

$$\|\psi\|^2_{\mathrm L^p(\mathbb S^1)} = \left(\int_{\mathbb S^1} \phi^{-\frac p2}\,\left(\phi\,|\psi|^2\right)^{\frac p2}\,d\sigma\right)^{2/p} \le \|\phi^{-1}\|_{\mathrm L^q(\mathbb S^1)} \int_{\mathbb S^1} \phi\,|\psi|^2\,d\sigma$$

with $q = p/(2-p)$, and with $\mu = \|\phi^{-1}\|^{-1}_{\mathrm L^q(\mathbb S^1)}$,

$$\begin{aligned}\int_{\mathbb S^1} |\psi' - i\,a\,\psi|^2\,d\sigma + \int_{\mathbb S^1} \phi\,|\psi|^2\,d\sigma &\ge \int_{\mathbb S^1} |\psi' - i\,a\,\psi|^2\,d\sigma + \mu\,\|\psi\|^2_{\mathrm L^p(\mathbb S^1)} \\ &\ge \lambda_{a,p}(\mu) \int_{\mathbb S^1} |\psi|^2\,d\sigma\,. \qquad (3.10)\end{aligned}$$

If ϕ is constant, then there is equality in Hölder's inequality. □

The spectral estimate (3.10) is of a different nature than (2.10) because the potential energy and the magnetic kinetic energy have the same sign. By considering the threshold case $\mu\,(2-p)+4\,a^2=1$, we obtain an interesting estimate.

Corollary 3.1. *Let $a\in[0,1/2]$, $p\in(1,2)$ and $q=p/(2-p)$. If ϕ is a nonnegative potential such that $\phi^{-1}\in\mathrm L^q(\mathbb S^1)$, then*

$$\int_{\mathbb S^1}|\psi'-i\,a\,\psi|^2\,d\sigma+\frac{1-4\,a^2}{2-p}\,\|\phi^{-1}\|_{\mathrm L^q(\mathbb S^1)}\int_{\mathbb S^1}\phi\,|\psi|^2\,d\sigma \geq\left(\frac{1-4\,a^2}{2-p}+a^2\right)\|\psi\|^2_{\mathrm L^2(\mathbb S^1)}\quad\forall\,\psi\in\mathrm H^1(\mathbb S^1)\,.$$

3.3.2. *Magnetic Hardy-type inequalities in dimensions two and three*

Let us denote by $\theta\in[-\pi,\pi)$ the angular coordinate associated with $x\in\mathbb R^2$. As in [11], we can deduce a Hardy-type inequality for Aharonov-Bohm magnetic potentials in dimension $d=2$.

Corollary 3.2. *Let $a\in[0,1/2]$, $p\in(1,2)$ and $q=p/(2-p)$. If ϕ is a nonnegative potential such that $\phi^{-1}\in\mathrm L^q(\mathbb S^1)$ with $\|\phi^{-1}\|_{\mathrm L^q(\mathbb S^1)}=1$, then for any complex valued function $\psi\in\mathrm H^1(\mathbb R^2)$ we have*

$$\int_{\mathbb R^2}|\nabla_{\mathbf A}\,\psi|^2\,dx+\frac{1-4\,a^2}{2-p}\int_{\mathbb R^2}\frac{\phi(\theta)}{|x|^2}\,|\psi(x)|^2\,dx\geq\left(\frac{1-4\,a^2}{2-p}+a^2\right)\int_{\mathbb R^2}\frac{|\psi|^2}{|x|^2}\,dx\,.$$

Let us consider cylindrical coordinates $(\rho,\theta,z)\in\mathbb R^+\times[0,2\pi)\times\mathbb R$ such that $|x|^2=\rho^2+z^2$. In this system of coordinates the magnetic kinetic energy is

$$\int_{\mathbb R^3}|\nabla_{\mathbf A}\,\psi|^2\,dx=\int_{\mathbb R^3}\left(\left|\tfrac{\partial\psi}{\partial\rho}\right|^2+\tfrac1{\rho^2}\left|\tfrac{\partial\psi}{\partial\theta}-i\,a\,\psi\right|^2+\left|\tfrac{\partial\psi}{\partial z}\right|^2\right)dx$$

where $d\mu:=\rho\,d\rho\,d\theta\,dz$. The following result was proved in [17, Section 2.2].

Lemma 3.7. *For any $\psi\in\mathrm H^1(\mathbb R^3)$, we have*

$$\iiint_{\mathbb R^+\times[0,2\pi)\times\mathbb R}\left(\left|\tfrac{\partial\psi}{\partial\rho}\right|^2+\left|\tfrac{\partial\psi}{\partial z}\right|^2\right)d\mu\geq\frac14\iiint_{\mathbb R^+\times[0,2\pi)\times\mathbb R}\frac{|\psi|^2}{\rho^2+z^2}\,d\mu\quad\forall\,\psi\in\mathrm H^1(\mathbb R^3)\,.$$

Proof. We give an elementary proof. Assume that ψ is smooth and has compact support. The inequality follows from the expansion of the square

$$\iiint_{\mathbb R^+\times[0,2\pi)\times\mathbb R}\left(\left|\tfrac{\partial\psi}{\partial\rho}+\tfrac{\rho\,\psi}{2\,(\rho^2+z^2)}\right|^2+\left|\tfrac{\partial\psi}{\partial z}+\tfrac{z\,\psi}{2\,(\rho^2+z^2)}\right|^2\right)d\mu\geq0\quad\forall\,\psi\in\mathrm H^1(\mathbb R^3)$$

and of an integration by parts of the cross terms. □

Lemma 3.7 is an improved version of the standard Hardy inequality in the sense that the left-hand side of the inequality does not involve the angular part of the kinetic energy. A consequence of Corollary 3.1 and Lemma 3.7 is a Hardy-like estimate in dimension $d=3$. For the angular part we argue as in Corollary 3.2. Details of the proof are left to the reader.

Theorem 3.3. *Let $a\in[0,1/2]$, $p\in(1,2)$ and $q=p/(2-p)$. If ϕ is a potential such that $\phi^{-1}\in\mathrm L^q(\mathbb S^1)$ with $\|\phi^{-1}\|_{\mathrm L^q(\mathbb S^1)}=1$, then for any complex valued function $\psi\in\mathrm H^1(\mathbb R^3)$ we have*

$$\int_{\mathbb R^3}|\nabla_{\mathbf A}\,\psi|^2\,dx+\frac{1-4\,a^2}{2-p}\int_{\mathbb R^3}\frac{\phi(\theta)}{\rho^2}\,|\psi(x)|^2\,dx \geq\frac14\int_{\mathbb R^3}\frac{|\psi|^2}{|x|^2}\,dx+\left(\frac{1-4\,a^2}{2-p}+a^2\right)\int_{\mathbb R^3}\frac{|\psi|^2}{\rho^2}\,dx\,.$$

A simple case is $\phi\equiv1$, for which we obtain that

$$\int_{\mathbb R^3}|\nabla_{\mathbf A}\,\psi|^2\,dx\geq\frac14\int_{\mathbb R^3}\frac{|\psi|^2}{|x|^2}\,dx+a^2\int_{\mathbb R^3}\frac{|\psi|^2}{|\rho|^2}\,dx\quad\forall\,\psi\in\mathrm H^1_{\mathbf A}(\mathbb R^3)\,.$$

4. Aharonov-Bohm magnetic interpolation inequalities in $\mathbb R^2$

4.1. *Magnetic interpolation inequalities without weights*

Let us consider on $\mathbb R^2$ the Aharonov-Bohm magnetic potential $\mathbf A(x)=a\,|x|^{-2}\,\mathbf e_\theta$, with the notations of the introduction. Using the *diamagnetic inequality*

$$|\nabla_{\mathbf A}\,\psi|^2\geq\big|\nabla|\psi|\big|^2\quad\text{a.e. in }\mathbb R^2$$

and, for any $p\in(2,\infty)$ and $\lambda>0$, the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|^2_{\mathrm L^2(\mathbb R^2)}+\lambda\,\|\psi\|^2_{\mathrm L^p(\mathbb R^2)}\geq\mathsf C_p\,\lambda^{\frac p2}\,\|\psi\|^2_{\mathrm L^2(\mathbb R^2)}\quad\forall\,\psi\in\mathrm H^1(\mathbb R^2)\cap\mathrm L^p(\mathbb R^2)\tag{4.1}$$

with optimal constant $\mathsf C_p$, we deduce that

$$\|\nabla_{\mathbf A}\,\psi\|^2_{\mathrm L^2(\mathbb R^2)}+\lambda\,\|\psi\|^2_{\mathrm L^2(\mathbb R^2)}\geq\mu_{a,p}(\lambda)\,\|\psi\|^2_{\mathrm L^p(\mathbb R^2)}\quad\forall\,\psi\in\mathrm H^1_a(\mathbb R^2)\,.\tag{4.2}$$

See [10, Section 3] for details. Here $\mu_{a,p}(\lambda)$ is the optimal constant in (4.2) for any given a, p and λ and, as a function of λ, $\mu_{a,p}(\lambda)$ is monotone increasing and concave. Notice that right-hand sides in (4.1) and (4.2) involve norms with respect to Lebesgue's measure. It turns out that $\mu_{a,p}(\lambda)$ is equal to the best constant of the non-magnetic problem.

Proposition 4.1. *Let $a\in\mathbb R$ and $p\in(2,\infty)$. The optimal constant in (4.2) is*

$$\mu_{a,p}(\lambda)=\mathsf C_p\,\lambda^{\frac p2}\quad\forall\,\lambda>0$$

and equality is not achieved on $\mathrm H^1(\mathbb R^2)\cap\mathrm L^p(\mathbb R^2)$ if $a\in\mathbb R\setminus\mathbb Z$.

Proof. By construction we know that $\mu_{a,p}(\lambda)\geq\mathsf C_p\,\lambda^{p/2}$. By taking an optimal function ψ for (4.1) and considering $\psi_n(x)=\psi(x+n\,\mathsf e)$ with $n\in\mathbb N$ and $\mathsf e\in\mathbb S^1$, we see that there is equality.

Let us prove by contradiction that equality is not achieved. If $\psi \in \mathrm{H}^1(\mathbb{R}^2) \cap \mathrm{L}^p(\mathbb{R}^2)$ is optimal, let $\phi = e^{\,i\,a\,\theta}\psi$. Since

$$\|\nabla_{\mathbf{A}}\,\psi\|^2_{\mathrm{L}^2(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\partial_r\psi|^2 + \frac{|\partial_\theta\phi|^2}{|x|^2}\,dx$$

and equality in (4.1) is achieved by functions with a constant phase only, this means that $\partial_\theta\phi = 0$ a.e., a contradiction with the periodicity of ψ with respect to $\theta \in [0, 2\pi)$ if $a \notin \mathbb{Z}$. □

Proposition 4.1 means that the Aharonov-Bohm magnetic potential plays no role in non-weighted interpolation inequalities. This is why it is natural to introduce weighted norms with adapted scaling properties.

4.2. *Magnetic Hardy-Sobolev interpolation inequalities*

The *Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^2} \frac{|\nabla v|^2}{|x|^{2\mathsf{a}}}\,dx \ge \mathsf{C}_{\mathsf{a}} \left(\int_{\mathbb{R}^2} \frac{|v|^p}{|x|^{\mathsf{b}\,p}}\,dx\right)^{2/p} \quad \forall\, v \in \mathcal{D}(\mathbb{R}^2) \tag{4.3}$$

has been established in [6] and, earlier, in [20]. The exponent $\mathsf{b} = \mathsf{a} + 2/p$ is determined by the scaling invariance and as p varies in $(2, \infty)$, the parameters a and b are such that $\mathsf{a} < \mathsf{b} \le \mathsf{a} + 1$ and $\mathsf{a} < 0$. The case $\mathsf{a} > 0$ can be considered in an appropriate functional space after a Kelvin-type transformation: see [7,13], but we will not consider this case here. As noticed for instance in [13], by considering $v(x) = |x|^{\mathsf{a}}\,u(x)$, Ineq. (4.3) is equivalent to the *Hardy-Sobolev inequality*

$$\int_{\mathbb{R}^2} |\nabla u|^2\,dx + \mathsf{a}^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x|^2}\,dx \ge \mathsf{C}_{\mathsf{a}} \left(\int_{\mathbb{R}^2} \frac{|u|^p}{|x|^2}\,dx\right)^{2/p} \quad \forall\, u \in \mathcal{D}(\mathbb{R}^2)\,. \tag{4.4}$$

The optimal functions for (4.3) are radially symmetric if and only if

$$\mathsf{b} \ge \mathsf{b}_{\mathrm{FS}}(\mathsf{a}) := \mathsf{a} - \frac{\mathsf{a}}{\sqrt{1+\mathsf{a}^2}}$$

according to [18,12]. We refer to [5] for more details and for the proof of the following magnetic Hardy-Sobolev inequality.

Theorem 4.1 ([5]). *Let $a \in [0, 1/2]$ and $p > 2$. For any $\lambda > -a^2$, there is an optimal function $\lambda \mapsto \mu(\lambda)$ which is monotone increasing and concave such that*

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx \ge \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p} \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)\,. \tag{4.5}$$

If $a \in [0, 1/2)$, the optimal function in (4.5) is

$$\psi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-\frac{2}{p-2}} \quad \forall\, x \in \mathbb{R}^2\,, \quad \textit{with} \quad \alpha = \frac{p-2}{2}\,\sqrt{\lambda + a^2}\,,$$

up to a scaling and a multiplication by a constant, if

$$\lambda \le \lambda_\star := 4\,\frac{1-4\,a^2}{p^2-4} - a^2\,.$$

Conversely, if $a\in[0,1/2]$ *and* $\lambda>\lambda_\bullet$ *with*

$$\lambda_\bullet := \frac{8\left(\sqrt{p^4-a^2\,(p-2)^2\,(p+2)\,(3\,p-2)}+2\right)-4\,p\,(p+4)}{(p-2)^3\,(p+2)} - a^2\,,$$

there is symmetry breaking, i.e., *the optimal functions are not radially symmetric.*

An explicit computation shows that $\lambda_\star<\lambda_\bullet$ for any $a\in(0,1/2)$, and so there is a zone where we do not know whether the optimal functions in (4.5) are symmetric or not. Nevertheless, as shown in [5], the values of $\lambda_\star$ and $\lambda_\bullet$ are numerically very close to each other. If $\lambda\le\lambda_\star$, the expression of $\mu(\lambda)$ is explicit and given by

$$\mu(\lambda)=\frac{p}{2}\,(2\,\pi)^{1-\frac{2}{p}}\,\big(\lambda+a^2\big)^{1+\frac{2}{p}}\left(\frac{2\,\sqrt{\pi}\,\Gamma\big(\frac{p}{p-2}\big)}{(p-2)\,\Gamma\big(\frac{p}{p-2}+\frac12\big)}\right)^{1-\frac{2}{p}}.$$

See [5, Appendix] for the details of the computation of the constant.

Inspired by the equivalence of (4.3) and (4.4), we prove that the magnetic Hardy-Sobolev inequality (4.5) is equivalent to an interpolation inequality of Caffarelli-Kohn-Nirenberg type in the presence of the Aharonov-Bohm magnetic field.

Corollary 4.1 (Magnetic Caffarelli-Kohn-Nirenberg inequality). *Assume that* $p\in(2,+\infty)$, $\mathbf{A}(x)=a\,|x|^{-2}\,\mathbf{e}_\theta$ *for some* $a\in[0,1/2]$ *and* $\mathsf{a}\le0$. *With* μ *as in Theorem 4.1, for any* $\gamma<\mathsf{a}^2+a^2$, *we have that*

$$\int_{\mathbb{R}^2}\frac{|\nabla_{\mathbf{A}}\,\phi|^2}{|x|^{2\mathsf{a}}}\,dx\ge\gamma\int_{\mathbb{R}^2}\frac{|\phi|^2}{|x|^{2\mathsf{a}+2}}\,dx+\mu(\mathsf{a}^2-\gamma)\left(\int_{\mathbb{R}^2}\frac{|\phi|^p}{|x|^{\mathsf{a}\,p+2}}\,dx\right)^{2/p}\quad\forall\,\phi\in\mathcal{D}(\mathbb{R}^2;\mathbb{C})$$

and $\mu(\lambda)$ *with* $\lambda=\mathsf{a}^2-\gamma$ *is the optimal constant.*

The cases of symmetry and symmetry breaking in Theorem 4.1 have their exact counterpart in Corollary 4.1. Details are left to the reader.

Proof. Let us consider the function $\phi(x)=|x|^{\mathsf{a}}\,\psi(x)$ and observe that

$$\int_{\mathbb{R}^2}\frac{|\nabla_{\mathbf{A}}\,\phi|^2}{|x|^{2\mathsf{a}}}\,dx=\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx+\mathsf{a}^2\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx$$

and conclude by applying (4.5) to ψ with $\lambda=\mathsf{a}^2-\gamma$. □

4.3. *A magnetic Hardy inequality in* $\mathbb{R}^2$

Another consequence of Theorem 4.1 is the following *magnetic Keller-Lieb-Thirring inequality*, which can be found in [5, Theorem 1]. Let $q=p/(p-2)$. The ground state energy λ_1 of the magnetic Schrödinger operator $-\Delta_{\mathbf{A}}-\phi$ on $\mathbb{R}^2$ is such that

$$\lambda_1(-\Delta_{\mathbf{A}}-\phi)\ge-\lambda\,(\mu)\quad\text{where}\quad\mu=\left(\int_{\mathbb{R}^2}|\phi|^q\,|x|^{2\,(q-1)}\,dx\right)^{1/q}\tag{4.6}$$

and $\mu \mapsto \lambda(\mu)$ is a convex monotone increasing function on $\mathbb{R}^+$ such that $\lim_{\mu\to 0_+} \lambda(\mu) = -a^2$, defined as the inverse of $\lambda \mapsto \mu(\lambda)$ of Theorem 4.1. Again $\lambda(\mu)$ is optimal in (4.6) and the cases of symmetry and symmetry breaking are in correspondence with the ones of Theorem 4.1.

Alternatively, let us consider a function ϕ on $\mathbb{R}^2$. We can estimate an associated magnetic Schrödinger energy from below by

$$\int_{\mathbb{R}^2} \left(|\nabla_{\mathbf{A}}\, \psi|^2 - \tau\, \frac{\phi}{|x|^2}\, |\psi|^2 \right) dx \ge \int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\, \psi|^2\, dx - \tau \left(\int_{\mathbb{R}^2} \frac{|\phi|^q}{|x|^2}\, dx \right)^{\frac{1}{q}} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\, dx \right)^{\frac{2}{p}}$$

by Hölder's inequality, with $q = p/(p-2)$, for an arbitrary parameter $\tau > 0$. For an appropriate choice of τ, we obtain the following result.

Theorem 4.2 (A magnetic Hardy inequality). *Assume that* $q \in (1,2)$, $\mathbf{A}(x) = a\,|x|^{-2}\,\mathbf{e}_\theta$ *for some* $a \in [0,1/2]$. *Then for any function* $\phi \in \mathrm{L}^q\left(\mathbb{R}^2\, |x|^{-2}\, dx\right)$, *we have*

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\, \psi|^2\, dx \ge \mu(0) \left(\int_{\mathbb{R}^2} \frac{|\phi|^q}{|x|^2}\, dx \right)^{-\frac{1}{q}} \int_{\mathbb{R}^2} \frac{\phi}{|x|^2}\, |\psi|^2\, dx \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)\,,$$

where $\mu(\cdot)$ *is the best constant in* (4.5). *Finally, when* $a^2 \le 4/(12+p^2)$, *we know the value of* $\mu(0)$ *explicitly:*

$$\mu(0) = \frac{p}{2}\, (2\,\pi)^{1-\frac{2}{p}}\, a^{2+\frac{4}{p}} \left(\frac{2\,\sqrt{\pi}\,\Gamma\big(\frac{p}{p-2}\big)}{(p-2)\,\Gamma\big(\frac{p}{p-2}+\frac{1}{2}\big)} \right)^{1-\frac{2}{p}}\,.$$

5. Aharonov-Bohm magnetic Hardy inequalities in $\mathbb{R}^3$

In this section we address Aharonov-Bohm magnetic potentials in dimension $d = 3$.

5.1. *An improved Hardy inequality with radial symmetry*

In [15, Section V.B], it is proved that *for all* $a > 0$, *there is a constant* $\mathcal{C}(a)$ *such that* $\mathcal{C}(a) = a^2$ *if* $a \in [0,1/2]$ *and*

$$\int_{\mathbb{R}^3} |\nabla_{\mathbf{A}}\, \psi|^2\, dx \ge \big(\tfrac{1}{4} + \mathcal{C}(a)\big) \int_{\mathbb{R}^3} \frac{|\psi|^2}{|x|^2}\, dx \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)\,. \tag{5.1}$$

If we allow for an angular dependence, we have the following result.

Theorem 5.1. *Let* $a \in [0,1/2]$ *and* $q \in (1,+\infty)$. *Then, for all* $\phi \in \mathrm{L}^q(\mathbb{S}^2)$,

$$\int_{\mathbb{R}^3} |\nabla_{\mathbf{A}}\, \psi|^2\, dx \ge \int_{\mathbb{R}^3} \left(\frac{1}{4} + \frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^2)}}\, \phi(\omega) \right) \frac{|\psi|^2}{|x|^2}\, dx \quad \forall\, \psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)\,.$$

Here $\omega = x/|x|$ *and* $\mu_{a,p}$ *is defined as in Proposition 2.2.*

In the case $a \in [0,1/2]$, according to Proposition 2.2, we find in the limit case as $p \to 2_+$ that $\mu_{a,2}(0) \ge \Lambda_a = a\,(a+1)$ and improve the estimate (5.1) to $\mathcal{C}(a) = a\,(a+1)$ if $\phi \equiv 1$.

Proof. Let us use spherical coordinates $(r,\omega)\in[0,+\infty)\times\mathbb{S}^2$. The result follows from an expansion of the square and an integration by parts in

$$0\le\int_0^{+\infty}\left|\partial_r\psi+\tfrac{1}{2\,r}\,\psi\right|^2\,r^2\,dr=\int_0^{+\infty}|\partial_r\psi|^2\,r^2\,dr-\tfrac14\int_0^{+\infty}|\psi|^2\,dr$$

for the radial part of the Dirichlet integral, and from Corollary 2.2 for the angular part. □

5.2. *An improved Hardy inequality with cylindrical symmetry*

The improved Hardy inequality (without angular kinetic energy) of Lemma 3.7 and (2.9) can be combined into the following improved Hardy inequality in presence of a magnetic potential.

Theorem 5.2. *Let $a\in[0,1/2]$, $p>2$, $q=p/(p-2)$ and $\phi\in\mathrm{L}^q(\mathbb{S}^1)$. For any $\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)$, we have*

$$\int_{\mathbb{R}^3}|\nabla_{\mathbf{A}}\,\psi|^2\,dx\ge\frac14\int_{\mathbb{R}^3}\frac{|\psi|^2}{|x|^2}\,dx+\frac{\mu_{a,p}(0)}{\|\phi\|_{\mathrm{L}^q(\mathbb{S}^1)}}\iiint_{\mathbb{R}^+\times[0,2\pi)\times\mathbb{R}}\frac{\phi(\theta)}{\rho^2}\,|\psi(\rho,\theta,z)|^2\,d\mu\,.$$

Notice that the inequality is a strict improvement upon the Hardy inequality without a magnetic potential combined with the diamagnetic inequality. A simple case which is particularly illuminating is $\phi\equiv1$ with $a^2\le1/(p+2)$ so that $\mu_{a,p}(0)=a^2$ according to Proposition 2.1, in which case we obtain that

$$\int_{\mathbb{R}^3}|\nabla_{\mathbf{A}}\,\psi|^2\,dx\ge\frac14\int_{\mathbb{R}^3}\frac{|\psi|^2}{|x|^2}\,dx+a^2\int_{\mathbb{R}^3}\frac{|\psi|^2}{|\rho|^2}\,dx\quad\forall\,\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^3)\,.$$

Acknowledgments

This research has been partially supported by the project *EFI*, contract ANR-17-CE40-0030 (D.B., J.D.) of the French National Research Agency (ANR), by the PDR (FNRS) grant T.1110.14F and the ERC AdG 2013 339958 "Complex Patterns for Strongly Interacting Dynamical Systems - COMPAT" grant (D.B.) and by the NSF grant DMS-1600560 (M.L.).

References

[1] Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in the quantum theory*, Physical Review, 115 (1959), pp. 485–491.

[2] D. Bakry and M. Émery, *Inégalités de Sobolev pour un semi-groupe symétrique*, C. R. Acad. Sci. Paris Sér. I Math., 301 (1985), pp. 411–413.

[3] W. Beckner, *Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality*, Ann. of Math. (2), 138 (1993), pp. 213–242.

[4] M.-F. Bidaut-Véron and L. Véron, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math., 106 (1991), pp. 489–539.

[5] D. Bonheure, J. Dolbeault, M. Esteban, A. Laptev, and M. Loss, *Symmetry results in two-dimensional inequalities for Aharonov-Bohm magnetic fields.* Preprint Hal: 02003872 or ArXiv: 1902.01065, Feb. 2019.

[6] L. Caffarelli, R. Kohn, and L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math., 53 (1984), pp. 259–275.

[7] F. Catrina and Z.-Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., 54 (2001), pp. 229–258.

[8] D. Chafaï, *Entropies, convexity, and functional inequalities: on Φ-entropies and Φ-Sobolev inequalities*, J. Math. Kyoto Univ., 44 (2004), pp. 325–363.

[9] J. Dolbeault, M. J. Esteban, and A. Laptev, *Spectral estimates on the sphere*, Anal. PDE, 7 (2014), pp. 435–460.

[10] J. Dolbeault, M. J. Esteban, A. Laptev, and M. Loss, *Interpolation inequalities and spectral estimates for magnetic operators*, Ann. Henri Poincaré, 19 (2018), pp. 1439–1463.

[11] J. Dolbeault, M. J. Esteban, A. Laptev, and M. Loss, *Magnetic rings*, Journal of Mathematical Physics, 59 (2018), p. 051504.

[12] J. Dolbeault, M. J. Esteban, and M. Loss, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*, Invent. Math., 206 (2016), pp. 397–440.

[13] J. Dolbeault, M. J. Esteban, M. Loss, and G. Tarantello, *On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities*, Adv. Nonlinear Stud., 9 (2009), pp. 713–726.

[14] J. Dolbeault and X. Li, *Φ-Entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations*, Mathematical Models and Methods in Applied Sciences, 28 (2018), pp. 2637–2666.

[15] T. Ekholm and F. Portmann, *A magnetic contribution to the Hardy inequality*, J. Math. Phys., 55 (2014), pp. 022101, 16.

[16] P. Exner, E. M. Harrell, and M. Loss, *Optimal eigenvalues for some Laplacians and Schrödinger operators depending on curvature*, in Mathematical results in quantum mechanics (Prague, 1998), vol. 108 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 1999, pp. 47–58.

[17] L. Fanelli, D. Krejcirik, A. Laptev, and L. Vega, *On the improvement of the Hardy inequality due to singular magnetic fields.* Preprint ArXiv: 1807.04430, July 2018.

[18] V. Felli and M. Schneider, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Differential Equations, 191 (2003), pp. 121–142.

[19] T. Hoffmann-Ostenhof and A. Laptev, *Hardy inequalities with homogeneous weights*, J. Funct. Anal., 268 (2015), pp. 3278–3289.

[20] V. P. Il'in, *Some integral inequalities and their applications in the theory of differentiable functions of several variables*, Mat. Sb. (N.S.), 54 (96) (1961), pp. 331–380.

[21] A. Laptev and T. Weidl, *Hardy inequalities for magnetic Dirichlet forms*, in Mathematical results in quantum mechanics (Prague, 1998), vol. 108 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 1999, pp. 299–305.

[22] E. H. Lieb and M. Loss, *Analysis*, vol. 14 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, second ed., 2001.

[23] G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.

Contents

February 14, 2019