Fractional hypocoercivity

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Abstract: This paper is devoted to kinetic equations without confinement. We investigate the large time behaviour induced by collision operators with fat tailed local equilibria. Such operators have an anomalous diffusion limit. In the appropriate scaling, the macroscopic equation involves a fractional diffusion operator so that the optimal decay rate is determined by a fractional Nash type inequality. At kinetic level we develop an L^2-hypocoercivity approach and establish a rate of decay compatible with the fractional diffusion limit.

Keywords: Hypocoercivity; linear kinetic equations; fat tail equilibrium; Fokker-Planck operator; anomalous diffusion; fractional diffusion limt; scattering operator; transport operator; micro/macro decomposition; Fourier modes decomposition; fractional Nash inequality; algebraic decay rates

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1. Introduction: from fractional diffusion limits to hypocoercivity

We study decay rates in kinetic equations when local equilibria have fat tails. Let us start by some heuristics in a simplified framework, in order to outline our strategy and explain why fractional diffusion limits play a crucial role. Our goal is to build an adapted Lyapunov functional and develop a L^2-hypocoercivity method. In this introduction, we shall insist on scalings and exponents. The reader interested in detailed results is invited to go directly to Section 2.

Let us consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f^{\mathrm{in}}(x,v) \tag{1}$$

for a distribution function $f(t,x,v)$ depending on a *position* variable $x \in \mathbb{R}^d$, on a *velocity* variable $v \in \mathbb{R}^d$, and on *time* $t \geq 0$. The *collision operator* L acts only on the v variable and, by assumption, its null space is spanned by a *local equilibrium* F. We shall also assume that F is a probability density with algebraic decay given for some $\gamma > 0$ by

$$\forall\, v \in \mathbb{R}^d, \quad F(v) = \frac{c_\gamma}{\langle v\rangle^{d+\gamma}} \quad \text{where} \quad \langle v\rangle := \sqrt{1+|v|^2}\,. \tag{2}$$

It is classical that the normalization constant c_γ is given by

$$c_\gamma = \frac{\Gamma\big(\frac{d+\gamma}{2}\big)}{\pi^{d/2}\,\Gamma\big(\frac{\gamma}{2}\big)}\,.$$

We shall also consider the measure

$$\mathrm{d}\mu = F^{-1}(v)\,\mathrm{d}v$$

and define for functions f and g of the variable $v \in \mathbb{R}^d$ a scalar product and a norm respectively by

$$\langle f, g\rangle := \int_{\mathbb{R}^d} \bar f\, g\,\mathrm{d}\mu \quad \text{and} \quad \|f\|^2 := \int_{\mathbb{R}^d} |f|^2\,\mathrm{d}\mu\,. \tag{3}$$

Here $\bar f$ denotes the complex conjugate of f, as we shall later allow for complex valued functions.

1.1. Decay rates of the homogeneous solution. If f is an *homogeneous solution* of (1), that is, a function which depends only on $v \in \mathbb{R}^d$, with initial datum $f^{\rm in} \in \mathrm{L}^1_+(\mathrm{d}v) \cap \mathrm{L}^2(\mathrm{d}\mu)$ such that $\int_{\mathbb{R}^d} f^{\rm in}\,\mathrm{d}v = 1$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\|f-F\|^2 = 2\,\langle f, \mathsf{L} f\rangle\,.$$

It is natural to ask whether such an estimate proves the convergence of the solution $f(t,\cdot)$ to F as $t \to +\infty$ and provides us with a rate of convergence. Let us assume that L is a self-adjoint operator on $\mathrm{L}^2(\mathrm{d}\mu)$ such that, for some $k \in (0,\gamma)$:
(i) the *interpolation inequality*

$$\int_{\mathbb{R}^d} |g|^2\,\mathrm{d}\mu \leq \mathcal{C}\,\Big(-\,\langle g, \mathsf{L} g\rangle\Big)^\theta \left(\int_{\mathbb{R}^d} |g|^2\,\langle v\rangle^k\,\mathrm{d}\mu\right)^{1-\theta} \tag{4}$$

holds if $\int_{\mathbb{R}^d} g\,\mathrm{d}\mu = 0$, for some $\theta \in (0,1)$ and $\mathcal{C} > 0$,
(ii) there is a constant $\mathcal{C}_k$ such that

$$\forall\, t \geq 0\,, \quad \int_{\mathbb{R}^d} |f(t,\cdot)|^2\,\langle v\rangle^k\,\mathrm{d}\mu \leq \mathcal{C}_k \int_{\mathbb{R}^d} \big|f^{\rm in}\big|^2\,\langle v\rangle^k\,\mathrm{d}\mu\,,$$

then an elementary computation shows the *algebraic decay rate*

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot)-F\|^2 \leq \big(\|f^{\rm in}-F\|^{-2\,a} + \kappa\,a\,t\big)^{-1/a}$$

with $a=(1-\theta)/\theta$ and $\kappa=2\,\mathcal{C}^{-1/\theta}\left(\mathcal{C}_k\int_{\mathbb{R}^d}|f^{\rm in}|^2\,\langle v\rangle^k\,{\rm d}\mu\right)^{-a}$. In this framework, the convergence rate to F is algebraic. This is already an indication that in the general case of (1), we can expect a similar bound on the rate of convergence to a local equilibrium, that is, locally in x. The bound depends on k and, of course, on the choice of L. For a general solution, the main difficulty is to understand the interplay of the transport operator $v\cdot\nabla_x$ and of the collision operator L: this question is the main issue of this paper.

1.2. Scalings and fractional diffusion limits. We consider the *non-homogeneous case* of (1), *i.e.*, solutions which explicitly depend on x, and specialize to solutions which have a finite total mass. Since there is no stationary solution, we expect that a nonnegative solution f of (1) with appropriate conditions on the initial datum is locally vanishing as $t\to+\infty$ and we aim at measuring its decay rate in a well-chosen norm. Our strategy is to adapt the L^2-hypocoercivity method of [11] to the case of local equilibria with fat tails and, in practice, to F. We expect some decoupling of the rate of convergence to local equilibria and the decay rate of the spatial density $\rho=\int_{\mathbb{R}^d}f\,{\rm d}v$ in a *micro/macro decomposition* perspective. We learn from [21,11,10] that *diffusion limits* are usually a convenient tool for uncovering the decay rate at the macroscopic scale, for the simple reason that the rate is uniform with respect to the scaling corresponding to this limit. This is not a surprise because the Lyapunov function in the standard L^2-hypocoercivity method is built by twisting the standard L^2-norm with the term which measures the macroscopic rate of convergence in the diffusion limit. A new difficulty arises from local equilibria with fat tails: in a certain range of γ, only *fractional diffusion equations* can be expected in the appropriate scaling. Let us explain at a formal level why.

In order to fix ideas, we consider the simple scattering operator defined by

$$\mathsf{L}f=Z^{-1}\int_{\mathbb{R}^d}\mathrm{b}(v,v')\,(f'\,F-f\,F')\,{\rm d}v'\quad\text{with}\quad\mathrm{b}(v,v')=\langle v\rangle^\beta\,\langle v'\rangle^\beta$$

with $Z:=\int_{\mathbb{R}^d}\langle v\rangle^\beta\,F(v)\,{\rm d}v$ and *local mass conservation* property: $\int_{\mathbb{R}^d}\mathsf{L}f\,{\rm d}v=0$. Let us investigate the diffusion limit as $\varepsilon\to0_+$ of the scaled kinetic equation written in Fourier variables as

$$\varepsilon^\alpha\,\partial_t\widehat f+i\,\varepsilon\,v\cdot\xi\,\widehat f=\mathsf{L}\widehat f \tag{5}$$

for some exponent α to be chosen. We can rewrite the scattering operator as

$$\mathsf{L}\widehat f=Z^{-1}\,\langle v\rangle^\beta\left(\mathsf{r}\,F-\,Z\,\widehat f\right)\quad\text{with}\quad\mathsf{r}(t,\xi):=\int_{\mathbb{R}^d}\langle v'\rangle^\beta\,\widehat f(t,\xi,v')\,{\rm d}v'\,.$$

As a consequence, the Fourier transform of the spatial density defined as

$$\rho(t,\xi):=\int_{\mathbb{R}^d}\widehat f(t,\xi,v)\,{\rm d}v$$

solves the continuity equation

$$\varepsilon^\alpha\,\partial_t\rho+i\,\varepsilon\int_{\mathbb{R}^d}\xi\cdot v\,\widehat f\,{\rm d}v=0\,.$$

The *fractional diffusion limit* as $\varepsilon \to 0_+$ has already been studied, for instance in [33,4] (more references will be given later). Let us perform a formal Hilbert expansion as in [36], in which only the case $\beta = 0$ is covered, and as in [19] where the collision frequency is $|v|^\beta$. We look for some g such that $\widehat{f} = Z^{-1}\,\mathsf{r}\,F + g$, so that (5) has to be replaced with

$$\varepsilon^\alpha\,(F\,\partial_t\mathsf{r} + \partial_t g) + i\,\varepsilon\,v\cdot\xi\,\big(Z^{-1}\,\mathsf{r}\,F + g\big) + \langle v\rangle^\beta\,g = 0\,.$$

If we assume that the $O(\varepsilon^\alpha)$ term is negligible compared to the other factors, this means that $g(t,\xi,v) \approx g_\varepsilon(t,\xi,v)$ up to lower order terms, where

$$g_\varepsilon(t,\xi,v) = -\,\frac{i\,\varepsilon\,v\cdot\xi}{i\,\varepsilon\,v\cdot\xi + \langle v\rangle^\beta}\,Z^{-1}\,\mathsf{r}(t,\xi)\,F(v)\,.$$

Hence we obtain at formal level that

$$Z\,\rho(t,\xi) = \mathsf{r}(t,\xi) + Z\int_{\mathbb{R}^d} g(t,\xi,v)\,\mathrm{d}v \approx \mathsf{r}(t,\xi) + Z\int_{\mathbb{R}^d} g_\varepsilon(t,\xi,v)\,\mathrm{d}v = \mathsf{a}_\varepsilon(\xi)\,\mathsf{r}(t,\xi)$$

and

$$i\,\varepsilon\,Z\int_{\mathbb{R}^d}\xi\cdot v\,\widehat{f}\,\mathrm{d}v \approx i\,\varepsilon\int_{\mathbb{R}^d}\xi\cdot v\,g_\varepsilon\,\mathrm{d}v = \mathsf{b}_\varepsilon(\xi)\,\mathsf{r}(t,\xi)$$

where

$$\mathsf{a}_\varepsilon(\xi) := \int_{\mathbb{R}^d}\frac{\langle v\rangle^\beta}{i\,\varepsilon\,v\cdot\xi + \langle v\rangle^\beta}\,F(v)\,\mathrm{d}v \quad\text{and}\quad \mathsf{b}_\varepsilon(\xi) := \int_{\mathbb{R}^d}\frac{\varepsilon^2\,(v\cdot\xi)^2}{i\,\varepsilon\,v\cdot\xi + \langle v\rangle^\beta}\,F(v)\,\mathrm{d}v\,.$$

In the limiting regime, the continuity equation becomes

$$\varepsilon^\alpha\,\partial_t\rho + \frac{\mathsf{b}_\varepsilon(\xi)}{\mathsf{a}_\varepsilon(\xi)}\,\rho \approx 0\,.$$

It is easy to check that $\lim_{\varepsilon\to 0_+}\mathsf{a}_\varepsilon(\xi) = 1$, so that

$$\mathsf{r}(t,\xi) \underset{\varepsilon\to 0_+}{\sim} Z\,\rho(t,\xi)\,.$$

If $\beta + \gamma > 2$, then $\lim_{\varepsilon\to 0_+}\varepsilon^{-2}\,\mathsf{b}_\varepsilon(\xi) = \kappa\,|\xi|^2$. With the choice $\alpha = 2$, we recover the standard diffusion limit as $\varepsilon \to 0_+$ and obtain that, in the diffusion limit, the spatial density ρ solves the heat equation written in Fourier variables,

$$\partial_t\rho + \kappa\,|\xi|^2\,\rho = 0$$

with diffusion coefficient $\kappa = c_\gamma\int_{\mathbb{R}^d}(v\cdot\mathsf{e})^2\,\langle v\rangle^{-(d+\beta+\gamma)}\,\mathrm{d}v$, where $\mathsf{e} = \xi/|\xi|$. Notice that κ is independent of $\mathsf{e} \in \mathbb{S}^{d-1}$.

Now let us consider the range $\beta + \gamma < 2$. As a subcase of (2), for *local equilibria with heavy tails* such that

$$F(v) := c_\gamma\,\langle v\rangle^{-(d+\gamma)} \quad\text{with}\quad \beta + \gamma < 2\,,\quad \beta < 1\,,$$

$\mathsf{b}_\varepsilon(\xi)$ diverges as $\varepsilon \to 0_+$ for $\xi \neq 0$. This is why we have to pick an appropriate value of $\alpha \neq 2$. After observing that $\mathsf{b}_\varepsilon(\xi) = \mathsf{b}_1(\varepsilon\,|\xi|\,\mathsf{e})$ with $\mathsf{e} = \xi/|\xi|$, a tedious but elementary computation inspired by [33] and [36, Proposition 2.1] shows that

$$\begin{aligned}\mathsf{b}_1(\varepsilon\,\mathsf{e}) &\sim \int_{|v|>1} \frac{\varepsilon^2\,(v\cdot\mathsf{e})^2}{i\,\varepsilon\,v\cdot\mathsf{e}+\langle v\rangle^\beta}\,F(v)\,\mathrm{d}v\\ &\sim \int_{|v|>1} \frac{\varepsilon^2\,(v\cdot\mathsf{e})^2\,\langle v\rangle^\beta}{(\varepsilon\,v\cdot\mathsf{e})^2+\langle v\rangle^{2\beta}}\,F(v)\,\mathrm{d}v\\ &\sim \varepsilon^{\frac{\gamma-\beta}{1-\beta}} \int_{|w|>\varepsilon^{\frac{1}{1-\beta}}} \frac{(w\cdot\mathsf{e})^2\,|w|^\beta}{(w\cdot\mathsf{e})^2+|w|^{2\beta}}\,\frac{c_\gamma}{|w|^{d+\gamma}}\,\mathrm{d}w\end{aligned}$$

using the change of variables $v=\varepsilon^{\frac{1}{\beta-1}}\,w$. This suggests to make the choice

$$\alpha=\frac{\gamma-\beta}{1-\beta}\,.$$

By taking the limit as $\varepsilon\to 0_+$, we expect that the spatial density ρ (in Fourier variables) solves the *fractional heat equation*

$$\partial_t\rho+\kappa\,|\xi|^\alpha\,\rho=0 \tag{6}$$

with $\kappa=\int_{\mathbb{R}^d}\frac{(w\cdot\mathsf{e})^2\,|w|^\beta}{(w\cdot\mathsf{e})^2+|w|^{2\beta}}\,\frac{c_\gamma}{|w|^{d+\gamma}}\,\mathrm{d}w$. The expression of α is going to play a key role in this paper.

1.3. Mode-by-mode L^2*-hypocoercivity.* We recall that

$$\lim_{\varepsilon\to 0_+}\varepsilon^{-\alpha}\,\mathsf{b}_\varepsilon(\xi)=\int_{\mathbb{R}^d}\frac{(v\cdot\xi)^2\,\langle v\rangle^\beta}{(v\cdot\xi)^2+\langle v\rangle^{2\beta}}\,\frac{c_\gamma}{\langle v\rangle^{d+\gamma}}\,\mathrm{d}v\,.$$

As in [11,10], our goal is to build a quadratic form $\widehat{f}\mapsto\mathcal{H}[\widehat{f}]$ that can be compared with its own t-derivative whenever f solves (1), and which is also equivalent to $\|\widehat{f}\|^2$, *without integrating on* $\xi\in\mathbb{R}^d$. Let us introduce some notation. We define the *transport operator* T in Fourier variables by

$$\mathsf{T}\widehat{f}:=i\,v\cdot\xi\,\widehat{f}$$

and the *orthogonal projection* Π on the subspace generated by F is given by

$$\Pi\,g=\rho_g\,F\quad\text{where}\quad\rho_g:=\int_{\mathbb{R}^d}g\,\mathrm{d}v\,.$$

In the *mode-by-mode approach of the* L^2*-hypocoercivity method*, in which ξ can be seen as a simple parameter, we define

$$\mathcal{H}_\xi[\widehat{f}]:=\|\widehat{f}\|^2+\delta\,\mathrm{Re}\left\langle\mathcal{A}_\xi\widehat{f},\widehat{f}\right\rangle\,,\quad\mathcal{A}_\xi:=\Pi\,\frac{(-\,i\,v\cdot\xi)\,\langle v\rangle^\beta}{(v\cdot\xi)^2+\langle v\rangle^{2\beta}}\,.$$

If f solves (1), then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}_\xi[\widehat{f}]=-\,\mathcal{D}_\xi[\widehat{f}]$$

with

$$\mathcal{D}_\xi[\widehat{f}] := -\,2\,\langle \mathsf{L}\widehat{f},\widehat{f}\rangle + \delta\,\langle \mathcal{A}_\xi \mathsf{T}\Pi\widehat{f}, f\rangle \\ -\,\delta\,\mathrm{Re}\langle \mathsf{T}\mathcal{A}_\xi\widehat{f},\widehat{f}\rangle + \delta\,\mathrm{Re}\langle \mathcal{A}_\xi\mathsf{T}(1-\Pi)\widehat{f},\widehat{f}\rangle - \delta\,\mathrm{Re}\langle \mathcal{A}_\xi\mathsf{L}\widehat{f},\widehat{f}\rangle\,.$$

We can expect that $-\langle \mathsf{L}\widehat{f},\widehat{f}\rangle$ controls $\|(1-\Pi)\widehat{f}\|^2$ by (4) and notice that

$$\langle \mathcal{A}_\xi\mathsf{T}\Pi\widehat{f},\widehat{f}\rangle = \mathsf{b}_1(\xi)\,\|\Pi\widehat{f}\|^2\,.$$

The technical point of the method is to prove that all other terms in $\mathcal{D}_\xi[\widehat{f}]$ can be estimated in terms of $-\langle \mathsf{L}\widehat{f},\widehat{f}\rangle$ and $\langle \mathcal{A}_\xi\mathsf{T}\Pi\widehat{f},\widehat{f}\rangle$.

Even if this is not straightforward, the expression of $\mathcal{H}_\xi[\widehat{f}]$ is compatible with a fractional diffusion limit and this is why one can expect to get a decay rate which corresponds to the decay of the solution of (6), given by the *fractional Nash inequality*

$$\|u\|_{\mathrm{L}^2(\mathrm{d}x)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|_{\mathrm{L}^1(\mathrm{d}x)}^{\frac{\alpha}{d+\alpha}}\,\||\xi|^{\frac{\alpha}{2}}\,\widehat{u}\|_{\mathrm{L}^2(\mathrm{d}\xi)}^{\frac{d}{d+\alpha}}\,. \tag{7}$$

Let $d\ge 2$ and assume that $\beta\in(0,1)$ and $\gamma\in(0,2)$ are such that $\beta<\gamma<2-\beta$. We shall prove that there exists a positive constant C such that, if f is a solution of (1) with initial condition $f^{\mathrm{in}}\in\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$, then

$$\forall\,t\ge 0\,,\quad \|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le C\,(1+t)^{-\frac{d}{\alpha}}\,\|f^{\mathrm{in}}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

Here and throughout this paper, we use the notation $\|f\|^2_{X\cap Y} := \|f\|^2_X + \|f\|^2_Y$. Detailed results and references will be given in Section 2 for a much wider range of parameters (covering the case $\beta\le 0$) and other collision operators L. For technical reasons that will be exposed later, we shall also use a slightly modified definition of the operator $\mathcal{A}_\xi$. Our main task is to relate the corresponding functionals $\mathcal{D}_\xi$ and $\mathcal{H}_\xi$ and to establish decay rates using a convenient extension of the fractional Nash inequality. The outline of the strategy and key technical results are given in Section 3.

2. Assumptions and main results

2.1. Three collision operators. We shall cover three cases of linear collision operators whose local equilibria are given by (2):

▷ the generalized *Fokker-Planck* operator with local equlibrium F

$$\mathsf{L}_1 f = \nabla_v\cdot\big(F\,\nabla_v\big(F^{-1}f\big)\big)\,, \tag{a}$$

▷ the *linear Boltzmann* operator, or *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot,v')\,\Big(f(v')\,F(\cdot) - f(\cdot)\,F(v')\Big)\,\mathrm{d}v'\,, \tag{b}$$

▷ the *fractional Fokker-Planck* operator

$$\mathsf{L}_3 f = \Delta_v^{\sigma/2} f + \nabla_v\cdot(E\,f)\,, \tag{c}$$

with $\sigma \in (0,2)$. In this latter case, we shall simply assume that the *friction force* $E = E(v)$ is radial and solves the equation

$$\Delta_v^{\sigma/2} F + \nabla_v \cdot (E\, F) = 0\,. \tag{8}$$

The operator $\Delta_v^{\sigma/2}$ has Fourier symbol $-|\xi|^\sigma$ and coincides with Δ_v if $\sigma = 2$ but Case (a) should not be considered as a limit of Case (c) when $\sigma \to 2_-$. Notice that $\Delta_v^{\sigma/2}$ is a shorthand notation for $-(-\Delta_v)^{\sigma/2}$.

In Case (b), for the linear Boltzmann operator, we have in mind a *collision kernel* b with either $\mathrm{b}(v,v') = Z^{-1} \langle v\rangle^\beta \langle v'\rangle^\beta$ with $Z := \int_{\mathbb{R}^d} \langle v\rangle^\beta F(v)\,\mathrm{d}v$ as in [33] and in Section 1, or $\mathrm{b}(v,v') = |v'-v|^\beta$. We shall assume that the *collision frequency* ν is positive, locally bounded and verifies

$$\nu(v) := \int_{\mathbb{R}^d} \mathrm{b}(v,v')\,F(v')\,\mathrm{d}v' \underset{|v|\to+\infty}{\sim} |v|^\beta \tag{H0}$$

for a given $\beta \in \mathbb{R}$. Inspired by our observations on the fractional diffusion limit of Section 1 and after noticing that the three above operators can formally be written as $\mathsf{B}[f] - \nu(v)\,f$, we define β as the exponent at infinity of the function ν. This means $\beta = -2$ in Case (a) and $\beta = \gamma - \sigma$ in Case (c), as a consequence of the fact that

$$E(v) = G(v)\,\langle v\rangle^\beta\,v\,,$$

where $G \in \mathrm{L}^\infty(\mathbb{R}^d)$ is a positive function such that $G^{-1} \in \mathrm{L}^\infty(B_0^c(1))$. This property is independent of the other results of the paper and will be proved in Proposition 4 of Section 6.1. Notice that $\beta = \gamma - \sigma$ in Case (c) does not approach $\beta = -2$ of Case (a) as $\sigma \to 2_-$: in view of rates, this limit is very singular.

In Case (b), additional assumptions are needed. The *local mass conservation* property is equivalent to

$$\int_{\mathbb{R}^d} \big(\mathrm{b}(v,v') - \mathrm{b}(v',v)\big)\,F(v')\,\mathrm{d}v' = 0\,. \tag{H1}$$

As in [10], we also assume the existence of constants $\beta \le 0$ and $\mathcal{B} > 0$ such that

$$\frac{1}{Z}\,\langle v\rangle^\beta \,\langle v'\rangle^\beta \le \mathsf{b}(v,v') \le \mathcal{B}\,|v-v'|^\beta\,. \tag{H2}$$

All these assumptions are verified for instance when

$$\begin{aligned} &\mathrm{b}(v',v) = Z^{-1}\,\langle v'\rangle^\beta \,\langle v\rangle^\beta && \text{with} \quad |\beta| \le \gamma\,,\\ &\mathrm{b}(v',v) = |v'-v|^\beta && \text{with} \quad \beta \in (-d/2, 0]\,. \end{aligned}$$

Summarizing, we shall say that *Assumption* (H) *holds* if L is one of the three operators corresponding to the cases (a), (b), or (c), and if additionally the above assumptions hold in Case (b), *i.e.*,

$$\mathsf{L} = \mathsf{L}_1\,,\ \mathsf{L}_2\,,\ \text{or}\ \mathsf{L}_3 \quad \text{and} \quad \text{(H0)–(H2) hold in Case (b)}\,. \tag{H}$$

2.2. Main results: decay rates. Our purpose is to consider a solution of (1) with finite mass and discuss its decay rates as $t \to +\infty$ in terms of β, $\gamma > 0$,

$$\alpha = \frac{\gamma - \beta}{1 - \beta} \quad \text{and} \quad \begin{cases} \alpha' = \alpha & \text{if} \quad \beta + \gamma < 2\,, \\ \alpha' = 2 & \text{if} \quad \beta + \gamma \geq 2\,. \end{cases} \tag{9}$$

Notice that $\alpha \in (0,2)$ if $\beta + \gamma < 2$. With this notation, our main result goes as follows.

Theorem 1. *Let $d \geq 2$, $\gamma > 0$ and assume that β and γ are such that*

$$\beta < \gamma\,, \quad \beta + \gamma \neq 2\,.$$

Under Assumption (H), for any $k \in (0, \gamma)$, there is a constant $C > 0$ such that, for any solution f of (1) with initial condition $f^{\rm in} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$ and for any $t \geq 0$,

$$\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,(1+t)^{-\frac{d}{\alpha'}}\,\|f^{\rm in}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \qquad \textit{if} \quad \beta \geq 0\,,$$

$$\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,(1+t)^{-\min\left\{\frac{d}{\alpha'},\frac{k}{|\beta|}\right\}}\,\|f^{\rm in}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)} \qquad \textit{if} \quad \beta < 0\,.$$

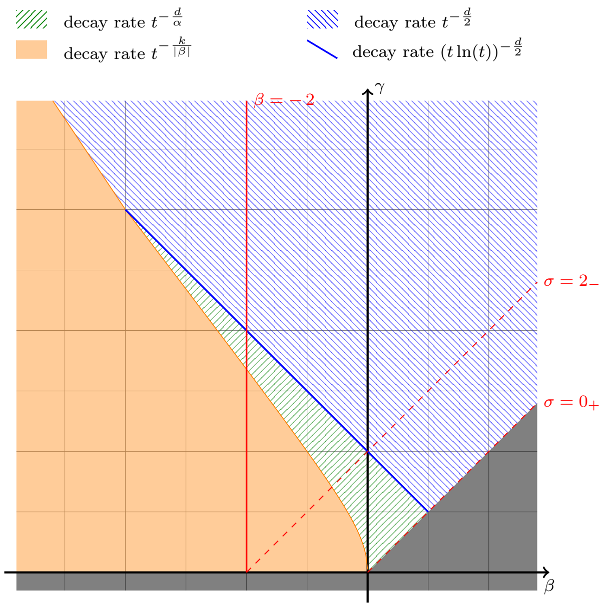

Fig. 1. As $t \to +\infty$, decay rates are at most $O(t^{-k/|\beta|})$ if $\beta < 0 < k < \gamma$ sufficiently close to γ and $\gamma < \gamma_\star(\beta)$, with $\gamma_\star$ given by (10), and otherwise either $O(t^{-d/\alpha})$ if $\max\{0,\beta\} < \gamma < 2-\beta$ or $O(t^{-d/2})$ if $\gamma > \max\{2-\beta,\beta\}$. The picture corresponds to Theorem 1 and 2 with $d = 3$. In Case (c), γ is limited to the strip enclosed between the two dashed red lines.

If $d \geq 2$ and $\beta \leq 0$, the threshold between the region with decay rate $O(t^{-k/|\beta|})$, with $k < \gamma$ but close enough to γ, and the region with decay rate

$O(t^{-d/\alpha'})$ is obtained by solving $\frac{d}{\alpha'}+\frac{k}{\beta}=0$ in the limit case $k=\gamma$. The corresponding curve is given by $\beta\mapsto\gamma_\star(\beta)$ defined as

$$\begin{aligned}\gamma_\star(\beta):=\max\left\{\tfrac12\left(\beta+\sqrt{(4\,d+1)\,\beta^2-4\,d\,\beta}\right),\tfrac{d}{2}\,|\beta|\right\}\quad&\text{if}\quad d\ge3\,,\\ \gamma_\star(\beta)=\tfrac12\left(\beta+\sqrt{\beta\,(9\,\beta-8)}\right)\quad&\text{if}\quad d=2\,.\end{aligned}\tag{10}$$

If $d\ge3$, notice that $\gamma_\star(\beta):=\big(\beta+\sqrt{(4\,d+1)\,\beta^2-4\,d\,\beta}\big)/2$ if $-4/(d-2)\le\beta<0$ and $\gamma_\star(\beta)=\frac{d}{2}\,|\beta|$ if $\beta\le-4/(d-2)$. See Figures 1 and 2.

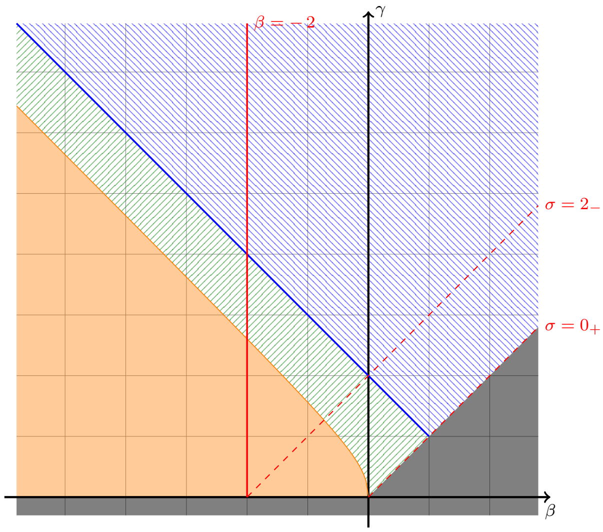

Fig. 2. Decay rates of Theorem 1 and 2 depending on β and γ in dimension $d=2$, as $t\to+\infty$. The caption convention is the same as for Figure 1. When $\beta\le0$, the upper threshold of the region with decay rate $O(t^{-k/|\beta|})$, with k close enough to γ, is $\gamma=\gamma_\star(\beta)$.

If $\beta+\gamma=2$, there is a logarithmic correction. The following result deals with this special case, in any dimension.

Theorem 2. *Let $d\ge1$, $\gamma>0$, $\beta=2-\gamma<1$ and f be a solution of (1) with initial condition $f^{\rm in}\in\mathrm L^1(\mathrm dx\,\mathrm dv)$. Assume that (H) holds. For any $k\in(0,\gamma)$ if $\beta\le0$ and for $k=0$ if $\beta>0$, if $f^{\rm in}\in\mathrm L^2(\langle v\rangle^k\,\mathrm dx\,\mathrm d\mu)$, then there is a constant $C>0$ such that, for any $t\ge0$,*

$$\|f\|^2_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}\le C\,\big((2+t)\,\log(2+t)\big)^{-\frac d2}\,\|f^{\rm in}\|^2_{\mathrm L^1(\mathrm dx\,\mathrm dv)\cap\mathrm L^2(\langle v\rangle^k\mathrm dx\,\mathrm d\mu)}\,,$$

under the additional condition $k\le\frac d2\,|\beta|$ if $d\ge3$. If $d\ge3$ and $k>\frac d2\,|\beta|$, then, for any $t\ge0$,

$$\|f\|^2_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}\le C\,(1+t)^{-\frac{k}{|\beta|}}\,\|f^{\rm in}\|^2_{\mathrm L^1(\mathrm dx\,\mathrm dv)\cap\mathrm L^2(\langle v\rangle^k\mathrm dx\,\mathrm d\mu)}\,.$$

If $d = 1$, the results when $\beta \neq 2 - \gamma$ slightly differs from Theorem 1. Let

$$\gamma_\star(\beta) = \max\left\{|\beta|, \tfrac{1}{2}\left(\beta + \sqrt{(5\,\beta - 4)\,\beta}\right)\right\}\,.$$

Notice that $\alpha < 0$ if $\beta + \gamma < 0$ and $\gamma_\star(\beta) = -\beta > 0$ if and only if $\beta \leq -1$.

Theorem 3. *Assume that* (H) *holds. Let* $d = 1$, $\gamma > \max\{0, \beta\}$ *and* f *be a solution of* (1) *with initial condition* $f^{\rm in} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$.

- *If* $\beta \geq 0$ *and* $\beta + \gamma \neq 2$ *and* $f^{\rm in} \in \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$, *there is a constant* $C > 0$ *such that, for any* $t \geq 0$,

$$\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,(1+t)^{-\frac{d}{\alpha'}}\,\|f^{\rm in}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

- *If* $f^{\rm in} \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ *and the parameters* β, γ *and* k *are in the range*

$$\beta < -1\,,\quad \gamma \in (1, -\beta)\,,\quad k \in \left(\tfrac{\gamma}{\alpha}, \gamma\right)\quad \textit{and}\quad 0 < \tau < \frac{k+\gamma}{k\,\alpha - \gamma + |\beta|\,(\alpha+1)}\,, \tag{11}$$

there is a constant $C > 0$ *such that, for any* $t \geq 0$,

$$\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,(1+t)^{-\tau}\,\|f^{\rm in}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

- *If* $\beta < 0$, $\gamma > 0$, $\gamma + \beta \neq 2$ *and* $k \in (0, \gamma)$ *but* $(\gamma, k) \notin (1, -\beta) \times (\frac{\gamma}{\alpha}, \gamma)$, *then there is a constant* $C > 0$ *such that, for any* $f^{\rm in} \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ *and any* $t \geq 0$,

$$\|f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \leq C\,(1+t)^{-\min\left\{\frac{d}{\alpha'}, \frac{k}{|\beta|}\right\}}\,\|f^{\rm in}\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)\cap\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

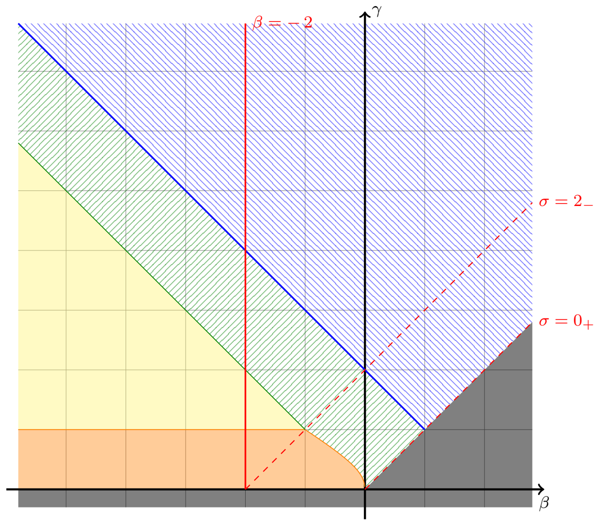

Fig. 3. Decay rates of Theorem 3 depending on β and γ in dimension $d = 1$, as $t \to +\infty$. When $\beta \leq 0$, k is chosen arbitrarily close to γ. The caption convention is the same as for Figure 1 except for $1 < \gamma < |\beta|$ which corresponds to the decay rate $O(t^{-\tau})$.

See Figure 3 for an illustration of Theorem 3 in dimension $d = 1$.

Our method for proving Theorems 1, 2 and 3 relies on a mode-by-mode analysis in Fourier variables based on the L^2-hypocoercivity method as in [11]. A detailed outline of the strategy and the sketch of the proof of the main results will be given in Section 3.

2.3. A brief review of the literature. *Fractional diffusion limits* of kinetic equations attracted a considerable interest in the recent years. The microscopic jump processes are indeed easy to encode in kinetic equations and the diffusion limit provides a simple procedure to justify the use of fractional operators at macroscopic level. Formal derivations are known for a long time, see for instance [38], but rigorous proofs are more recent. In the case of linear scattering operators like those of Case (b), we refer to [33,32,36,4] for some early results and to [25] for a closely related work on Markov chains. Numerical schemes which are asymptotically preserving have been obtained in [18,19]. Beyond the classical paper [20], we also refer to [33,32,36,4] for a discussion of earlier results on standard, *i.e.*, non-fractional, diffusion limits. Concerning the generalized Fokker-Planck operators of Case (a), such that local equilibria have fat tails, the problem has recently been studied in [31] in dimension $d = 1$ by spectral methods and, from a probabilistic point of view, in [23]. Depending on the range of the exponents, various regimes corresponding to Brownian processes, stable processes or integrated symmetric Bessel processes are obtained and described in [23] as well as the threshold cases (some were already known, see for instance [15]). Higher dimensional results have recently been obtained in [22]. Concerning Case (c), the fractional diffusion limit of the fractional Vlasov-Fokker-Planck equation, or Vlasov–Lévy–Fokker–Planck equation, has been studied in [16,1,2] when the friction force is proportional to the velocity. Here our Case (c) is slightly different, as we pick forces giving rise to collision frequencies of the order of $|v|^\beta$ as $|v| \to +\infty$.

In the *homogeneous case*, that is, when there is no x-dependence, it is classical to introduce a function $\Phi(v) = -\log F(v)$, where F denotes the local equilibrium but is not necessarily of the form (2), and classify the possible behaviors of the solution f to (1) according to the growth rate of Φ. Assume that the collision operator is either the generalized Fokker-Planck operator of Case (a) or the scattering operator of Case (b). Schematically, if

$$\Phi(v) = \langle v\rangle^\zeta \ ,$$

we obtain that $\|f(t,\cdot) - M\,F\|_{\mathrm{L}^2(\mathrm{d}\mu)}$ decays exponentially if $\zeta \ge 1$, with $M = \int_{\mathbb{R}^d} f\,\mathrm{d}v$. In the range $\zeta \in (0,1)$, the Poincaré inequality of Case (a) has to be replaced by a *weak Poincaré* or a *weighted Poincaré inequality*: see [37,27,10] and rates of convergence are typically algebraic in t. Summarizing, the lowest is the rate of growth of Φ as $|v| \to +\infty$, the slowest is the rate of convergence of f to $M\,F$. Now let us focus on the limiting case as $\zeta \to 0_+$. The turning point precisely occurs for the minimal growth which guarantees that F is integrable, at least for solutions of the homogeneous equation with initial data in $\mathrm{L}^1(\mathrm{d}v)$. Hence, if we consider

$$\Phi(v) = \eta\,\log\langle v\rangle\,,$$

with $\eta < d$, then diffusive effects win over confinement and the unique local equilibrium with finite mass is 0. To measure the sharp rate of decay of f towards 0, one can replace the Poincaré inequality and the weak Poincaré or the the weighted Poincaré inequalities by *weighted Nash inequalities*. See [12] for details. In this paper, we consider the case $\eta = \gamma + d > d$, which guarantees that F is integrable. Standard diffusion limits can be invoked if $\beta + \gamma > 2$, but here we are interested in the regime corresponding to fractional diffusion limits, with $\beta + \gamma \leq 2$.

As explained in Section 1, standard diffusion limits provide an interesting insight into the *micro/macro decomposition* which is the key of the L^2-*hypocoercive approach* of [21]. Another parameter can be taken into account: the confinement in the spatial variable x. In presence of a confining potential $V = V(x)$ with sufficient growth and when F has fast decay, typically for $\zeta \geq 1$, the rate of convergence is found to be exponential. A milder growth of V gives a slower convergence rate as analyzed in [14]. If e^{-V} is not integrable, the diffusion wins in the hypocoercive picture, and the rate of convergence of a finite mass solution of (1) towards 0 can be captured by Nash and related Caffarelli-Kohn-Nirenberg inequalities: see [11, 12].

A typical regime for fractional diffusion limits is given by local equilibria with fat tails which behave according to (2) with $\gamma \in (0, 2 - \beta)$: F is integrable but has no standard diffusion limit. Whenever fractional diffusion limits can be obtained, it was expected that rates of convergence can also be obtained by an adapted L^2-hypocoercive approach. To simplify the exposition, we shall consider only the case $V = 0$ and measure the decay rate. In view of [28] (also see references therein), it is natural to expect that a fractional Nash type approach has to play the central role, and this is indeed what happens. The mode-by-mode hypocoercivity estimate shows that rates are of the order of $|\xi|^\alpha$ as $\xi \to 0$ which results in the expected time decay. In this direction, let us mention that the spectral information associated with $|\xi|^\alpha$ is very natural in connection with the fractional heat equation as was recently observed in [5]. As far as we know, asymptotic rates for (1) have not been studied so far, to the exception of the very recent results of [2] which deal with the Vlasov–Lévy–Fokker–Planck equation in the case of a spatial variable in the flat torus $\mathbb{T}^d$ by an H^1-hypocoercivity method and the simplest version ($\beta = 0$) of the scattering collision operator: see Section 8.2 for more details. Preliminary versions of the present paper can be found in [29] and [9, v1].

3. Mode by mode hypocoercivity method and outline of the method

3.1. Definitions and preliminary observations. Let us consider the measure $\mathrm{d}\mu = F^{-1}(v)\,\mathrm{d}v$ and the Fourier transform of f in x defined by

$$\widehat{f}(t,\xi,v) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\,x\cdot\xi} f(t,x,v)\,\mathrm{d}x\,.$$

If f solves (1), then the equation satisfied by $\widehat{f}$ is

$$\partial_t \widehat{f} + \mathsf{T}\widehat{f} = \mathsf{L}\widehat{f}\,, \quad \widehat{f}(0,\xi,v) = \widehat{f}^{\mathrm{in}}(\xi,v)\,, \quad \mathsf{T}\widehat{f} = i\,v\cdot\xi\,\widehat{f}\,,$$

where $\xi \in \mathbb{R}^d$ can be seen as a parameter, so that for each Fourier mode ξ, we can study the decay of f. For this reason why we call it a *mode-by-mode analysis*, as in [11].

For any given $\xi \in \mathbb{R}^d$, taken as a parameter, we consider $(t, v) \mapsto \widehat{f}(t, \xi, v)$ on the complex valued Hilbert space $\mathrm{L}^2(\mathrm{d}\mu)$ with scalar product and norm given by (3). We also recall that Π denotes the orthogonal projection on the subspace generated by F and observe that the property

$$\Pi\mathsf{T}\Pi = 0$$

holds as a consequence of the radial symmetry of F. Let us define the operator A_ξ by

$$\mathsf{A}_\xi := \frac{1}{\langle v\rangle^2}\,\Pi\,\frac{(-\,i\,v\cdot\xi)\,\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2}$$

and the *entropy functional* by

$$\mathsf{H}_\xi[f] := \|\widehat{f}\|^2 + \delta\,\mathrm{Re}\left\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\right\rangle .$$

These definitions are reminiscent of the considerations in Section 1.3 on the quadratic form $\mathcal{H}_\xi$ and the operator $\mathcal{A}_\xi$. Up to the weight $\langle v\rangle^{-2}$, we may notice that $\mathcal{A}_\xi$ and A_ξ have the same scaling structure with respect to (v, ξ) for any $\beta \le 1$. The first elementary result is the observation that A_ξ is a bounded operator and that $\mathsf{H}_\xi[f]$ is equivalent to $\|f\|^2$ if $\delta > 0$ is not too large.

Lemma 1. *With the above notation, for any $\delta \in (0,2)$ and $f \in \mathrm{L}^2(\mathrm{d}\mu)$, we have*

$$|\,\langle \mathsf{A}_\xi f, f\rangle\,| \le \frac{1}{2}\,\|f\|^2 \quad \textit{and} \quad (2-\delta)\,\|f\|^2 \le 2\,\mathsf{H}_\xi[f] \le (2+\delta)\,\|f\|^2 .$$

We shall use the notation

$$\varphi(\xi, v) := \frac{\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2} \quad \text{and} \quad \psi(v) := \langle v\rangle^{-2}$$

and may notice that $\mathsf{A}_\xi \widehat{f} = \psi\,\Pi\mathsf{T}^*\,\varphi\,\widehat{f}$, where T^* denotes the dual of T acting on $\mathrm{L}^2(\mathrm{d}\mu)$.

Proof (Proof of Lemma 1). With these definitions, we obtain $|\psi| \le 1$ and $|(v\cdot\xi)\,\varphi(\xi, v)| \le 1/2$, so that the Cauchy-Schwarz inequality yields

$$|\langle \mathsf{A}_\xi f, f\rangle|^2 \le \int_{\mathbb{R}^d} |\psi(v)|^2\,|f(\xi, v)|^2\,\mathrm{d}v \int_{\mathbb{R}^d} |(v\cdot\xi)\,\varphi(\xi, v)|^2\,|f(\xi, v)|^2\,\mathrm{d}v \le \frac{1}{4}\,\|f\|^4 ,$$

which completes the proof of Lemma 1. □

We observe that

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}_\xi[\widehat{f}] = \mathsf{D}_\xi[\widehat{f}] := -\,2\,\langle \mathsf{L}\widehat{f}, \widehat{f}\rangle + \,\delta\,\mathsf{R}_\xi[\widehat{f}]$$

if f solves (1), where $\mathsf{R}_\xi[\widehat{f}] = -\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\,\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle$. Our goal is to relate $\mathsf{H}_\xi[\widehat{f}]$ and $\mathsf{D}_\xi[\widehat{f}]$. Any decay rate of $\mathsf{H}_\xi[\widehat{f}]$ obtained by a Grönwall estimate gives us a decay rate for $\|f\|^2$ by Lemma 1 and, using an inverse Fourier transform, in $\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$.

More notation will be needed. Let us define the weighted norms

$$\|g\|_k^2 := \int_{\mathbb{R}^d} |g|^2 \, \langle v\rangle^k \, \mathrm{d}\mu \,,$$

so that in particular $\|g\| = \|g\|_0$. A crucial observation, which will be used repeatedly, is the fact that for any constant $\kappa > 0$,

$$\|g - \kappa\, F\|_k^2 = \|g\|_k^2 + \kappa^2 \int_{\mathbb{R}^d} \langle v\rangle^k \, F \,\mathrm{d}x - 2\,\kappa \int_{\mathbb{R}^d} \langle v\rangle^k \, g \,\mathrm{d}x \geq \|(1 - \Pi_k)\, g\|_k^2$$

where

$$\Pi_k\, g := \frac{\int_{\mathbb{R}^d} \langle v\rangle^k \, g \,\mathrm{d}v}{\int_{\mathbb{R}^d} \langle v\rangle^k \, F \,\mathrm{d}v}\, F \,.$$

This is easily shown by optimizing the l.h.s. of the inequality on $\kappa \in \mathbb{R}$. Notice that $\Pi_0 = \Pi$.

The parameters β and γ are chosen as in Theorems 1, 2 or 3 (see Section 2.2) while α and α' are given by (9): $\alpha' = \alpha$ if $\beta + \gamma < 2$ and $\alpha' = 2$ if $\beta + \gamma \geq 2$. For simplicity, we shall not keep track of all constants and simply write that $\mathsf{a} \lesssim \mathsf{b}$ and $\mathsf{a} \gtrsim \mathsf{b}$ if there is a positive constant c such that, respectively, $\mathsf{a} \leq \mathsf{b}\,\mathsf{c}$ and $\mathsf{a} \geq \mathsf{b}\,\mathsf{c}$. We define $\omega_d := |\mathbb{S}^{d-1}|$ where $\mathbb{S}^{d-1}$ denotes the unit sphere in $\mathbb{R}^d$.

3.2. Outline of the method and key intermediate estimates. Assume that f is a finite mass solution of (1) on $\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$. Our goal is to relate

$$\mathsf{H}[f] := \int_{\mathbb{R}^d} \mathsf{H}_\xi[\widehat{f}]\,\mathrm{d}\xi$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = -\,2 \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathsf{L} f \,\mathrm{d}x\,\mathrm{d}\mu + \delta \int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi$$

by a differential inequality and use a Grönwall estimate. According to Lemma 1, the decay rate of $\|f\|^2$ is the same as for $\mathsf{H}_\xi[\widehat{f}]$. Under Assumption (H), we consider a solution f of (1) with initial condition $f^{\rm in} \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$. The main steps of our method are as follows:

$\triangleright$ *The solution is bounded in a weighted* L^2 *space.* We shall prove the following result in Section 4.

Proposition 1. *Assume that* (H) *holds. Let* $d \geq 1$, $\gamma > 0$, $\gamma \geq \beta$, $k \in (0, \gamma)$ *and* f *be a solution of* (1) *with initial condition* $f^{\rm in} \in \mathrm{L}^2(\langle v\rangle^k \,\mathrm{d}x\,\mathrm{d}\mu)$*. Then, there exists a positive constant* $\mathcal{C}_k$ *depending on* d, γ, β *and* k *such that*

$$\forall\, t \geq 0\,, \quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)} \leq \mathcal{C}_k\, \|f^{\rm in}\|_{\mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}\,.$$

$\triangleright$ *The collision term controls the distance to the local equilibrium.* We have the following microscopic coercivity estimate.

Proposition 2. *Let $d \ge 1$, $\gamma > 0$, $\gamma \ge \beta$, $\eta \in [\beta, \gamma)$ and $k \in (0, \gamma)$. Assume that $\beta = -2$ if $\mathsf{L} = \mathsf{L}_1$, that Assumptions (H1) and (H2) hold if $\mathsf{L} = \mathsf{L}_2$, and that $\sigma \in (0,2)$, $\beta = \gamma - \sigma$ if $\mathsf{L} = \mathsf{L}_3$. Then there exists a positive constant $\mathcal{C}$ depending on $\|f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}$ such that for any $f \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$,*

$$\mathcal{C}\,\|(1-\Pi_\eta)f\|^{2\,\frac{k-\beta}{k-\eta}}_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\eta\mathrm{d}\mu)}\,\|f\|^{-2\,\frac{\eta-\beta}{k-\eta}}_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^k\mathrm{d}\mu)} \le -\iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathsf{L}f\,\mathrm{d}x\,\mathrm{d}\mu\,.$$

This estimate is the extension of (4) to the non-homogeneous case. The proof is done in Section 5. We shall use Proposition 2 with $\eta = \beta$ if $\beta + \gamma > 0$ and for some $\eta \in (-\gamma, 0)$ if $\beta + \gamma \le 0$. The case $\eta \ge 0$ is needed only in Step 4 of the proof of Proposition 3.

▷ Our proofs require the computation of a large number of coefficients and various estimates which are collected in Section 6. There we also prove bounds on E if $\mathsf{L} = \mathsf{L}_3$, in Case (c).

▷ *A microscopic coercivity estimate* is established in Section 7.1, which goes as follows. Let us define the function

$$\mathcal{L}(\xi) := \frac{|\xi|^{\alpha'}}{\langle\xi\rangle^{\alpha'}} \quad \text{if} \quad \beta+\gamma \neq 2\,, \quad \mathcal{L}(\xi) := \frac{|\xi|^2\,\big|\log|\xi|\big|}{1+|\xi|^2\,\log|\xi|} \quad \text{if} \quad \beta+\gamma = 2\,.$$

Proposition 3. *Let $\gamma > \max\{0,\beta\}$ and $\eta \in (-\gamma,\gamma)$ such that $\eta \ge \beta$. Under Assumption (H), there exists a positive, bounded function $\xi \mapsto \mathcal{K}(\xi)$ such that*

$$\mathsf{R}_\xi[\widehat{f}] \gtrsim \mathcal{L}(\xi)\,\|\Pi\widehat{f}\|^2 - \mathcal{K}(\xi)\,\|(1-\Pi)\widehat{f}\|^2_\eta\,.$$

In Section 7.2, inspired by *fractional Nash inequalities*, we deduce from Proposition 3 an estimate on the distance in the direction which is orthogonal to the local equilibria.

Corollary 1. *Under Assumption (H), we have*

$$\int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi \;\gtrsim\; \|\Pi f\|^{2\,(1+\frac{\alpha'}{d})}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} - \|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\beta\,\mathrm{d}\mu)} \quad \textit{if} \quad \beta+\gamma\neq 2\,,$$

$$\int_{\mathbb{R}^d} \mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi \;\gtrsim\; \|\Pi f\|^{2\,(1+\frac{\alpha'}{d})}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\,\log\left(\frac{\|\Pi f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}}{\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu)}}\right) - \|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\beta\,\mathrm{d}\mu)} \quad \textit{if} \quad \beta+\gamma = 2\,.$$

The proof is a straightforward consequence of Lemma 14 if $\beta+\gamma\neq 2$ and of Lemma 15 if $\beta+\gamma = 2$. See details in Section 7.2 and 7.3.

3.3. Sketch of the proof of the main results. The difficult part of the paper is the proof of Propositions 1, 2 and 3, and Corollary 1. If $\beta+\gamma\le 0$, we have to take $\eta\neq\beta$ and use additional interpolation estimates: see Section 8. Otherwise, the proof of Theorems 1, 2 and 3 is not difficult if $\beta+\gamma>0$ and can be done as follows.

Under Assumption (H), a solution of (1) is such that

$$-\frac{1}{2}\,\frac{\mathrm d}{\mathrm dt}\mathsf H[f]=-\iint_{\mathbb R^d\times\mathbb R^d} f\,\mathsf L f\,\mathrm dx\,\mathrm d\mu-\delta\int_{\mathbb R^d}\mathsf R_\xi[\widehat f]\,\mathrm d\xi\,.$$

Let us assume that $\beta+\gamma\neq 2$ and $\beta+\gamma>0$. We rely on Proposition 2.

• If $\beta\ge 0$, with $\eta=\beta$, we find that

$$\int_{\mathbb R^d}\mathsf R_\xi[\widehat f]\,\mathrm d\xi\gtrsim\|\mathsf\Pi f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{\alpha'}{d})}-\int_{\mathbb R^d}\|(1-\mathsf\Pi)f\|_\beta^2\,\mathrm dx\,.$$

We obtain

$$\begin{aligned}-\frac{1}{2}\,\frac{\mathrm d}{\mathrm dt}\mathsf H[f]&\gtrsim(1-\delta)\int_{\mathbb R^d}\|(1-\mathsf\Pi)f\|_\beta^2\,\mathrm dx+\delta\,\|\mathsf\Pi f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{\alpha'}{d})}\\&\gtrsim(1-\delta)\,\|(1-\mathsf\Pi)f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^2\,\mathrm dx+\delta\,\|\mathsf\Pi f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{\alpha'}{d})}\\&\gtrsim\mathsf H[f]^{2\,(1+\frac{\alpha'}{d})}\end{aligned}$$

using the simple observation that $\|(1-\mathsf\Pi)f\|_\beta^2\ge\|(1-\mathsf\Pi)f\|_{\mathrm L^2(\mathrm d\mu)}^2$ if $\beta\ge 0$.

• If $\beta\in(-\gamma,0)$ and $2\neq\beta+\gamma>0$, again with $\eta=\beta$, we find that

$$-\frac{1}{2}\,\frac{\mathrm d}{\mathrm dt}\mathsf H[f]\gtrsim(1-\delta)\int_{\mathbb R^d}\|(1-\mathsf\Pi)f\|_\beta^2\,\mathrm dx+\delta\,\|\mathsf\Pi f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{\alpha'}{d})}\,.$$

Using Hölder's inequality

$$\|(1-\mathsf\Pi)f\|^2\le\|(1-\mathsf\Pi)f\|_\beta^{\frac{2\,k}{k-\beta}}\,\|(1-\mathsf\Pi)f\|_k^{\frac{2\,\beta}{\beta-k}}\,,$$

we conclude that

$$-\frac{1}{2}\,\frac{\mathrm d}{\mathrm dt}\mathsf H[f]\gtrsim(1-\delta)\|(1-\mathsf\Pi)f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{|\beta|}{k})}+\delta\,\|\mathsf\Pi f\|_{\mathrm L^2(\mathrm dx\,\mathrm d\mu)}^{2\,(1+\frac{\alpha'}{d})}\,.$$

• If $d\ge 1$, $\beta\le 0$ and $\beta+\gamma=2$, $\alpha'=2$ but there is a logarithmic correction in the expression of $\int_{\mathbb R^d}\mathsf R_\xi[\widehat f]\,\mathrm d\xi$, which is responsible for the $O(\log t)$ correction of Theorem 2 as $t\to+\infty$.

• For integrability reasons, the case $\beta+\gamma\le 0$ requires further estimates involving some $\eta\in(-\gamma,0)$ that will be dealt with in Sections 5.4 and 8.1. Except in this case, the proofs of Theorems 1, 2 and 3 are complete.

4. Estimates in weighted $\mathrm L^2$ spaces

In this section, we assume that $\beta\le 0$.

4.1. A result in weighted L^2 *spaces.* Let us prove Proposition 1, *i.e.*, the propagation of weighted norms $\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ with power law of order $k\in(0,\gamma)$.

The conservation of weighted norms has also been used in [10] when F has a sub-exponential form. In that case, any value of k was authorized, and this was implicitly a consequence of the fact that such a local equilibrium F had finite weighted norms $\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ for any $k\in\mathbb{R}^+$. For a local equilibrium given by (2), there is a limitation on k as we cannot expect a global propagation of higher moments than those of F.

For any function $h\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$, one can notice that

$$\|h\|_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}=\|F^{-1}h\|_{\mathrm{L}^2(F\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\,.$$

In other words, it is equivalent to control the semi-group $e^{(\mathsf{L}-\mathsf{T})t}$ in $\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ and $F^{-1}\,e^{(\mathsf{L}-\mathsf{T})t}$ in $\mathrm{L}^2(F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$. Since $\mathrm{L}^2(\langle v\rangle^k\,F\,\mathrm{d}x\,\mathrm{d}v)$ is a space interpolating between $\mathrm{L}^1(F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$ and $\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)$ (see [39, Theorem (2.9)]), we shall establish the result of Proposition 1 by proving that $F^{-1}\,e^{(\mathsf{L}-\mathsf{T})t}$ is bounded onto $\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)$ in Section 4.2 and onto $\mathrm{L}^1(F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$ in Section 4.5. In order to prove this last estimate, as in [27,28,10], we shall use a Lyapunov function method in Section 4.3 and a splitting of the operator in Section 4.4.

4.2. The boundedness in $\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)$*.*

Lemma 2. *Let* $d\ge1$ *and* $\gamma>0$*. If* (H) *holds, then*

$$\forall\,t\ge0\,,\quad\|F^{-1}e^{t(\mathsf{L}-\mathsf{T})}\|_{\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)\to\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)}\le1\,.$$

Proof. This is a consequence of the maximum principle in Case (a). In Case (b), $h^\#(t,x,v)=F^{-1}(v)\,f(t,x+v\,t,v)$ solves

$$\partial_t h^\#+\nu(v)\,h^\#=\int_{\mathbb{R}^d}\mathrm{b}(v,v')\,F(v')\,h^\#(t,x,v')\,\mathrm{d}v'\,,$$

which is clearly a positivity preserving equation. The positivity of

$$(t,x,v)\mapsto\|h(0,\cdot,\cdot)\|_{\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)}-h^\#(t,x,v)$$

is also preserved, as it solves the same equation, which proves the claim. Case (c) is less standard as it relies on the maximum principle for fractional operators. As this is out of the scope of the present paper, we will only sketch the main steps of a proof. First of all, the results of [28] can be adapted to E as defined by (8), thus proving that the evolution according to $\partial_t-F\,\mathsf{L}_3(F^{-1}\cdot)$ preserves L^∞ bounds. This is also the case of $\partial_t-\mathsf{T}$. We can then conclude using a time-splitting approximation scheme of evolution and a Trotter formula. ◻

4.3. A Lyapunov function method. The boundedness of the operator $F^{-1}e^{t(\mathsf{L}-\mathsf{T})}$ in $\mathrm{L}^1(F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$ is equivalent to the boundedness of the operator $e^{t(\mathsf{L}-\mathsf{T})}$ in $\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$. To obtain such a bound, we rely on a Lyapunov function estimate.

Lemma 3. *Let $d\ge 1$ and $\gamma>0\ge\beta$. If (H) holds, then for any $k\in[0,\gamma-\beta)$, there exists $(a,b,R)\in\mathbb{R}\times\mathbb{R}^+\times\mathbb{R}^+$ such that for any $f\in\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$,*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\tfrac{f}{|f|}\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(a\,\mathbb{1}_{B_R}-b\,\langle v\rangle^\beta\Big)\,|f|\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v\,.$$

As a special case corresponding to $k=0$, we have $\iint_{\mathbb{R}^d\times\mathbb{R}^d}\frac{f}{|f|}\,\mathsf{L}f\,\mathrm{d}x\,\mathrm{d}v\le 0$.

Here by convention, we shall write that $\frac{f}{|f|}=0$ if $f=0$.

Proof. First assume that $f\ge 0$. Then one may write,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathsf{L}f\,F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu$$
$$=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathsf{L}^*(F\,\langle v\rangle^k)\,f\,\mathrm{d}x\,\mathrm{d}\mu\,.$$

• In Case (a), we notice that L is self-adjoint on $\mathrm{L}^2(\mathrm{d}\mu)$, recall that $\beta=-2$ and compute

$$\begin{aligned}F^{-1}\,\mathsf{L}_1\big(F\,\langle\cdot\rangle^k\big)(v)&=\langle v\rangle^{d+\gamma}\,\nabla_v\cdot\Big(\langle v\rangle^{-d-\gamma}\,\nabla_v\,\langle v\rangle^k\Big)\\&=k\,\langle v\rangle^{d+\gamma}\,\nabla_v\cdot\Big(\langle v\rangle^{-d-\gamma+k-2}\,v\Big)\\&=k\,(d+\gamma-k+2)\,\langle v\rangle^{k-4}-k\,(\gamma+2-k)\,\langle v\rangle^{k-2}\end{aligned}$$

and obtain the result for any $k<\gamma-\beta=\gamma+2$.

• In Case (b), by Assumption (H1) one obtains that

$$\begin{aligned}F^{-1}\,\mathsf{L}_2^*\big(F\,\langle\cdot\rangle^k\big)(v)&=\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,\Big(\langle v'\rangle^k\,F(v')-\langle v\rangle^k\,F(v')\Big)\,\mathrm{d}v'\\&=\left(\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,\frac{\langle v'\rangle^k}{\langle v\rangle^k}\,F(v')\,\mathrm{d}v'-\nu(v)\right)\langle v\rangle^k\,.\end{aligned}$$

By Assumption (H2), we know that

$$\mathcal{C}_\mathrm{b}(k):=\sup_{v\in\mathbb{R}^d}\langle v\rangle^{-\beta}\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,\langle v'\rangle^k\,F(v')\,\mathrm{d}v'$$

is finite for any $k\in(0,\gamma-\beta)$, and as a consequence, we know that

$$\forall\,v\in\mathbb{R}^d\,,\quad\nu(v)\le\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,\langle v'\rangle^k\,F(v')\,\mathrm{d}v'\le\mathcal{C}_\mathrm{b}(k)\,\langle v\rangle^\beta\,.$$

This yields

$$F^{-1}\,\mathsf{L}_2^*\big(F\,\langle\cdot\rangle^k\big)(v)\le\left(\frac{\mathcal{C}_{\mathrm{b}}(b)}{\langle v\rangle^k}-\frac{\nu(v)}{\langle v\rangle^\beta}\right)\langle v\rangle^\beta\,.$$

We conclude that Inequality (3) holds for any $k\in(0,\gamma-\beta)$ by Assumption (H0).

- In Case (c), it is elementary to compute L_3^* and observe that

$$\begin{aligned}F^{-1}\,\mathsf{L}_3^*\big(F\,\langle\cdot\rangle^k\big)(v)&=\varDelta_v^{\sigma/2}\,\langle v\rangle^k-E(v)\cdot\nabla_v\,\langle v\rangle^k\\&=\Big[\langle v\rangle^{-k}\,\varDelta_v^{\sigma/2}\,\langle v\rangle^k-k\,(v\cdot E)\,\langle v\rangle^{-2}\Big]\,\langle v\rangle^k\,,\\&\le\Big[\langle v\rangle^{-k}\,\varDelta_v^{\sigma/2}\,\langle v\rangle^k-C\,\langle v\rangle^\beta\Big]\,\langle v\rangle^k\,,\end{aligned}$$

where the estimate $k\,(v\cdot E)\,\langle v\rangle^{-2}\ge C\,\langle v\rangle^\beta$ for some $C>0$ arises as a consequence of Proposition 4. According to [7, Lemma 3.1] (also see [6,28]), we have

$$\forall\,v\in\mathbb{R}^d\,,\quad\varDelta_v^{\sigma/2}\,\langle v\rangle^k\lesssim\langle v\rangle^{k-\sigma}\,,$$

under the condition that $k<\sigma=\gamma-\beta$. This again completes the proof of Inequality (3).

When f changes sign, it is possible to reduce the problem to the case $f\ge0$ as follows. In Case (a), we use Kato's inequality to assert that

$$\frac{f}{|f|}\,\varDelta_vf\le\varDelta_v|f|$$

in the sense of Radon measures (see [26, Lemma A] or, for instance, [13, Theorem 1.1]). Case (b) relies on the elementary observation that

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\tfrac{f}{|f|}\,\mathsf{L}_2f\,\langle v\rangle^k\,\mathrm{d}v\,\mathrm{d}v'&=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\,f'\,\frac{f}{|f|}\,F\,\langle v\rangle^k\,\mathrm{d}v\,\mathrm{d}v'-\int_{\mathbb{R}^d}\nu\,|f|\,\mathrm{d}v\\&\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\,|f'|\,F\,\langle v\rangle^k\,\mathrm{d}v\,\mathrm{d}v'-\int_{\mathbb{R}^d}\nu\,|f|\,\mathrm{d}v\,.\end{aligned}$$

In Case (c), the result follows from Kato's inequality extended to the fractional Laplacian as follows. Let us consider $\varphi_\varepsilon(s)=\sqrt{\varepsilon^2+s^2}$ and notice that

$$\begin{aligned}&\Big(\varDelta_v^{\sigma/2}\varphi_\varepsilon(f)\Big)(v)-\varphi_\varepsilon'(f(v))\,\Big(\varDelta_v^{\sigma/2}f\Big)(v)\\&\qquad=C_{d,\sigma}\iint_{\mathbb{R}^d}\frac{\varphi_\varepsilon(f(v'))-\varphi_\varepsilon(f(v))-\varphi_\varepsilon'(f(v))\,(f(v')-f(v))}{|v'-v|^{d+\sigma}}\,\mathrm{d}v\ge0\end{aligned}$$

because φ_ε is convex since $\varphi_\varepsilon''(s)=\varepsilon^2\,(\varepsilon^2+s^2)^{-3/2}$ and according for example to [30, Chapter 2]

$$C_{d,\sigma}=-\,\frac{2^\sigma}{\pi^{d/2}}\,\frac{\varGamma\big(\frac{d+\sigma}{2}\big)}{\varGamma\big(-\frac{\sigma}{2}\big)}>0\,.\tag{12}$$

By passing to the limit as $\varepsilon\to0$, we obtain

$$\frac{f}{|f|}\,\varDelta_v^{\sigma/2}f\le\varDelta_v^{\sigma/2}|f|\,.$$

In all cases, with $\mathsf{L} = \mathsf{L}_i$, $i = 1, 2, 3$, we have

$$\int_{\mathbb{R}^d} \frac{f}{|f|}\,\mathsf{L}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v \le \int_{\mathbb{R}^d} (\mathsf{L}|f|)\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v$$

and the problem is reduced to the case of a nonnegative distribution function f. □

4.4. A splitting of the evolution operator. We rely on the strategy of [24,27,34] by writing $\mathsf{L}-\mathsf{T}$ as the sum of a dissipative part C and a bounded part B such that $\mathsf{L}-\mathsf{T} = \mathsf{B}+\mathsf{C}$.

Lemma 4. *Under the assumptions of Lemma 3, let* $(k,k_*) \in (0,\gamma)\times(0,\gamma-\beta)$ *be such that* $k_* > k-\beta$, $a = \max\{a_k, a_{k_*}\}$, $R = \min\{R_k, R_{k_*}\}$, $\mathsf{C} := a\,\mathbb{1}_{B_R}$ *and* $\mathsf{B} := \mathsf{L}-\mathsf{T}-\mathsf{C}$. *Then for any* $t\in\mathbb{R}^+$, *we have:*

(i) $\|\mathsf{C}\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu)\to\mathrm{L}^1\left(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}\mu\right)} \le a\,(1+R^2)^{k_*/2}$,

(ii) $\|e^{t\mathsf{B}}\|_{\mathrm{L}^1\left(\langle v\rangle^{k}\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^1\left(\langle v\rangle^{k}\mathrm{d}x\,\mathrm{d}\mu\right)} \le 1$,

(iii) $\|e^{t\mathsf{B}}\|_{\mathrm{L}^1\left(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^1\left(\langle v\rangle^{k}\mathrm{d}x\,\mathrm{d}\mu\right)} \le c\,(1+t)^{\frac{k_*-k}{\beta}}$ *for some* $c>0$.

Proof. Property (i) is a consequence of the definition of C. Property (ii) follows from Lemma 3. Indeed, for any $g\in\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g}{|g|}\,\mathsf{B}\,g\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v \le \iint_{\mathbb{R}^d\times\mathbb{R}^d}\Big(a_k\,\mathbb{1}_{B_{R_k}} - a\,\mathbb{1}_{B_R} - b_k\,\langle v\rangle^{\beta}\Big)\,|g|\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v$$
$$\le -\,b_k\,\|g\|_{\mathrm{L}^1\left(\langle v\rangle^{k+\beta}\,\mathrm{d}x\,\mathrm{d}v\right)}\,.$$

To prove (iii), define $g := e^{t\mathsf{B}}\,g^{\mathrm{in}}$. By Hölder's inequality, we get

$$\|g\|_{\mathrm{L}^1\left(\langle v\rangle^k\mathrm{d}v\,\mathrm{d}x\right)} \le \|g\|^{\frac{k_*-k}{k_*-k-\beta}}_{\mathrm{L}^1\left(\langle v\rangle^{k+\beta}\,\mathrm{d}x\,\mathrm{d}v\right)}\,\|g^{\mathrm{in}}\|^{\frac{|\beta|}{k_*-k-\beta}}_{\mathrm{L}^1\left(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}v\right)}$$

and, as a consequence of the above contraction property,

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{g}{|g|}\,\mathsf{B}\,g\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v \le -\,b_k\,\left(\|g\|_{\mathrm{L}^1\left(\langle v\rangle^k\mathrm{d}v\,\mathrm{d}x\right)}\right)^{1+\frac{|\beta|}{k_*-k}}\,\|g^{\mathrm{in}}\|^{-\frac{|\beta|}{k_*-k}}_{\mathrm{L}^1\left(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}v\right)}\,,$$

so that by Grönwall's lemma, we obtain

$$\|g\|_{\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)} \le \left(\|g^{\mathrm{in}}\|^{-\frac{|\beta|}{k_*-k}}_{\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)} + \tfrac{b_k\,|\beta|}{k_*-k}\,\|g^{\mathrm{in}}\|^{-\frac{|\beta|}{k_*-k}}_{\mathrm{L}^1(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}v)}\,t\right)^{-\frac{k_*-k}{|\beta|}}$$
$$\le \left(1+\tfrac{k_*-k}{b_k\,|\beta|}\,t\right)^{-\frac{k_*-k}{|\beta|}}\,\|g^{\mathrm{in}}\|_{\mathrm{L}^1(\langle v\rangle^{k_*}\,\mathrm{d}x\,\mathrm{d}v)}\,.$$

□

4.5. The boundedness in $\mathrm{L}^1(F\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$.

Lemma 5. *Let* $d\ge1$, $\gamma>0\ge\beta$, $k\in(0,\gamma)$ *and assume that* (H) *holds. There exists a positive constant* $\mathcal{C}_k$ *such that, for any solution* f *of* (1) *with initial condition* $f^{\rm in}\in\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)$,

$$\forall\,t\ge0\,,\quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)}\le\mathcal{C}_k\,\|f^{\rm in}\|_{\mathrm{L}^1(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}v)}\,.$$

Proof. Let us consider the Duhamel formula

$$e^{t(\mathsf{L}-\mathsf{T})}=e^{t\mathsf{B}}+\int_0^t\mathsf{e}^{(t-s)\mathsf{B}}\,\mathsf{C}\,e^{s(\mathsf{L}-\mathsf{T})}\,\mathrm{d}s\,.$$

By Lemma 3, we know that

$$\|e^{t(\mathsf{L}-\mathsf{T})}\|_{\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)\to\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\le a\,(1+R^2)^{k_*/2}\,.$$

Using the estimates of Lemma 4, we get

$$\|e^{t(\mathsf{L}-\mathsf{T})}\|_{\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)\to\mathrm{L}^1(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}v)}\le1+a\,c\,(1+R^2)^{k_*/2}\int_0^t(1+s)^{\frac{k_*-k}{\beta}}\,\mathrm{d}s\,,$$

which is bounded uniformly in time with the choice $k_*-k>-\beta=|\beta|$. □

5. Interpolation inequalities

We refer to [41] for a general strategy for proving (4) which applies in particular to L_3 in Case (c). However, for the operators considered in this paper, direct estimates can be obtained as follows.

5.1. Hardy-Poincaré inequality and consequences.

Lemma 6. *Let* $d\ge1$ *and* $\gamma>0$. *We have the* Hardy-Poincaré inequality

$$\forall\,h\in\mathrm{L}^2(\langle v\rangle^{-2}\,F\,\mathrm{d}v)\,,\quad\int_{\mathbb{R}^d}|\nabla_vh|^2\,F\,\mathrm{d}v\ge2\,(d+\gamma)\int_{\mathbb{R}^d}|h-\overline{h}_{-2}|^2\,\langle v\rangle^{-2}\,F\,\mathrm{d}v \tag{13}$$

with $\overline{h}_{-2}:=\frac{\int_{\mathbb{R}^d}h\,\langle v\rangle^{-2}F\,\mathrm{d}v}{\int_{\mathbb{R}^d}\langle v\rangle^{-2}F\,\mathrm{d}v}$.

See [8] for a proof. We deduce the following interpolation inequality.

Corollary 2. *Let* $d\ge1$, $\gamma>0$ *and* $k\in(0,\gamma)$. *There exists a positive constant* $\mathcal{C}_1$ *such that, for any* $f\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ *such that* $\nabla_vh\in\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$ *where* $h=f/F$, *we have the inequality*

$$\mathcal{C}_1\,\|(1-\Pi)f\|^{2+\frac4k}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\le\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_vh|^2\,F\,\mathrm{d}x\,\mathrm{d}v\right)\|f\|^{\frac4k}_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

Proof. Let $\overline{h}_0 := \int_{\mathbb{R}^d} h\,F\,\mathrm{d}v$ and observe that

$$\int_{\mathbb{R}^d} |h-\overline{h}_0|^2\,F\,\mathrm{d}v = \inf_{H\in\mathbb{R}} \int_{\mathbb{R}^d} |h-H|^2\,F\,\mathrm{d}v \le \int_{\mathbb{R}^d} |h-\overline{h}_{-2}|^2\,F\,\mathrm{d}v.$$

Setting $g = h-\overline{h}_{-2}$, we deduce on the one hand from the Cauchy-Schwarz inequality that

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |h-\overline{h}_{-2}|^2\,F\,\mathrm{d}x\,\mathrm{d}v &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\,F\,\mathrm{d}x\,\mathrm{d}v \\ &\le \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\,\frac{F}{\langle v\rangle^2}\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{k}{k+2}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\,\langle v\rangle^k\,F\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{2}{k+2}},\end{aligned}$$

and we deduce that

$$|g|^2 \le \frac{1}{2}\left(|h|^2+\overline{h}_{-2}^2\right) \le \frac{1}{2}\left(|h|^2+\frac{\overline{h}_0^2}{\int_{\mathbb{R}^d}\langle v\rangle^{-2}\,F\,\mathrm{d}v}\right)$$

using $\langle v\rangle \ge 1$ and the definition of $\overline{h}_{-2}$ on the other hand. Collecting these estimates with the result of Lemma 6 shows that

$$\frac{2^{1+\frac{2}{k}}\,(d+\gamma)\,\|(1-\Pi)f\|^{2+\frac{4}{k}}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}}{\left(\|f\|^2_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}+\mathsf{c}_k\,\|\Pi f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\right)^{\frac{2}{k}}} \le \int_{\mathbb{R}^d} |\nabla_v h|^2\,F\,\mathrm{d}v$$

where $\mathsf{c}_k := \int_{\mathbb{R}^d}\langle v\rangle^k\,F\,\mathrm{d}v / \int_{\mathbb{R}^d}\langle v\rangle^{-2}\,F\,\mathrm{d}v$. This completes the proof after observing that

$$\|\Pi f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \|f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \|f\|_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}\,.$$

□

5.2. A gap inequality for the scattering operator. Let $\mathsf{L} = \mathsf{L}_2$ be the scattering operator of Case (b).

Lemma 7. *Let $\gamma > \max\{0,\beta\}$. Assume that (H1) holds and that $\mathrm{b}(v,v') \ge Z^{-1}\,\langle v\rangle^\beta\,\langle v'\rangle^\beta$ for any v, $v'\in\mathbb{R}^d$. Then we have*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,(h-h')^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v' \ge \Lambda \int_{\mathbb{R}^d} \left|h-\overline{h}_\beta\right|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v$$

for any $h\in\mathrm{L}^2(\mathrm{d}v)$, with $\Lambda := \frac{2}{Z}\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v$ and $\overline{h}_\beta := \frac{\int_{\mathbb{R}^d} h\,F\,\langle v\rangle^\beta\,\mathrm{d}v}{\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v}$.

Notice that here we do not assume the upper bound in (H2) and consider any $\beta\in(-\infty,\gamma)$.

Proof. Using (H1), we have

$$
\begin{aligned}
&2\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,(h'-h)\,h\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&=\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,\left(2\,h'h-h^2\right)\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'-\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v',v)\,h^2F\,F'\,\mathrm{d}v\,\mathrm{d}v'\,.
\end{aligned}
$$

Exchanging variables v and v' gives

$$
\begin{aligned}
-\iint_{\mathbb{R}^d\times\mathbb{R}^d} &\mathrm{b}(v,v')\,(h'-h)\,h\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&=\frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,(h-h')^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\,.
\end{aligned}
$$

By assumption, we know that $\mathrm{b}(v,v')\ge Z^{-1}\,\langle v\rangle^\beta\,\langle v'\rangle^\beta$ and observe that

$$
\begin{aligned}
&\iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle v\rangle^\beta\,\langle v'\rangle^\beta\,(h-h')^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\
&\quad=2\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v\int_{\mathbb{R}^d}\frac{|f|^2}{F}\,\langle v\rangle^\beta\,\mathrm{d}\mu-2\left(\int_{\mathbb{R}^d} f\,\langle v\rangle^\beta\,\mathrm{d}v\right)^2\\
&\quad=2\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v\int_{\mathbb{R}^d}\left|\frac{f}{F}-\frac{\int_{\mathbb{R}^d} f\,\langle v\rangle^\beta\,\mathrm{d}v}{\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v}\right|^2\langle v\rangle^\beta\,F\,\mathrm{d}v\\
&\quad=\varLambda\,Z\int_{\mathbb{R}^d}\left|h-\overline{h}_\beta\right|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v\,.
\end{aligned}
$$

□

Next we deduce the following interpolation inequality.

Corollary 3. *Under the assumptions of Lemma 7, for any $k\in(0,\gamma)$, there exists a positive constant $\mathcal{C}_2$ such that, for any $f\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$,*

$$
\begin{aligned}
&\mathcal{C}_2\,\|(1-\mathsf{\Pi})f\|^{2+2\frac{|\beta|}{k}}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\le\left(-\int_{\mathbb{R}^d} f\,\mathsf{L}_2 f\,\mathrm{d}\mu\right)\|f\|^{2\frac{|\beta|}{k}}_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}\quad\text{if}\quad\beta<0\,,\\
&\mathcal{C}_2\,\|(1-\mathsf{\Pi})f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\le-\int_{\mathbb{R}^d} f\,\mathsf{L}_2 f\,\mathrm{d}\mu\quad\text{if}\quad\beta\ge0\,.
\end{aligned}
$$

Proof. If $\beta\ge0$, the result is a straightforward consequence of Lemma 7 using $\langle v\rangle^\beta\ge1$ and

$$
\begin{aligned}
\int_{\mathbb{R}^d}|h-\overline{h}_\beta|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v\ge&\int_{\mathbb{R}^d}|h-\overline{h}_\beta|^2\,F\,\mathrm{d}v\\
&\ge\int_{\mathbb{R}^d}|h-\overline{h}_0|^2\,F\,\mathrm{d}v=\|(1-\mathsf{\Pi})f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}\,.
\end{aligned}
$$

Assume next that $\beta < 0$ and let $g := h - \overline{h}_\beta$. We deduce from the Cauchy-Schwarz inequality that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\, F\,\mathrm{d}x\,\mathrm{d}v \le \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{k}{k-\beta}} \left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |g|^2\,\langle v\rangle^k\,F\,\mathrm{d}x\,\mathrm{d}v\right)^{\frac{|\beta|}{k-\beta}}.$$

The next point is to observe that

$$\int_{\mathbb{R}^d} |h-\overline{h}_0|^2\,F\,\mathrm{d}v = \inf_{H\in\mathbb{R}} \int_{\mathbb{R}^d} |h-H|^2\,F\,\mathrm{d}v \le \int_{\mathbb{R}^d} |h-\overline{h}_\beta|^2\,F\,\mathrm{d}v = \int_{\mathbb{R}^d} |g|^2\,F\,\mathrm{d}v$$

on the one hand, and that

$$|g|^2 \le \frac{1}{2}\left(|h|^2+\overline{h}_\beta^2\right) \le \frac{1}{2}\left(|h|^2 + \frac{\overline{h}^2}{\int_{\mathbb{R}^d}|g|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v}\right)$$

using $\langle v\rangle \ge 1$ and the definition of $\overline{h}_\beta$ on the other hand. Collecting these estimates with the result of Lemma 7 completes the proof.

With $h = f/F$, we recall that, as a consequence of Lemma 7,

$$-\int_{\mathbb{R}^d} f\,\mathsf{L}_2 f\,\mathrm{d}\mu = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{b}(v,v')\,(h-h')^2\,F\,F'\,\mathrm{d}v\,\mathrm{d}v' \ge \frac{\Lambda}{2}\int_{\mathbb{R}^d} |g|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v\,.$$

Using $\|\mathsf{\Pi} f\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \|f\|_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)}$, the result follows by collecting the estimates. □

5.3. Fractional Fokker-Planck operator: an interpolation inequality. Let us compute $\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu$. We recall that in Case (c), $\mathsf{L}_3 f = \Delta_v^{\sigma/2} f + \nabla_v\cdot(E\,f)$. With $h = f/F$, we have

$$\int_{\mathbb{R}^d} f\,\Delta_v^{\sigma/2} f\,\mathrm{d}\mu = C_{d,\sigma}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{h^2 - h\,h'}{|v-v'|^{d+\sigma}}\,F\,\mathrm{d}v\,\mathrm{d}v'.$$

On the other hand, we know that $h\,\nabla_v\cdot(f\,E) = h\,\nabla_v\cdot(h\,F\,E) = \frac{1}{2}\nabla_v(h^2)\cdot(F\,E) + h^2\,\nabla_v\cdot(F\,E)$ and after an integration by parts, we obtain

$$\int_{\mathbb{R}^d} h\,\nabla_v\cdot(f\,E)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d} h^2\,\nabla_v\cdot(F\,E)\,\mathrm{d}v = \frac{1}{2}\int_{\mathbb{R}^d} h^2\,\Delta^{\sigma/2}F\,\mathrm{d}v\,.$$

After exchanging the variables v and v', we arrive at

$$\int_{\mathbb{R}^d} h\,\nabla_v\cdot(f\,E)\,\mathrm{d}v = -\frac{C_{d,\sigma}}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{h^2-(h')^2}{|v-v'|^{d+\sigma}}\,F\,\mathrm{d}v\,\mathrm{d}v'.$$

Altogether, this means that

$$-\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu = \frac{C_{d,\sigma}}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|h-h'|^2}{|v-v'|^{d+\sigma}}\,F\,\mathrm{d}v\,\mathrm{d}v' = \frac{C_{d,\sigma}}{4}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|h-h'|^2}{|v-v'|^{d+\sigma}}\,(F+F')\,\mathrm{d}v\,\mathrm{d}v'\,.$$

Corollary 4. *Let $d \ge 1$, $\gamma > 0$, $\sigma \in (0,2)$, $\beta = \gamma - \sigma$ and $k \in (0,\gamma)$. With the notation of Corollary 3, there exists a positive constant $\mathcal{C}_3$ such that, for any $f \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$, we have the inequality*

$$\mathcal{C}_3\,\|(1-\Pi)f\|^{2+2\,\frac{|\beta|}{k}}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \left(-\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu\right)\|f\|^{2\,\frac{|\beta|}{k}}_{\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)} \quad \textit{if} \quad \beta < 0\,,$$
$$\mathcal{C}_3\,\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le -\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu \quad \textit{if} \quad \beta \ge 0\,.$$

Proof. From the elementary estimate

$$\forall\,(v,v') \in \mathbb{R}^d \times \mathbb{R}^d\,, \quad \langle v\rangle^\beta\,\langle v'\rangle^\beta\,F\,F' \le \kappa\,\frac{F+F'}{|v-v'|^{d+\sigma}}$$

with $\kappa = c_\gamma\,\sup_{(v,v')\in\mathbb{R}^d\times\mathbb{R}^d} \frac{\langle v\rangle^{d+\gamma}+\langle v'\rangle^{d+\gamma}}{\langle v\rangle^{d+\sigma}\,\langle v'\rangle^{d+\sigma}}\,|v-v'|^{d+\sigma}$, we deduce that

$$-\int_{\mathbb{R}^d} f\,\mathsf{L}_2 f\,\mathrm{d}\mu \le -\kappa\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu$$

and conclude by Corollary 3 with $\mathcal{C}_3 = \mathcal{C}_2/\kappa$. □

As a side result, let us observe that we obtain a *fractional Poincaré inequality* as in [40, Corollary 1.2, (1)], with an explicit constant, that goes as follows.

Corollary 5. *Under the same assumptions as in Corollary 4,*

$$-\int_{\mathbb{R}^d} f\,\mathsf{L}_3 f\,\mathrm{d}\mu \ge \kappa\,\Lambda \int_{\mathbb{R}^d} \left|h - \overline{h}_\beta\right|^2\,\langle v\rangle^\beta\,F\,\mathrm{d}v$$

where κ is as in the proof of Corollary 4 and Λ is the constant of Lemma 7.

5.4. Convergence to the local equilibrium: microscopic coercivity. We can summarize Lemma 6, Lemma 7 and Corollary 5 as

$$\mathcal{C}\,\|f - h_\beta\,F\|^2_\beta \le -\langle f\,\mathsf{L} f\rangle$$

for positive constant $\mathcal{C}$, where $\overline{h}_\beta = \int_{\mathbb{R}^d} h\,F\,\langle v\rangle^\beta\,\mathrm{d}v/\int_{\mathbb{R}^d} F\,\langle v\rangle^\beta\,\mathrm{d}v$ and $h = f/F$. Here $\mathsf{L} = \mathsf{L}_1$, L_2 or L_3 respectively in Cases (a), (b) or (c). In the homogeneous case, an additional Hölder inequality establishes Inequality (4) of Section 1. The same strategy can be applied in the non-homogeneous case after integrating with respect to $x \in \mathbb{R}^d$.

Proof (Proof of Proposition 2). It is a straightforward consequence of Proposition 1 on the one hand, and of Corollaries 2, 3 and 4 if, respectively, $\mathsf{L} = \mathsf{L}_1$, L_2 or L_3. □

As a an alternative formulation of Proposition 2 and in preparation for the case $\beta+\gamma\le 0$ (see Section (8.1)), let us collect some additional observations. The inequality

$$\|(1-\Pi)f\|_\eta^2 \le \left(-\langle f,\mathsf{L}f\rangle\right)^\theta \,\|(1-\Pi)f\|_k^{2\,(1-\theta)}$$

with $\theta=\frac{k-\eta}{k-\beta}$ can be rewritten with $\mathsf{x}_\zeta:=\|(1-\Pi)f\|_\zeta^2$ and $\mathsf{z}:=\left(-\langle f,\mathsf{L}f\rangle\right)$ as

$$\mathsf{x}_\eta\le \mathsf{z}^\theta\,\mathsf{x}_k^{1-\theta}=\left(R^{-1/\theta}\,\mathsf{z}\right)^\theta\left(R^{1/(1-\theta)}\mathsf{x}_k\right)^{1-\theta}\le\theta\,R^{-1/\theta}\,\mathsf{z}+(1-\theta)\,R^{1/(1-\theta)}\mathsf{x}_k$$

for any $R>0$, by Young's inequality. This amounts to

$$\mathsf{z}\ge\frac{1}{\theta}\,R^{\frac{1}{\theta}}\,\mathsf{x}_\eta-\frac{1-\theta}{\theta}\,R^{\frac{1}{\theta}+\frac{1}{1-\theta}}\,\mathsf{x}_k=r\,\mathsf{x}_\eta-(1-\theta)\,\theta^{\frac{\theta}{1-\theta}}\,r^{\frac{1}{1-\theta}}\,\mathsf{x}_k\,.$$

An integration with respect to x shows the following result.

Corollary 6. *Let $\theta=\frac{k-\eta}{k-\beta}$. Under the assumptions of Proposition 2, we have*

$$-\int_{\mathbb{R}^d}\langle f,\mathsf{L}f\rangle\,\mathrm{d}x\ge r\,\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\eta\mathrm{d}v)}
-(1-\theta)\,\theta^{\frac{\theta}{1-\theta}}\,r^{\frac{1}{1-\theta}}\,\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^k\mathrm{d}v)}\tag{14}$$

for any $f\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$ and for any $r>0$.

6. Technical estimates

This section is devoted to estimates that can be skipped at first reading. With the notation $\|g\|_\eta^2:=\int_{\mathbb{R}^d}|g|^2\,\langle v\rangle^\eta\,\mathrm{d}\mu$, we want bound μ_L and λ_L defined by

$$\mu_\mathsf{L}(\xi):=\|\mathsf{L}^*\big((v\cdot\xi)\,\varphi(\xi,v)\,F\big)\|_\eta^2\quad\lambda_\mathsf{L}:=\|\mathsf{L}^*(\psi\,F)\|_\eta^2\,,$$

some parameter $\eta\in(-\gamma,\gamma)$, were L^* denotes the dual of L in $\mathrm{L}^2(\mathrm{d}\mu)$, and

$$\begin{aligned}
\lambda_k&:=\textstyle\int_{\mathbb{R}^d}|v\cdot\xi|^k\,\langle v\rangle^{-2}\,F\,\mathrm{d}v=\left\langle F,\,|\mathsf{T}|^k\,\psi\,F\right\rangle\,,\\
\mu_k&:=\textstyle\int_{\mathbb{R}^d}|v\cdot\xi|^k\,\varphi(v)\,F\,\mathrm{d}v=\left\langle F,\,|\mathsf{T}|^k\,\varphi\right\rangle\,,\\
\tilde\lambda_k&:=\left\||v\cdot\xi|^k\,\psi\,F\right\|_\eta\,,\\
\tilde\mu_k&:=\left\||v\cdot\xi|^k\,\varphi\,F\right\|_\eta\,.
\end{aligned}\tag{15}$$

Notice that only the case $\beta=\eta$ will be needed if $\beta+\gamma>0$. When $\beta+\gamma\le 0$, we shall assume that $\eta\in(-\gamma,0)$. See Section 8.1 for consequences. In Case (c) with $\mathsf{L}=\mathsf{L}_3$, we need a bound on E. Let us start by this estimate.

6.1. Steady states and force field for the fractional Laplacian with drift. This appendix is devoted to the Case (c) of the collision operator L, that is, to $\mathsf{L}_3 f = \Delta_v^{\sigma/2} f + \nabla_v \cdot (E\,f)$. Our goal here is to prove that the *collision frequency* $\nu(v)$ behaves like $|v|^\beta$ with $\beta = \gamma - \sigma$ as $|v| \to +\infty$, as claimed in Section 2.1. By Definition (8) of the force field E, we know that

$$\nabla_v \cdot (E\,F) = -\Delta_v^{\sigma/2} F = -\,\nabla_v \cdot \Big(\nabla_v (-\Delta_v)^{\frac{\sigma-2}{2}} F\Big)\,,$$

and this implies that, up to an additive constant,

$$E\,F = -\,\nabla_v (-\Delta_v)^{\frac{\sigma-2}{2}} F = -\,\nabla_v \left(\frac{C_{d,\sigma}}{|v|^{d+\sigma-2}} * \frac{c_\gamma}{\langle v\rangle^{d+\gamma}}\right)$$

where c_γ and $C_{d,\sigma}$ are given respectively by (2) and (12).

Proposition 4. *Assume that $\gamma > 0$, $\sigma \in (0,2)$ and let $\beta = \gamma - \sigma$. There is a positive function $G \in \mathrm{L}^\infty(\mathbb{R}^d)$ with $1/G \in \mathrm{L}^\infty(B_0^c(1))$ such that E is given by*

$$\forall\, v \in \mathbb{R}^d\,, \quad E(v) = G(v)\,\langle v\rangle^\beta\, v\,.$$

Proof. Let $u(v) = -\nabla_v \big(\frac{1}{|v|^{d+\sigma-2}} * \frac{1}{\langle v\rangle^{d+\gamma}}\big)(v)$ so that $E(v) = C_{d,\sigma}\,\langle v\rangle^{d+\gamma}\,u(v)$. Since

$$u(v) = (d+\gamma)\left(\frac{1}{|v|^{d+\sigma-2}} * \frac{v}{\langle v\rangle^{d+\gamma+2}}\right)$$

where $v\,\langle v\rangle^{-(d+\gamma+2)} \in C^\infty(\mathbb{R}^d) \cap \mathrm{L}^1(\mathrm{d}v)$, and $\sigma < 2$, one has $u \in C^1_{\mathrm{loc}}(\mathbb{R}^d)$ and $u(0) = 0$ which proves the result in $B_1(0)$. We look for an estimate of $u(v) \cdot v$ from above and below on $B_0^c(1)$. Notice that u can also be written as

$$u(v) = (d+\sigma-2)\left(\frac{v}{|v|^{d+\sigma}} * \frac{1}{\langle v\rangle^{d+\gamma}}\right). \tag{16}$$

Depending on the integrability at infinity of $v/|v|^{d+\sigma}$, that is, whether $\sigma \in (0,1)$ or not, we have to distinguish two cases.

• **Case** $\sigma \in (0,1)$**.** Using (16), we have the estimates

$$\begin{aligned}
\left|\int_{|w|\ge\langle v\rangle/2} \frac{w}{|w|^{d+\sigma}}\,\frac{\mathrm{d}w}{\langle w-v\rangle^{d+\gamma}}\right| &\le \frac{2^{d+\sigma-1}}{\langle v\rangle^{d+\sigma-1}} \int_{\mathbb{R}^d} \frac{\mathrm{d}w}{\langle w\rangle^{d+\gamma}}\,,\\
\left|\int_{|w|<\langle v\rangle/2} \frac{w}{|w|^{d+\sigma}}\,\frac{\mathrm{d}w}{\langle w-v\rangle^{d+\gamma}}\right| &\le \left(\int_{|w|<\langle v\rangle/2} \frac{\mathrm{d}w}{|w|^{d+\sigma-1}}\right) \frac{2^{d+\gamma}}{|v|^{d+\gamma}}\\
&\le \frac{2^{d+\sigma-1}\,\omega_d}{(1-\sigma)\,|v|^{d+\gamma+\sigma-1}}\,,
\end{aligned}$$

and obtain

$$\forall\, v \in \mathbb{R}^d\,, \quad |u(v)\cdot v| \le |u(v)|\,|v| \lesssim |v|^{-(d+\sigma-2)}\,.$$

To get a bound from below on $u(v)\cdot v$, we cut the integral in two pieces and use the fact that $|v|>1$ and $|w-v|<1/2$ implies $w\cdot v>0$. First

$$\int_{\substack{|w-v|>1/2\\|w+v|>1/2}} \frac{w\cdot v}{|w|^{d+\sigma}}\,\frac{\mathrm{d}w}{\langle w-v\rangle^{d+\gamma}} = \left(\int_{\substack{|w-v|>1/2\\w\cdot v>0}} + \int_{\substack{|w+v|>1/2\\w\cdot v<0}}\right)\frac{w\cdot v}{|w|^{d+\sigma}}\,\frac{\mathrm{d}w}{\langle w-v\rangle^{d+\gamma}}$$
$$= \int_{\substack{|w-v|>1/2\\w\cdot v>0}} \left(\frac{1}{\langle w-v\rangle^{d+\gamma}} - \frac{1}{\langle w+v\rangle^{d+\gamma}}\right)\frac{w\cdot v}{|w|^{d+\sigma}}\,\mathrm{d}w\,,$$

which is positive since $\langle w+v\rangle^2 - \langle w-v\rangle^2 = 2\,w\cdot v \ge 0$. The remaining terms are dealt with as follows

$$\int_{\substack{|w-v|\le 1/2\\ \text{or}\\ |w+v|\le 1/2}} \frac{w\cdot v}{|w|^{d+\sigma}}\,\frac{\mathrm{d}w}{\langle w-v\rangle^{d+\gamma}}$$
$$= \int_{|w-v|<\frac12} \left(\frac{1}{\langle w-v\rangle^{d+\gamma}} - \frac{1}{\langle w+v\rangle^{d+\gamma}}\right)\frac{w\cdot v}{|w|^{d+\sigma}}\,\mathrm{d}w$$
$$\ge \left((4/5)^{d+\gamma} - (2/5)^{d+\gamma}\right)\int_{|w-v|<\frac12}\frac{w\cdot v}{|w|^{d+\sigma}}\,\mathrm{d}w\,,$$

since $|w+v|\ge 2\,|v|-|w-v|\ge\frac32$. Finally, if $|v|>1$ and $|w-v|<\frac12$, we get

$$2\,w\cdot v = |v|^2+|w|^2-|w-v|^2 \ge |v|^2-\frac12 \ge \frac{|v|^2}{2}\,,$$
$$|w|\le|v|+|w-v|\le 2\,|v|\,,$$

so that

$$\int_{|w-v|<\frac12}\frac{w\cdot v}{|w|^{d+\sigma}}\,\mathrm{d}w \ge \frac{|B_0(1/2)|}{2^{d+\sigma+2}}\,\frac{1}{|v|^{d+\sigma-2}}\,.$$

This implies $u(v)\cdot v\ge C\,|v|^{-(d+\sigma-2)}$ for some $C>0$. Since u is radial, we proved that

$$u(v) = G(v)\,\frac{v}{|v|^{d+\sigma}}$$

where $G\in \mathrm{L}^\infty(\mathbb{R}^d)$ and $G^{-1}\in\mathrm{L}^\infty(B_0^c(1))$ and the conclusion holds with $\beta=\gamma-\sigma$.

• **Case** $\sigma\in[1,2)$. The gradient of $v\mapsto|v|^{2-d-\sigma}$ is a distribution of order 1 that can be defined as a *principal value*. Indeed, in the sense of distributions, for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$, we have

$$\left\langle\nabla_v|v|^{2-d-\sigma},\varphi\right\rangle_{\mathcal{D}',\mathcal{D}} = -\int_{\mathbb{R}^d}\frac{\nabla_v\varphi(v)}{|v|^{d+\sigma-2}}\,\mathrm{d}v = -\int_{\mathbb{R}^d}\frac{\nabla_v(\varphi(v)-\varphi(0))}{|v|^{d+\sigma-2}}\,\mathrm{d}v$$
$$= (d+\sigma-2)\int_{\mathbb{R}^d}\frac{v}{|v|^{d+\sigma}}\,(\varphi(v)-\varphi(0))\,\mathrm{d}v$$
$$=: (d+\sigma-2)\left\langle \mathrm{pv}\left(\frac{v}{|v|^{d+\sigma}}\right),\varphi\right\rangle_{\mathcal{D}',\mathcal{D}}\,.$$

Identity (16) is replaced by

$$\frac{u(v)}{d+\sigma-2}=\mathrm{pv}\left(\frac{v}{|v|^{d+\sigma}}\right)*\frac{1}{\langle v\rangle^{d+\gamma}}=\int_{\mathbb{R}^d}\frac{w}{|w|^{d+\sigma}}\left(\frac{1}{\langle v-w\rangle^{d+\gamma}}-\frac{1}{\langle v\rangle^{d+\gamma}}\right)\mathrm{d}w\,,$$

so that, after computations like the ones in the proof of Lemma 10,

$$\frac{|u(v)|}{d+\sigma-2}\le\int_{\mathbb{R}^d}\frac{1}{|w-v|^{d+\sigma-1}}\left|\frac{1}{\langle w\rangle^{d+\gamma}}-\frac{1}{\langle v\rangle^{d+\gamma}}\right|\mathrm{d}w\lesssim\frac{1}{\langle v\rangle^{d+\sigma-2}}\,.$$

Now estimate $u(v)\cdot v$ by

$$\int_{|w|\ge\frac12}\frac{w\cdot v}{|w|^{d+\sigma}}\left(\frac{1}{\langle v-w\rangle^{d+\gamma}}-\frac{1}{\langle v\rangle^{d+\gamma}}\right)\mathrm{d}w\\
=\int_{|w|\ge\frac12}\frac{w\cdot v}{|w|^{d+\sigma}}\frac{1}{\langle v-w\rangle^{d+\gamma}}\,\mathrm{d}w\gtrsim\frac{1}{\langle v\rangle^{d+\sigma-2}}\,.$$

and

$$\left|\int_{|w|<\frac12}\frac{w\cdot v}{|w|^{d+\sigma}}\left(\frac{1}{\langle v-w\rangle^{d+\gamma}}-\frac{1}{\langle v\rangle^{d+\gamma}}\right)\mathrm{d}w\right|\\
\le\sup_{w\in B_v(1/2)}\frac{(d+\gamma)\,|v|}{\langle w\rangle^{d+\gamma+1}}\int_{|w|<\frac12}\frac{\mathrm{d}w}{|w|^{d+\sigma-2}}\lesssim\frac{1}{\langle v\rangle^{d+\gamma}}\,.$$

The result follows from the fact that $d+\gamma>d>d+\sigma-2$. □

6.2. Quantitative estimates of μ_{L} and λ_{L}.

Proposition 5. *Under Assumption (H), if $\eta\in(-\gamma,\gamma)$ is such that $\eta\ge\beta$, then λ_{L} is finite and*

$$\forall\,\xi\in\mathbb{R}^d\,,\quad\mu_{\mathsf{L}}(\xi)\lesssim\frac{|\xi|^{\alpha'}}{\langle\xi\rangle^{\alpha'}}\,.$$

6.2.1. Generalized Fokker-Planck operators.

Lemma 8. *With $\mathsf{L}=\mathsf{L}_1$, we have*

$$\mu_{\mathsf{L}}\lesssim|\xi|^{\min\left\{2,\frac{\gamma+4+\eta}{3}\right\}}\mathbb{1}_{|\xi|\le1}+|\xi|^{-2}\,\mathbb{1}_{|\xi|\ge1}\quad\textit{and}\quad\lambda_{\mathsf{L}}\lesssim1\,.$$

We recall that L_1 is self-adjoint.

Proof. Let us start by estimating μ_{L}. With

$$F^{-1}\,\mathsf{L}\big((v\cdot\xi)\,\varphi\,F\big)=\nabla_v\cdot\Big(F\,\nabla_v\,\big((v\cdot\xi)\,\varphi\big)\Big)\\
=\Delta_v\big((v\cdot\xi)\,\varphi\big)-(d+\gamma)\,\frac{v}{\langle v\rangle^2}\cdot\nabla_v\big((v\cdot\xi)\,\varphi\big)$$

and $\nabla_v\big((v\cdot\xi)\,\varphi\big)=\varphi\,\xi+(v\cdot\xi)\,\nabla_v\varphi$, $\Delta_v\big((v\cdot\xi)\,\varphi\big)=2\,\xi\cdot\nabla_v\varphi+(v\cdot\xi)\,\Delta_v\varphi$, we end up with

$$F^{-1}\,\mathsf{L}\big((v\cdot\xi)\,\varphi\big)=2\,\xi\cdot\nabla_v\varphi+(v\cdot\xi)\left(\Delta_v\varphi-\frac{(d+\gamma)}{\langle v\rangle^2}\,(\varphi+v\cdot\nabla_v\varphi)\right).$$

We recall that $\beta=-\,2$ and

$$\varphi(\xi,v)=\frac{\langle v\rangle^2}{A(\xi,v)}\quad\text{where}\quad A(\xi,v):=1+\langle v\rangle^6\;|\xi|^2\,,$$

so that

$$\nabla_v\varphi=\Big(2\,A^{-1}-6\,\langle v\rangle^6\,|\xi|^2A^{-2}\Big)\,v=2\,\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\frac{v}{A^2}$$

and

$$\xi\cdot\nabla_v\varphi=2\,\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\frac{v\cdot\xi}{A^2}\,,\quad v\cdot\nabla_v\varphi=2\,\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\frac{|v|^2}{A^2}\,.$$

Using $\langle v\rangle^6\,|\xi|^2\le A$, we can readily estimate

$$|\xi\cdot\nabla_v\varphi|\lesssim|v\cdot\xi|A^{-1},\quad |v\cdot\nabla_v\varphi|\lesssim\langle v\rangle^2\,A^{-1}\,.$$

The last part to estimate is

$$\begin{aligned}\Delta_v\varphi&=2\,\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\nabla_v\cdot\Big(\frac{v}{A^2}\Big)+2\,\nabla_v\cdot\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\frac{v}{A^2}\\&=\frac{2}{A^2}\,\Big(1-2\,\langle v\rangle^6\,|\xi|^2\Big)\,\Big(d+12\,|v|^2\,\langle v\rangle^4\,|\xi|^2\,A^{-1}\Big)-24\,|v|^2\,\langle v\rangle^4\,|\xi|^2\,A^{-2}\,,\end{aligned}$$

from which we deduce that

$$|\Delta_v\varphi|\lesssim A^{-1}\,.$$

Combining previous estimates, we thus end up with

$$\big|F^{-1}\,\mathsf{L}\big((v\cdot\xi)\,\varphi\big)\big|\lesssim|v\cdot\xi|\,A^{-1}\,.$$

This provides us with the estimate

$$\mu_{\mathsf{L}}(\xi)=\|\mathsf{L}\,((v\cdot\xi)\,\varphi(\xi,\cdot)\,F)\|^2_{\mathrm{L}^2(\langle v\rangle^\eta\,\mathrm{d}\mu)}\lesssim\int_{\mathbb{R}^d}\frac{|v\cdot\xi|^2}{\big(1+\langle v\rangle^6\,|\xi|^2\big)^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}}$$

which allows us to conclude by elementary computations. Similar computations will be detailed in the proof of Lemma 11.

Next we have to estimate λ_{L}. After recalling that $\psi=\langle v\rangle^{-2}$, we observe that

$$\big|F^{-1}\,\mathsf{L}\,(\psi\,F)\big|=\Big|\Delta_v\psi-(d+\gamma)\,\frac{v}{\langle v\rangle^2}\cdot\nabla_v\psi\Big|$$

is a bounded quantity. Since $\langle v\rangle^\eta\,F\in\mathrm{L}^1(\mathbb{R}^d)$, we conclude that λ_{L} is bounded. □

6.2.2. Scattering collision operators.

Lemma 9. *Assume that* (H) *holds. With* $\mathsf{L} = \mathsf{L}_2$*, we have*

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\min\left\{1,1+\frac{\gamma-\eta-2}{|1-\beta|}\right\}}\,\mathbb{1}_{|\xi|<1} + |\xi|^{-2}\,\mathbb{1}_{|\xi|\ge 1} \quad \textit{and} \quad \lambda_{\mathsf{L}} \lesssim 1\,.$$

Proof. To estimate μ_{L}, we write

$$\begin{aligned} F^{-1}\,\mathsf{L}^*\left((v\cdot\xi)\,\varphi\,F\right) &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\big((v'\cdot\xi)\,\varphi(v') - (v\cdot\xi)\,\varphi(v)\big)\,F(v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,(v'\cdot\xi)\,\varphi(v')\,F(v')\,\mathrm{d}v' - (v\cdot\xi)\,\varphi(v)\,\nu(v) \end{aligned}$$

The Cauchy-Schwarz inequality yields

$$\begin{aligned} &\int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} \mathrm{b}(v',v)\,(v'\cdot\xi)\,\varphi(v')\,F'\,\mathrm{d}v'\right|^2 \langle v\rangle^\eta\,F\,\mathrm{d}v \\ &\le \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \big|(v'\cdot\xi)\,\varphi(v')\big|^2\,\langle v\rangle^\beta\,F'\,\mathrm{d}v'\right)\left(\int_{\mathbb{R}^d} \frac{\mathrm{b}(v',v)^2}{\nu(v')}F'\,\mathrm{d}v'\right)\langle v\rangle^\eta\,F\,\mathrm{d}v \\ &\le \mathcal{C}_{\mathrm{b}} \int_{\mathbb{R}^d} \big|\nu(v)\,(v\cdot\xi)\,\varphi(v)\big|^2\,\langle v\rangle^\eta\,F\,\mathrm{d}v\,, \end{aligned}$$

where, by Assumption (H2), $\mathcal{C}_{\mathrm{b}} = \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\mathrm{b}(v',v)^2}{\nu(v')\,\nu(v)}F\,F'\,\mathrm{d}v\,\mathrm{d}v'$ is finite. Hence

$$\int_{\mathbb{R}^d} \big|\nu(v)\,(v\cdot\xi)\,\varphi\big|^2\,\langle v\rangle^\eta\,F\,\mathrm{d}v \le C \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{\left(1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2\right)^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}}$$

for some positive constant C, which provides us with the result.

The estimate for λ_{L} arises from

$$\begin{aligned} F^{-1}\,\mathsf{L}^*\,(\psi\,F) &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\big(\psi(v') - \psi(v)\big)\,F(v')\,\mathrm{d}v' \\ &= \int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\psi(v')\,F(v')\,\mathrm{d}v' - \nu(v)\,\psi(v)\,. \end{aligned}$$

Again, the Cauchy-Schwarz inequality yields

$$\begin{aligned} \left|\int_{\mathbb{R}^d} \mathrm{b}(v',v)\,\psi(v')\,F'\,\mathrm{d}v'\right|^2 &\le \int_{\mathbb{R}^d} |\psi(v')|^2\,\langle v'\rangle^\beta\,F'\,\mathrm{d}v' \int_{\mathbb{R}^d} \frac{\mathrm{b}(v',v)^2}{\nu(v')}F'\,\mathrm{d}v' \\ &\le \mathcal{C}_{\mathrm{b}}\,|\nu(v)\,\psi(v)|^2\,, \end{aligned}$$

so that $\left|F^{-1}\,\mathsf{L}^*\,(\psi\,F)\right| \le \left(\mathcal{C}_{\mathrm{b}}^{1/2}+1\right)\,|\nu(v)\,\psi(v)|$. It follows from

$$\int_{\mathbb{R}^d} |\nu(v)\,\psi(v)|^2\,\langle v\rangle^\eta\,F\,\mathrm{d}v \le C \int_{\mathbb{R}^d} \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta-2\beta+4}}$$

that $\lambda_{\mathsf{L}} \lesssim 1$ because $\gamma-\eta-2\,\beta+4 > \gamma-\beta+\eta-\beta > 0$. □

6.2.3. Fractional Fokker-Planck operators.

Lemma 10. *For any $\sigma \in (0,2)$, we have*

$$|\Delta^{\sigma/2}\big((v\cdot\xi)\,\varphi\big)| \lesssim |\xi|^{\frac{\alpha'}{2}}\,\mathbb{1}_{|\xi|\le 1} + \mathbb{1}_{|\xi|\ge 1}\,.$$

Proof. Let us introduce the notation

$$\forall\, v\in\mathbb{R}^d,\quad m(v) := (v\cdot\xi)\,\varphi(v)$$

and estimate the fractional Laplacian by $I_1 + I_2$ where

$$I_1 := \int_{|v-v'|<\langle v\rangle/2} \frac{m(v') - m(v) - (v'-v)\cdot\nabla m(v)}{|v-v'|^{d+\sigma}}\,\mathrm{d}v'\,,$$
$$I_2 := \int_{|v-v'|\ge\langle v\rangle/2} \frac{m(v') - m(v)}{|v-v'|^{d+\sigma}}\,\mathrm{d}v'\,.$$

• ***Step 1 : a bound of*** I_1.

We perform a second order Taylor expansion. From

$$\nabla_v\varphi = -\Big(\beta + (\beta + 2\,|1-\beta|)\,\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2\Big)\,\langle v\rangle^{\beta-2}\,\varphi^2\,v\,,$$

we deduce that $|\nabla_v\varphi| \lesssim \langle v\rangle^{-1}\varphi$. In order to estimate the Hessian of φ, we write

$$\begin{aligned}\big|\nabla_v^2\varphi(v)\big| &= \Big|\nabla_v\,\Big(\Big(\beta + (\beta+2\,|1-\beta|)\,\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2\Big)\,\langle v\rangle^{\beta-2}\,\varphi^2\,v\Big)\Big|\\ &\lesssim \langle v\rangle^{-2}\,\varphi^2 + \langle v\rangle^{-1}\,|\nabla_v\varphi|\,,\end{aligned}$$

from which we deduce that $|\nabla_v^2\varphi| \lesssim \langle v\rangle^{-2}\varphi$. It turns out that

$$\big|\nabla_v^2\big((v\cdot\xi)\,\varphi\big)\big| \lesssim |\nabla_v\varphi(v)|\,|\xi| + |v\cdot\xi|\,\big|\nabla_v^2\,(\varphi(v))\big| \lesssim |\xi|\,\langle v\rangle^{-1}\,\varphi$$

because $\nabla_v\big((v\cdot\xi)\,\varphi(v)\big) = \varphi(v)\,\xi + (v\cdot\xi)\,\nabla_v\varphi(v)$. Therefore,

$$\begin{aligned}|I_1| &\le \int_{|z|\le\langle v\rangle/2} \frac{\|\nabla_v^2 m\|_{\mathrm{L}^\infty(B(v,\langle v\rangle/2))}}{|z|^{d+\sigma-2}}\,\mathrm{d}z\\ &\qquad\le \frac{2^{\sigma-2}\,\omega_d\,|\xi|}{(2-\sigma)\,\langle v\rangle^{\sigma-2}}\,\|\langle\cdot\rangle^{-1}\,\varphi\|_{\mathrm{L}^\infty(B(v,\langle v\rangle/2))} \lesssim \frac{|\xi|\,\varphi(v)}{(2-\sigma)\,\langle v\rangle^{\sigma-1}}\end{aligned}$$

because $\langle v'\rangle$ is comparable to $\langle v\rangle$ uniformly on $B(v,\langle v\rangle/2)$.

• ***Step 2: a bound of*** I_2.

We distinguish two cases, $|\xi|\le 1$ and $|\xi|\ge 1$.

• Assume that $|\xi|\ge 1$. We estimate I_2 by the three integrals

$$\int_{\substack{|v-v'|\ge\langle v\rangle/2\\ |v'|<\langle v\rangle}} \frac{|m(v')|}{|v-v'|^{d+\sigma}}\,\mathrm{d}v'\,,\quad \int_{\substack{|v-v'|\ge\langle v\rangle/2\\ |v'|\ge\langle v\rangle}} \frac{|m(v')|}{|v-v'|^{d+\sigma}}\,\mathrm{d}v'$$

$$\text{and}\quad \int_{|v-v'|\ge\langle v\rangle/2} \frac{|m(v)|}{|v-v'|^{d+\sigma}}\,\mathrm{d}v'$$

so that

$$|I_2| \leq \frac{2^{d+\sigma}}{\langle v\rangle^{d+\sigma}} \|m\|_{\mathrm{L}^1(B_0(\langle v\rangle))} + \frac{2^\sigma \omega_d \|m\|_{\mathrm{L}^\infty(B_0^c(\langle v\rangle))}}{\sigma \langle v\rangle^\sigma} + \frac{2^\sigma \omega_d |m(v)|}{\sigma \langle v\rangle^\sigma}$$
$$\lesssim \langle v\rangle^{-\sigma} \Big(\langle v\rangle^{-d} \|m\|_{\mathrm{L}^1(B_0(\langle v\rangle))} + \|m\|_{\mathrm{L}^\infty(B_0^c(\langle v\rangle))} + |m(v)| \Big) .$$

To proceed further, we have to estimate $\langle v\rangle^{-d} \|m\|_{\mathrm{L}^1(B_0(\langle v\rangle))}$ and $\|m\|_{\mathrm{L}^\infty(B_0^c(\langle v\rangle))}$ for any $v \in \mathbb{R}^d$ and this is where $|\xi| \geq 1$ will help. Let us observe that

$$\langle v\rangle^{-d} \|m\|_{\mathrm{L}^1(B_0(\langle v\rangle))} \leq \begin{cases} \langle v\rangle^{-d} |\xi|^{-1} & \text{if } \beta + 2|1-\beta| - 1 > d\,, \\ \langle v\rangle^{-d} \langle v\rangle^{d+1-\beta-2|1-\beta|} |\xi|^{-1} & \text{if } \beta + 2|1-\beta| - 1 < d\,, \\ 2|\xi| \langle v\rangle \varphi(v) & \text{if } \beta + 2|1-\beta| - 1 < d\,. \end{cases}$$

For any $v' \in B_0^c(\langle v\rangle)$, we have

$$|(v'\cdot\xi)\,\varphi(v')| \leq \langle v'\rangle |\xi| \frac{\langle v'\rangle^{-\beta}}{1 + \langle v'\rangle^{2|1-\beta|} |\xi|^2} \leq \frac{\langle v'\rangle^{1-\beta} |\xi|}{1 + \langle v'\rangle^{2|1-\beta|} |\xi|^2}$$
$$\leq \frac{\langle v\rangle^{1-\beta} |\xi|}{1 + \langle v\rangle^{2|1-\beta|} |\xi|^2} = \langle v\rangle\, |\xi|\, \varphi(v)\,,$$

where we have used that $\langle v'\rangle \mapsto \frac{\langle v'\rangle^{1-\beta}|\xi|}{1+\langle v'\rangle^{2|1-\beta|}|\xi|^2}$ is decreasing for $\langle v'\rangle \geq \langle v\rangle$. Indeed, when $1-\beta \leq 0$ this is straightforward and when $1-\beta \geq 0$ it results from the fact that $\langle v'\rangle^{|1-\beta|} |\xi| \geq 1$ because $|\xi| \geq 1$. Hence

$$|I_2| \lesssim \begin{cases} \langle v\rangle^{-\sigma} \Big(\langle v\rangle^{-d} |\xi|^{-1} + \langle v\rangle\, |\xi|\, \varphi(v) \Big) & \text{if } \beta + 2|1-\beta| - 1 > d\,, \\ \langle v\rangle^{-\sigma} \langle v\rangle\, |\xi|\, \varphi(v) & \text{if } \beta + 2|1-\beta| - 1 < d\,. \end{cases}$$

• Assume now that $|\xi| < 1$. Let us write

$$|I_2| \leq \int_{|z|\geq\langle v\rangle/2} \sup_{|v-v'|>\langle v\rangle/2} \left(\frac{|m(v)-m(v')|}{|v-v'|^\ell} \right) \frac{\mathrm{d}z}{|z|^{d+\sigma-\ell}}$$
$$\leq \frac{2^{\sigma-\ell}\,\omega_d}{(\sigma-\ell)\,\langle v\rangle^{\sigma-\ell}} \sup_{|v-v'|>\langle v\rangle/2} \left(\frac{|m(v)-m(v')|}{|v-v'|^\ell} \right)$$

where ℓ will be chosen later. The next step is to estimate the $\ell-$Hölder semi-norm of m. For $\beta < 1$ and any $w \in \mathbb{R}^d$, we may write

$$|m(w)| \leq |\xi|\, \langle w\rangle\, \varphi = \frac{\langle w\rangle^{1-\beta} |\xi|}{1 + \langle w\rangle^{2|1-\beta|} |\xi|^2}$$
$$\leq |\xi|^{\frac{\alpha'}{2}} \langle w\rangle^{\frac{\alpha'(1-\beta)}{2}} \frac{\langle w\rangle^{\frac{(1-\beta)(2-\alpha')}{2}} |\xi|^{\frac{2-\alpha'}{2}}}{1 + \langle w\rangle^{2|1-\beta|} |\xi|^2} \lesssim |\xi|^{\frac{\alpha'}{2}} \langle w\rangle^\ell\,,$$

with $\ell = \alpha'(1-\beta)/2 \in (0,1)$. For any (v,v') such that $|v-v'| > \langle v\rangle/2$, we deduce that

$$|m(v)-m(v')| \lesssim |\xi|^{\frac{\alpha'}{2}} \left(2\langle v\rangle^{\ell} + |v'-v|^{\ell}\right) \lesssim |\xi|^{\frac{\alpha'}{2}} |v-v'|^{\ell}\,,$$

and finally obtain

$$|I_2| \le |\xi|^{\frac{\alpha'}{2}} \frac{2^{\sigma-\ell}\omega_d}{(\sigma-\ell)\langle v\rangle^{\sigma-\ell}}\,.$$

In the case $\beta \ge 1$, the estimate can be performed exactly as for $|\xi| \ge 1$ and we do not repeat the argument. □

Proposition 6. *Let $\gamma > |\beta|$. With $\mathsf{L} = \mathsf{L}_3$, we have*

$$\mu_{\mathsf{L}} \lesssim |\xi|^{\alpha'}\,\mathbb{1}_{|\xi|\le 1} + \mathbb{1}_{|\xi|\ge 1}\,.$$

Proof. We recall that $F^{-1}\,\mathsf{L}^*(F\,\cdot) = \Delta_v^{\sigma/2} - E\,\cdot\nabla_v$ and compute

$$\begin{aligned}\mu_{\mathsf{L}}^2 = \int_{\mathbb{R}^d} \left|\Delta_v^{\sigma/2}\big((v\cdot\xi)\,\varphi\big) - E\,\cdot\nabla_v\big((v\cdot\xi)\,\varphi\big)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\eta}}\\ \le 2\int_{\mathbb{R}^d} \left|\Delta_v^{\sigma/2}\big((v\cdot\xi)\,\varphi\big)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\eta}}\\ + 2\int_{\mathbb{R}^d} \left|E\,\cdot\nabla_v\big((v\cdot\xi)\,\varphi\big)\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma+\eta}}\,.\end{aligned}$$

We have to estimate the two integrals of the latter r.h.s. The first one follows from Lemma 10. As for the second one, using Proposition 4, we obtain

$$\begin{aligned}\left|E\cdot\nabla_v\big((v\cdot\xi)\,\varphi\big)\right| &\lesssim |E\cdot\xi\,\varphi| + |(v\cdot\xi)\,E\cdot\nabla_v\varphi|\\ &\lesssim |v\cdot\xi|\,\langle v\rangle^{\beta}\,\varphi + |v\cdot\xi|\,\varphi\,\langle v\rangle^{-2}\,\langle v\rangle^{\beta}\,|v|^2\\ &\lesssim |v\cdot\xi|\,\langle v\rangle^{\beta}\,\varphi\,,\end{aligned}$$

so that

$$\begin{aligned}\|E\cdot\nabla_v\big((v\cdot\xi)\,\varphi\big)\|^2_{\mathrm{L}^2(\langle v\rangle^{\eta}F\,\mathrm{d}v)} &\lesssim \int_{\mathbb{R}^d} \frac{|v\cdot\xi|^2}{(1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2)^2}\frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}}\\ &\lesssim |\xi|^{\alpha'-\frac{\beta+\eta}{1-\beta}}\,\mathbb{1}_{|\xi|\le 1} + \mathbb{1}_{|\xi|\ge 1}\,.\end{aligned}$$

□

Proposition 7. *Let $\gamma \ge \beta_+$ and consider $\mathsf{L} = \mathsf{L}_3$. There exists a constant $C>0$, independent of ξ, such that*

$$\lambda_{\mathsf{L}} \le C\,.$$

Proof. We follow the same steps as in the proof of Proposition 6. We have

$$\begin{aligned}\lambda_{\mathsf{L}}^2 &= \int_{\mathbb{R}^d} \left(\Delta_v^{\sigma/2}\,\psi - E\cdot\nabla_v\psi\right)^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}}\\ &\le 2\int_{\mathbb{R}^d} \left|\Delta_v^{\sigma/2}\,\psi\right|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}} + 2\int_{\mathbb{R}^d} |E\cdot\nabla_v\psi|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}}\,.\end{aligned}$$

Since $\Delta_v^{\sigma/2}\psi$ is a bounded function, the first integral of the r.h.s. is bounded because $\gamma-\eta>0$. For the second integral, we simply observe that

$$\int_{\mathbb{R}^d} |E\cdot\nabla_v\psi|^2 \frac{\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta}} \le \int_{\mathbb{R}^d} \frac{|v|^2\,\mathrm{d}v}{\langle v\rangle^{d+\gamma-\eta-2\,\beta+8}}$$

is bounded because $\gamma-\eta-2\,\beta+6>0$. □

6.3. Two technical estimates. We recall that the coefficient μ_2 and $\tilde\mu_k$ have been defined in (15). The coefficients $\tilde\mu_1$ and $\tilde\mu_2$ are well defined when $\beta\le\eta<\gamma$ since $(\beta-1)+|\beta-1|=2\,(\beta-1)_+$ so that, for any $k\le2$,

$$\eta+\gamma+2\,\beta+4\,|\beta-1| = (\eta-\beta)+(\gamma-\beta)+8\,(\beta-1)_+ + 4 > 2\,k\,.$$

The notation $a\simeq b$ means that there exists a constant $C>0$ such that $a/C\le b\le C\,a$. Our first result investigates the dependence of μ_2 and $\tilde\mu_k$ in $\xi\in\mathbb{R}^d$.

Lemma 11. *For $\beta\le\eta<\gamma$ with $\gamma>0$, the coefficient μ_2 is bounded from above and below for large values of $|\xi|$ and satisfies*

$$\begin{aligned}\mu_2(\xi) &\underset{\xi\to0}{\simeq} |\xi|^{\min\left\{2,2+\frac{\beta+\gamma-2}{|1-\beta|}\right\}} && \text{if}\quad \beta+\gamma\neq2\,,\\ \mu_2(\xi) &\underset{\xi\to0}{\sim} -\frac{1}{d\,|1-\beta|}\,|\xi|^2\,\log|\xi| && \text{if}\quad \beta+\gamma=2\,.\end{aligned}$$

If $\eta+\gamma+2\,\beta+4\,|1-\beta|>2\,k$, then $\tilde\mu_k(\xi)\underset{|\xi|\to+\infty}{\simeq}|\xi|^{k-2}$ and

$$\begin{aligned}\tilde\mu_k(\xi) &\underset{\xi\to0}{\simeq} |\xi|^{\min\left\{k,k+\frac{\gamma+\eta+2\,\beta-2k}{2\,|1-\beta|}\right\}} && \text{if}\quad \gamma+2\,\beta+\eta\neq2\,k\,,\\ \tilde\mu_k(\xi) &\underset{\xi\to0}{\simeq} -|\xi|^k\,\log|\xi| && \text{if}\quad \gamma+2\,\beta+\eta=2\,k\,.\end{aligned}$$

Proof. We start by considering $\xi\to0$. Let $c:=|1-\beta|\ge0$. If $\beta+\gamma>2$, then

$$\mu_2(\xi)\underset{\xi\to0}{\sim}|\xi|^2\int_{\mathbb{R}^d}\frac{c_\gamma\,|v_1|^2}{\langle v\rangle^{d+\beta+\gamma}}\,\mathrm{d}v\,.$$

• If $\beta+\gamma<2$, then $\beta<1$ and $c>0$. With the change of variables $v=u\,|\xi|^{-1/c}$, we observe that

$$\mu_2(\xi)\underset{\xi\to0}{\sim}|\xi|^{2+\frac{\beta+\gamma-2}{c}}\int_{\mathbb{R}^d}\frac{|u_1|^2}{1+|u|^{2\,c}}\,\frac{c_\gamma\,\mathrm{d}u}{|u|^{d+\beta+\gamma}}$$

using $\langle u\,|\xi|^{-1/c}\rangle \underset{\xi\to 0}{\sim} |u|\,|\xi|^{-1/c}$ for any $u\in\mathbb{R}^d\backslash\{0\}$.

• If $\beta+\gamma=2$ and $\gamma>\beta$, then $\beta<1$, $c=1-\beta$ is positive and

$$\mu_2=|\xi|^2\int_{\mathbb{R}^d}\frac{c_\gamma\,|v_1|^2}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}\,.$$

With the change of variables $v=u\,|\xi|^{-1/c}$, we have that

$$I_0:=\int_{|v|\ge|\xi|^{-\frac{1}{c}}}\frac{|v_1|^2}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}\underset{\xi\to 0}{\sim}\int_{|u|\ge 1}\frac{|u_1|^2}{1+|u|^{2\,c}}\,\frac{\mathrm{d}u}{|u|^{d+2}}$$

is finite. Using the invariance under rotation with respect to $\xi\in\mathbb{R}^d$,

$$d\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{|v_1|^2}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}=\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{|v|^2}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}$$

can be splitted, using $|v|^2=-1+\langle v\rangle^2$, into

$$-\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{1}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}\underset{\xi\to 0}{\sim}-\int_{\mathbb{R}^d}\frac{\mathrm{d}v}{\langle v\rangle^{d+2}}$$

and

$$\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{1}{1+\langle v\rangle^{2\,c}\,|\xi|^2}\,\frac{\mathrm{d}v}{\langle v\rangle^{d}}=\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{\mathrm{d}v}{\langle v\rangle^{d}}-\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{1}{1+\langle v\rangle^{-2\,c}\,|\xi|^{-2}}\,\frac{\mathrm{d}v}{\langle v\rangle^{d}}$$

using $\frac{1}{1+X}=1-\frac{1}{1+1/X}$ with $X=\langle v\rangle^{2\,c}\,|\xi|^2$.

$$\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{\mathrm{d}v}{\langle v\rangle^{d}}\underset{\xi\to 0}{\sim}-\frac{\omega_d}{c}\,\log|\xi|$$

and

$$\int_{|v|<|\xi|^{-\frac{1}{c}}}\frac{1}{1+\langle v\rangle^{-2\,c}\,|\xi|^{-2}}\,\frac{\mathrm{d}v}{\langle v\rangle^{d}}\underset{\xi\to 0}{\sim}\int_{|u|<1}\frac{|u|^{2c}}{1+|u|^{2c}}\,\frac{\mathrm{d}u}{|u|^{d}}$$

by the change of variables $v=|\xi|^{-1/c}\,u$. After collecting terms, this yields

$$\mu_2\underset{\xi\to 0}{\sim}-\frac{\omega_d}{c\,d}\,|\xi|^2\,\log|\xi|\,.$$

On the other hand, when $|\xi|\to+\infty$, we have

$$\mu_2(\xi)=\int_{\mathbb{R}^d}\frac{|v\cdot\xi|^2\,\langle v\rangle^{\beta}}{\langle v\rangle^{2\beta}+\langle v\rangle^{2}\,|\xi|^2}\,\frac{c_\gamma\,\mathrm{d}v}{\langle v\rangle^{d+\gamma}}\underset{|\xi|\to+\infty}{\sim}\int_{\mathbb{R}^d}\frac{c_\gamma\,|v_1|^2}{\langle v\rangle^{d+\gamma+2-\beta}}\,\mathrm{d}v\,.$$

The claim on μ_2 is now completed. All other estimates follow from similar computations and we shall omit further details. □

The coefficients λ_0, λ_1, $\tilde\lambda_0$ and $\tilde\lambda_1$ have also been defined in (15). Our second technical estimate goes as follows.

Lemma 12. *The coefficients λ_0 and λ_1 are well defined for any $\gamma>0$. The coefficients $\tilde\lambda_0$ and $\tilde\lambda_1$ are also well defined if $\gamma>0$ and $\eta>-\gamma$.*

The proof is straightforward and left to the reader.

7. Hypocoercivity estimates

7.1. A macroscopic coercivity estimate. We recall that $\mathsf{R}_\xi[\widehat{f}] = -\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Re}\,\langle \mathsf{A}_\xi \widehat{f}, \widehat{f}\rangle$ if f solves (1) where

$$\mathsf{A}_\xi = \frac{1}{\langle v\rangle^2}\,\Pi\,\frac{(-\,i\,v\cdot\xi)\,\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2} = \psi\,\Pi\mathsf{T}^*\,\varphi\,\widehat{f}$$

with $\varphi(\xi,v) = \frac{\langle v\rangle^{-\beta}}{1+\langle v\rangle^{2\,|1-\beta|}\,|\xi|^2}$ and $\psi(v) := \langle v\rangle^{-2}$. In this section, our goal is to establish an estimate of $\mathsf{R}_\xi[\widehat{f}]$. In this section, we use the notation (15) and prove Proposition 3.

Proof (Proof of Proposition 3). Since φ and ψ commute with T and $\Pi\mathsf{T}\Pi = 0$, we get

$$\mathsf{A}_\xi\,\Pi = -\,\psi\,\Pi\mathsf{T}\Pi\,\varphi = 0, \quad \mathsf{A}_\xi^*\mathsf{T}\Pi = \varphi\,\mathsf{T}\Pi\mathsf{T}\Pi\,\psi = 0\,.$$

Moreover, $\mathsf{L}\Pi = 0$. With these identities, using the micro-macro decomposition $\widehat{f} = \Pi\widehat{f} + (1-\Pi)\widehat{f}$, we find that

$$\mathsf{R}_\xi[\widehat{f}] := \mathsf{I}_1 + \mathsf{I}_2 + \mathsf{I}_3 + \mathsf{I}_4 + \mathsf{I}_5 + \mathsf{I}_6 + \mathsf{I}_7$$

where

$$\mathsf{I}_1 := \left\langle \mathsf{A}_\xi\,\mathsf{T}\Pi\widehat{f}, \Pi\widehat{f}\right\rangle\,, \quad \mathsf{I}_2 := \left\langle \mathsf{A}_\xi\,\mathsf{T}\Pi\widehat{f}, (1-\Pi)\widehat{f}\right\rangle\,, \quad \mathsf{I}_3 := \left\langle \mathsf{A}_\xi\,\mathsf{T}(1-\Pi)\widehat{f}, \Pi\widehat{f}\right\rangle\,,$$

$$\mathsf{I}_4 := \left\langle \mathsf{A}_\xi\,\mathsf{T}(1-\Pi)\widehat{f}, (1-\Pi)\widehat{f}\right\rangle\,, \quad \mathsf{I}_5 := \left\langle \mathsf{A}_\xi(1-\Pi)\widehat{f}, \mathsf{T}(1-\Pi)\widehat{f}\right\rangle\,,$$

$$\mathsf{I}_6 := -\left\langle \mathsf{A}_\xi\,\mathsf{L}(1-\Pi)\widehat{f}, \widehat{f}\right\rangle\,, \quad \mathsf{I}_7 := -\left\langle \mathsf{A}_\xi(1-\Pi)\widehat{f}, \mathsf{L}(1-\Pi)\widehat{f}\right\rangle\,.$$

- ***Step 1: macroscopic coercivity.***

  Since $\int_{\mathbb{R}^d} F\,\mathrm{d}v = 1$ and $\Pi\widehat{f}(\xi,v) = \rho_{\widehat{f}}(\xi)\,F(v)$, we first notice that

$$|\rho_{\widehat{f}}(\xi)|^2 = \int_{\mathbb{R}^d} |\rho_{\widehat{f}}(\xi)\,F|^2\,\mathrm{d}\mu = \|\Pi\widehat{f}\|^2\,.$$

and

$$\mathsf{I}_1 = \langle \mathsf{A}_\xi\,\mathsf{T}F, F\rangle\,|\rho_{\widehat{f}}|^2 = \left\langle \psi\,\Pi\,|\mathsf{T}|^2\,\varphi\,F, F\right\rangle\,\|\Pi\widehat{f}\|^2 = \lambda_0\,\mu_2\,\|\Pi\widehat{f}\|^2$$

by definition of A_ξ.

- ***Step 2: micro-macro terms.***

  By definition of A_ξ,

$$\mathsf{I}_2 = \left\langle \psi\,\Pi\mathsf{T}^*\,\varphi\,\mathsf{T}\Pi\widehat{f}, (1-\Pi)\widehat{f}\right\rangle = \left\langle F, |\mathsf{T}|^2\,\varphi\,F\right\rangle\,\rho_{\widehat{f}}\left\langle \psi\,F, (1-\Pi)\widehat{f}\,\right\rangle$$

can be estimated using $|\rho_{\widehat{f}}| = \|\Pi\widehat{f}\|$ and the Cauchy-Schwarz inequality

$$\left\langle \psi\,F, (1-\Pi)\widehat{f}\,\right\rangle \le \|\psi\,F\|_{-\eta}\,\|(1-\Pi)\widehat{f}\|_\eta$$

by

$$|\mathsf{I}_2| \le \tilde\lambda_0\,\mu_2\,\|\Pi\widehat f\|\,\|(1-\Pi)\widehat f\|_\eta\,.$$

By similar estimates, we obtain

$$\begin{aligned}|\mathsf{I}_3| &\le \lambda_0\,\tilde\mu_2\,\|\Pi\widehat f\|\,\|(1-\Pi)\widehat f\|_\eta\,,\\ |\mathsf{I}_4| &\le \tilde\lambda_0\,\tilde\mu_2\,\|(1-\Pi)\widehat f\|_\eta^2\,,\\ |\mathsf{I}_5| &\le \tilde\lambda_1\,\tilde\mu_1\,\|(1-\Pi)\widehat f\|_\eta^2\,.\end{aligned}$$

To get a bound on I_6, we use the fact that $\mathsf{T}^*=-\mathsf{T}$ to obtain

$$\begin{aligned}\mathsf{I}_6 &= \left\langle \psi\,\Pi\mathsf{T}\varphi\,\mathsf{L}(1-\Pi)\widehat f,\widehat f\right\rangle = \left\langle F,\mathsf{T}\varphi\,\mathsf{L}(1-\Pi)\widehat f\right\rangle \left\langle F\,\psi,\widehat f\right\rangle\\ &= -\left\langle \mathsf{L}^*\mathsf{T}\,\varphi\,F,(1-\Pi)\widehat f\right\rangle \left\langle F\,\psi,\widehat f\right\rangle\,.\end{aligned}$$

By the micro-macro decomposition $\widehat f = \Pi\widehat f + (1-\Pi)\widehat f$, we have

$$\left\langle F\,\psi,\widehat f\right\rangle = \lambda_0\,\rho_{\widehat f} + \left\langle F\,\psi,(1-\Pi)\widehat f\right\rangle$$

and the Cauchy-Schwarz inequality gives

$$\left\langle \mathsf{L}^*\mathsf{T}\,\varphi\,F,(1-\Pi)\widehat f\right\rangle \le \mu_{\mathsf{L}}\,\|(1-\Pi)\widehat f\|_\eta\,.$$

This yields

$$|\mathsf{I}_6| \le \lambda_0\,\mu_{\mathsf{L}}\,\|\Pi\widehat f\|\,\|(1-\Pi)\widehat f\|_\eta + \tilde\lambda_0\,\mu_{\mathsf{L}}\,\|(1-\Pi)\widehat f\|_\eta^2\,.$$

In the same way, we get

$$|\mathsf{I}_7| \le \lambda_{\mathsf{L}}\,\tilde\mu_1\,\|(1-\Pi)\widehat f\|_\eta^2\,.$$

- ***Step 3: cross terms.***

With $X:=\|\Pi\widehat f\|$ and $Y:=\|(1-\Pi)\widehat f\|_\eta$, we collect all above estimates into

$$\begin{aligned}\mathsf{R}_\xi[\widehat f] \le -\lambda_0\,\mu_2\,X^2 &+ \left(\tilde\lambda_0\,\mu_2+\lambda_0\,\tilde\mu_2+\lambda_0\,\mu_{\mathsf{L}}\right)XY\\ &+\left(\tilde\lambda_0\,\tilde\mu_2+\tilde\lambda_1\,\tilde\mu_1+\tilde\lambda_0\,\mu_{\mathsf{L}}+\lambda_{\mathsf{L}}\,\tilde\mu_1\right)Y^2\,,\end{aligned}$$

which by Young's inequality leads to

$$\begin{aligned}\mathsf{R}_\xi[\widehat f] \le &\left(\frac{a}{2}\left(\tilde\lambda_0\,\mu_2+\lambda_0\,\tilde\mu_2+\lambda_0\,\mu_{\mathsf{L}}\right)-\lambda_0\,\mu_2\right)X^2\\ &+\left(\tilde\lambda_0\,\tilde\mu_2+\tilde\lambda_1\,\tilde\mu_1+\tilde\lambda_0\,\mu_{\mathsf{L}}+\lambda_{\mathsf{L}}\,\tilde\mu_1+\frac{\tilde\lambda_0\,\mu_2+\lambda_0\,\tilde\mu_2+\lambda_0\,\mu_{\mathsf{L}}}{2\,a}\right)Y^2\,.\end{aligned}$$

With the choice

$$a=\frac{\lambda_0\,\mu_2}{\tilde\lambda_0\,\mu_2+\lambda_0\,\tilde\mu_2+\lambda_0\,\mu_{\mathsf{L}}}\,,$$

we get

$$\mathsf{R}_\xi[\widehat{f}] \le -\frac{1}{2}\,\lambda_0\,\mu_2\,\|\Pi\widehat{f}\|^2 + \mathcal{K}(\xi)\,\|(1-\Pi)\widehat{f}\|_\eta^2\,,$$

with

$$\mathcal{K}(\xi) = \tilde\lambda_0\,\tilde\mu_2 + \tilde\lambda_1\,\tilde\mu_1 + \tilde\lambda_0\,\mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\,\tilde\mu_1 + \frac{\left(\tilde\lambda_0\,\mu_2 + \lambda_0\,\tilde\mu_2 + \lambda_0\,\mu_{\mathsf{L}}\right)^2}{2\,\lambda_0\,\mu_2}\,.$$

• ***Step 4: A uniform bound on*** $\mathcal{K}(\xi)$***.***

According to Lemma 12, λ_0 and $\tilde\lambda_0$ are independent of ξ and take finite positive values, so that

$$\mathcal{K}(\xi) \lesssim \tilde\mu_2 + \tilde\lambda_1\,\tilde\mu_1 + \mu_{\mathsf{L}} + \lambda_{\mathsf{L}}\,\tilde\mu_1 + \mu_2 + \frac{\tilde\mu_2^2}{\mu_2} + \frac{\mu_{\mathsf{L}}^2}{\mu_2}\,.$$

We also deduce from their definitions in (15) that μ_2, $\tilde\mu_2$, $|\xi|\,\tilde\mu_1$ and $\tilde\lambda_1/|\xi|$ have finite, positive limits as $|\xi| \to +\infty$. By Proposition 5, μ_{L} and λ_{L} are bounded from above, so that

$$\forall\,\xi \in \mathbb{R}^d \text{ such that } |\xi| \ge 1\,, \quad \mathcal{K}(\xi) \lesssim 1 + \mu_{\mathsf{L}} + \frac{\lambda_{\mathsf{L}}}{|\xi|} + \mu_{\mathsf{L}}^2 \lesssim 1\,.$$

It remains to investigate the behaviour of $\mathcal{K}(\xi)$ as $\xi \to 0$ and we shall distinguish two main cases:

• if $\beta < 1$, under the assumption that $\beta+\gamma \ne 2$, $\gamma+\eta+2\,\beta \ne 4$ and $\gamma+\eta+2\,\beta \ne 2$, for some positive constants C_1, C_2, $\widetilde{C}_1$, $\widetilde{C}_2$, we have

$$\mu_2 \underset{\xi\to0}{\sim} C_2\,|\xi|^{\alpha'}\,, \quad \tilde\mu_2 \underset{\xi\to0}{\sim} \widetilde{C}_2\,|\xi|^{\min\left\{2,\,\frac{\gamma-2\,\beta+\eta}{2\,(1-\beta)}\right\}}\,,$$

$$\tilde\mu_1 \underset{\xi\to0}{\sim} C_1\,|\xi|^{\min\left\{1,\,\frac{\gamma+\eta}{2\,(1-\beta)}\right\}}\,, \quad \tilde\lambda_1 \underset{\xi\to0}{\sim} \widetilde{C}_1\,|\xi|\,,$$

where $\alpha' = \min\left\{2, \frac{\gamma-\beta}{1-\beta}\right\}$ as in (9). Since $\eta \ge \beta \ge 2\,\beta-\gamma$ and $\eta \ge -\gamma$, this implies by Proposition 5 that

$$\forall\,\xi \in \mathbb{R}^d \text{ such that } |\xi| \le 1\,, \quad \mathcal{K}(\xi) \lesssim 1 + \mu_{\mathsf{L}} + \lambda_{\mathsf{L}} + \frac{\tilde\mu_2^2}{\mu_2} + \frac{\mu_{\mathsf{L}}^2}{\mu_2} \lesssim 1 + \frac{\tilde\mu_2^2}{\mu_2}\,.$$

Then $\frac{\tilde\mu_2^2}{\mu_2} = O\left(|\xi|^\varepsilon\right)$ is bounded as $\xi \to 0$ either if $\beta+\gamma < 2$ because

$$\varepsilon = \min\left\{\tfrac{\eta-\beta}{1-\beta}, \tfrac{4-3\,\beta-\gamma}{1-\beta}\right\}\,, \quad \eta \ge \beta\,, \quad 4-3\,\beta-\gamma = 2\,(1-\beta) + 2 - (\beta+\gamma) \ge 0\,,$$

or if $\beta+\gamma > 2$ because

$$\varepsilon = \min\left\{2, \tfrac{\gamma+\eta-2}{1-\beta}\right\} \quad \text{and} \quad \eta \ge \beta > 2-\gamma\,.$$

• if $\beta > 1$, under the assumption that $\gamma + 2\,\beta + \eta - 4 \neq 0$ and $\gamma + 2\,\beta + \eta - 2 \neq 0$, for some positive constants C_1, C_2, $\widetilde{C}_1$, $\widetilde{C}_2$, we have

$$\mu_2 \underset{\xi\to 0}{\sim} C_2\,|\xi|^2\,,\quad \tilde{\mu}_2 \underset{\xi\to 0}{\sim} \widetilde{C}_2\,|\xi|^{\min\left\{2,\,\frac{\gamma+\eta+6\,\beta-8}{2\,(\beta-1)}\right\}}\,,$$

$$\tilde{\mu}_1 \underset{\xi\to 0}{\sim} C_1\,|\xi|^{\min\left\{1,\,\frac{\gamma+\eta+4\,\beta-4}{2\,(\beta-1)}\right\}}\,,\quad \tilde{\lambda}_1 \underset{\xi\to 0}{\sim} \widetilde{C}_1\,|\xi|\,.$$

Since $\eta \geq \beta > 1$, we get $\gamma + \eta + 6\,\beta - 8 > 0$ and $\gamma + \eta + 4\,\beta - 4 > 0$, so that

$$\mathcal{K}(\xi) \lesssim 1 + \frac{\tilde{\mu}_2^2}{\mu_2}\,.$$

where $\frac{\tilde{\mu}_2^2}{\mu_2} = O\left(|\xi|^\varepsilon\right)$ is bounded as $\xi \to 0$ because

$$\varepsilon = \min\left\{2,\,\tfrac{\gamma+\eta+4\,\beta-6}{\beta-1}\right\} \quad \text{and} \quad \gamma + \eta + 4\,\beta - 6 > 0\,.$$

In the critical cases when a $\log|\xi|$ appears in the expression of μ_2, $\tilde{\mu}_1$ or $\tilde{\mu}_2$, we obtain expressions of the form $|\xi|^\varepsilon\,\big|\log|\xi|\big|$ for some $\varepsilon > 0$, so that all terms also remain bounded. We conclude that in all cases, $\mathcal{K}(\xi)$ is bounded from above uniformly with respect to ξ. This ends the proof of Proposition 3. □

7.2. A fractional Nash inequality and consequences. For any $a > 0$, let us define the function $\mathcal{N}_a$ by

$$\forall\, s \geq 0\,,\quad \mathcal{N}_a(s) := \frac{s^a}{(1+s^2)^{a/2}}$$

and the quadratic form

$$\mathcal{Q}_a[u] := \int_{\mathbb{R}^d} \mathcal{N}_a\big(|\xi|\big)\,|\hat{u}(\xi)|^2\,\mathrm{d}\xi$$

where $\hat{u}$ denotes the Fourier transform of a function $u \in \mathrm{L}^2(\mathrm{d}x)$ given by

$$\hat{u}(\xi) = (2\pi)^{-d/2}\int_{\mathbb{R}^d} e^{-\,i\,x\cdot\xi}\,u(x)\,\mathrm{d}x\,.$$

We recall that by Plancherel's formula, $\|u\|^2_{\mathrm{L}^2(\mathrm{d}x)} = \|\hat{u}\|^2_{\mathrm{L}^2(\mathrm{d}\xi)}$.

Lemma 13. *Let $d \geq 1$ and $a \in (0,2]$. There is a monotone increasing function $\Phi_a : \mathbb{R}^+ \to \mathbb{R}^+$ with $\Phi_a(s) \sim s^{d/(d+a)}$ as $s \to 0_+$ such that*

$$\forall\, u \in \mathcal{D}(\mathbb{R}^d)\,,\quad \|u\|^2_{\mathrm{L}^2(\mathrm{d}x)} \leq \|u\|^2_{\mathrm{L}^1(\mathrm{d}x)}\,\Phi_a\left(\frac{\mathcal{Q}_a[u]}{\|u\|^2_{\mathrm{L}^1(\mathrm{d}x)}}\right).$$

Proof. We rely on a simple argument based on Fourier analysis inspired by the proof of Nash's inequality in [35, page 935], which goes as follows. Since $\|\hat{u}\|_{\mathrm{L}^\infty(\mathrm{d}\xi)} \le \|u\|_{\mathrm{L}^1(\mathrm{d}x)}$, we obtain

$$\begin{aligned}\|u\|^2_{\mathrm{L}^2(\mathrm{d}x)} = \|\hat{u}\|^2_{\mathrm{L}^2(\mathrm{d}\xi)} &\le \int_{|\xi|\le R} |\hat{u}(\xi)|^2\,\mathrm{d}\xi + \int_{|\xi|>R} |\hat{u}(\xi)|^2\,\mathrm{d}\xi \\ &\le \frac{1}{d}\,\omega_d\,\|u\|^2_{\mathrm{L}^1(\mathrm{d}x)}\,R^d + \frac{1}{\mathcal{N}_a(R)}\,\mathcal{Q}_a[u]\end{aligned}$$

for any $R>0$, using the monotonicity of $s \mapsto \mathcal{N}_a(s)$.

Let us consider the function

$$f(x,R) := \frac{1}{d}\,R^d + \frac{x}{a}\,(1+R^{-2})^{a/2}$$

and notice that, as a function of R, f has a unique minimum $R = R(x)$ such that

$$R^{d+a}\,(1+R^2)^{1-\frac{a}{2}} = x$$

for any $x>0$. With $a \in (0,2]$, it is clear that $x \mapsto R(x)$ is monotone increasing and such that $R(x) \le x^{1/(d+2)}\,\big(1+o(1)\big)$ as $x \to +\infty$ and $R(x) = x^{1/(d+a)}\,\big(1+o(1)\big)$ as $x \to 0_+$. Altogether, for the optimal value $R = R(x)$, we obtain that $\phi(x) = f\big(x, R(x)\big)$ is such that

$$\begin{aligned}\phi(x) &= \big(\tfrac{1}{d} + \tfrac{1}{a}\big)\,x^{\frac{d}{d+a}}\,\big(1+o(1)\big) \quad \text{as} \quad s \to 0_+\,,\\ \phi(x) &= \frac{x}{a}\,\big(1+o(1)\big) \quad \text{as} \quad s \to +\infty\,.\end{aligned}$$

The proof is concluded with $\varPhi_a(s) = \omega_d\,\phi\big(\frac{a\,s}{\omega_d}\big)$. □

Let us consider the Fourier transform with respect to x of a distribution function f depending on x and v and define

$$\mathsf{Q}_a[f] := \int_{\mathbb{R}^d} \mathcal{Q}_a[f]\,\mathrm{d}\mu\,.$$

Lemma 14. *Let $d \ge 1$ and $a \in (0,2]$. With the above notation, we have*

$$\forall\, f \in \mathrm{L}^1 \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)\,, \quad \|\mathsf{\Pi} f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \le \|f\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}\,\varPhi_a\left(\frac{\mathsf{Q}_a[\mathsf{\Pi} f]}{\|f\|^2_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}}\right),$$

where the function $\varPhi_a$ is defined in Lemma 13.

Proof. We apply the strategy of Lemma 13 to $\mathsf{\Pi} f = \rho_f\,F$ and bound

$$\|\rho_f\|_{\mathrm{L}^2(\mathrm{d}x)} = \|\mathsf{\Pi} f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} = \|\widehat{\mathsf{\Pi} f}\|^2_{\mathrm{L}^2(\mathrm{d}\xi\,d\mu)}$$

by

$$\begin{aligned}\iint_{|\xi|\le R} |\widehat{\mathsf{\Pi} f}(\xi,v)|^2\,\mathrm{d}\xi\,\mathrm{d}\mu = \frac{1}{d}\,\omega_d\,R^d\,\|\rho_f\|^2_{\mathrm{L}^1(\mathrm{d}x)} \int_{\mathbb{R}^d} F^2\,\mathrm{d}\mu \\ = \frac{1}{d}\,\omega_d\,R^d\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}\end{aligned}$$

and

$$\iint_{|\xi|>R} |\widehat{\Pi f}(\xi,v)|^2\,d\xi\,d\mu \le \frac{\mathcal{Q}_a[\rho_f]}{\mathcal{N}_a(R)} \int_{\mathbb{R}^d} F^2\,d\mu = \frac{\mathsf{Q}_a[\Pi f]}{\mathcal{N}_a(R)}\,.$$

From this point, the computations are exactly the same as in the proof of Lemma 13. □

7.3. A limit case of the fractional Nash inequality. In the case when $\beta+\gamma=2$, we recall that

$$\mathsf{R}_\xi[\widehat{f}] \gtrsim \Lambda(\xi)\,\|\Pi\widehat{f}\|^2 - \mathcal{C}\,\|(1-\Pi)\widehat{f}\|_\beta^2$$

by Proposition 3, where $\Lambda(\xi)=h(|\xi|)$ and $h(r)=r^2\,|\log r|/(1+r^2\,\log r)$. The function $h:[0,1/\sqrt{e})\to\mathbb{R}$ is monotone increasing. Define

$$\Phi(x) := \frac{1}{d}\,x^{1+\frac{2}{d}}\;|\log x|\;.$$

The proof of Corollary 1 relies on the following result.

Lemma 15. *Let $d\ge 1$ and assume that $\beta+\gamma=2$. With the above notation, there exists a positive constant* A *such that, if*

$$\frac{\|\Pi f\|^2_{L^2(dx\,d\mu)}}{\|f\|^2_{L^1(dx\,dv)}} \le \mathsf{A}\,,$$

then

$$\|\Lambda^{\frac12}\,\Pi\widehat{f}\|^2_{L^2(dx\,d\mu)} \ge \tfrac{\omega_d}{2\,d}\,\|f\|^2_{L^1(dx\,d\mu)}\,\Phi\left(\frac{\|\Pi f\|^2_{L^2(dx\,d\mu)}}{\|f\|^2_{L^1(dx\,dv)}}\right).$$

Proof. As in the case $\beta+\gamma\neq 2$, we use

$$\begin{aligned}\|\Pi f\|^2_{L^2(d\xi\,d\mu)} = \|\rho_f\|^2_{L^2(dx)} &= \int_{|\xi|<R}|\widehat{\rho_f}|^2\,d\xi + \int_{|\xi|\ge R}|\widehat{\rho_f}|^2\,d\xi\\ &\le \tfrac{\omega_d}{d}\,R^d\,\|\widehat{\rho_f}\|^2_{L^\infty(d\xi)} + \frac{1}{h\,(R)}\int_{|\xi|\ge R}\Lambda\,|\widehat{\rho_f}|^2\,d\xi\\ &\le \tfrac{\omega_d}{d}\,R^d\,\|f\|^2_{L^1(dx\,d\mu)} + \frac{1}{h\,(R)}\,\|\Lambda^{\frac12}\,\Pi\widehat{f}\|^2_{L^2(dx\,d\mu)}\end{aligned}$$

for some $R>0$, small enough. The last inequality can be written as

$$X \le R^d\,a + \frac{b}{h(R)}$$

with $a=\frac{\omega_d}{d}\,\|f\|^2_{L^1(dx\,d\mu)}$, $b=\|\Lambda^{\frac12}\,\Pi\widehat{f}\|^2_{L^2(dx\,d\mu)}$ and $X=\|\Pi f\|^2_{L^2(d\xi\,d\mu)}$. There is a unique $R>0$, small, such that $R^{d+2}\,|\log R|\sim R^d\,h(R)=b/a$ if b/a is small enough, from which we deduce that $X\le 2\,a\,R^d$, *i.e*,

$$\frac{b}{a} \gtrsim \Phi\left(\frac{X}{2\,a}\right) \quad \text{where} \quad \Phi(x) := \frac{1}{d}\,x^{1+\frac{2}{d}}\;|\log x|\;.$$

The conclusion holds for some $\mathsf{A} < R^d\,h(R)$ with $R=1/\sqrt{e}$, whose detailed expression is inessential. □

7.4. An extension of Corollary 1 when $\beta+\gamma\le 0$. We do not have a good control of $\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\beta\,\mathrm{d}\mu)}$ when $\beta\le-\gamma$, but we claim that the issue can be solved if we consider $\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\eta\mathrm{d}\mu)}$.

Corollary 7. *Let $\beta+\gamma\le 0$ and $\eta\in(-\gamma,0)$. Under Assumption (H), if f is a solution of (1), then for any $t\ge 0$,*

$$\int_{\mathbb{R}^d}\mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi\;\gtrsim\;\|\Pi f\|^{2\,(1+\frac{\alpha}{d})}_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)}-\|(1-\Pi)f\|^2_{\mathrm{L}^2(\mathrm{d}x\,\langle v\rangle^\eta\mathrm{d}\mu)}\,.$$

8. Completion of the proofs and extension

8.1. The case $\beta+\gamma\le 0$. Let us define

$$X_\zeta:=\iint_{\mathbb{R}^d\times\mathbb{R}^d}|(1-\Pi)f|^2\,\langle v\rangle^\zeta\,\mathrm{d}x\,\mathrm{d}\mu\quad\text{and}\quad Y:=\iint_{\mathbb{R}^d\times\mathbb{R}^d}|\Pi f|^2\,\mathrm{d}x\,\mathrm{d}\mu\,.$$

With this notation, Inequality (14) in Corollary 6 can be written as

$$\forall\,r>0\,,\quad-\int_{\mathbb{R}^d}\langle f,\mathsf{L}f\rangle\,\mathrm{d}x\ge r\,X_\eta-(1-\theta)\,\theta^{\frac{\theta}{1-\theta}}\,r^{\frac{1}{1-\theta}}\,X_k \tag{17}$$

with $\theta=\frac{k-\eta}{k-\beta}$, $\beta\le-\gamma<\eta<0<k<\gamma$, while Corollary 7 simply means

$$\int_{\mathbb{R}^d}\mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi\;\gtrsim\;Y^{1+\frac{\alpha}{d}}-X_\eta\,. \tag{18}$$

Let us consider

$$\mathcal{H}:=X_0+Y+\delta(t)\,\mathrm{Re}\left(\int_{\mathbb{R}^d}\langle\mathsf{A}f,f\rangle\,\mathrm{d}x\right)\text{with}\quad\delta(t)=\delta_0\,(1+\varepsilon\,t)^{-\mathsf{a}}\,,$$

for some constant numbers $\mathsf{a}\in(0,1)$, $\delta_0>0$ and $\varepsilon>0$, to be chosen. The major difference with the case $\beta+\gamma>0$ considered in Section 3.3 is that we allow δ to depend on t and that we shall actually make an explicit choice of this dependence.

We know that

$$\left(1-\tfrac{\delta}{2}\right)(X_0+Y)\le\mathcal{H}\le\left(1+\tfrac{\delta}{2}\right)(X_0+Y)$$

and compute

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}=-\,2\iint_{\mathbb{R}^d\times\mathbb{R}^d}f\,\mathsf{L}f\,\mathrm{d}x\,\mathrm{d}\mu+\delta(t)\int_{\mathbb{R}^d}\mathsf{R}_\xi[\widehat{f}]\,\mathrm{d}\xi+\delta'(t)\,\mathrm{Re}\left(\int_{\mathbb{R}^d}\langle\mathsf{A}f,f\rangle\,\mathrm{d}x\right).$$

Using (17) and (18), we get the estimate

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t}\gtrsim\delta\,Y^{1+\frac{\alpha}{d}}-\delta\,X_\eta+r\,X_\eta-r^{\frac{k-\beta}{\eta-\beta}}-\tfrac{\delta\,\varepsilon}{1+\varepsilon\,t}\,\mathcal{H}\,.$$

We recall that $\gtrsim$ means that the inequality holds up to a positive, finite constant, which changes from line to line. Next we choose $r=2\,\delta$, $\delta_0>0$ and $\varepsilon>0$ small enough so that the above r.h.s. of the inequality is positive. However, we shall still do some further reductions before fixing the values of δ_0 and ε. The decay rate of $\mathcal{H}$ is governed by

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \gtrsim \delta\, Y^{1+\frac{\alpha}{d}} + \delta\, X_\eta - \delta^{\frac{k-\beta}{\eta-\beta}} - \tfrac{\delta\,\varepsilon}{1+\varepsilon\,t}\,\mathcal{H}\,,$$

with a positive r.h.s. at $t=0$. Using Hölder's inequality

$$X_0 \le X_\eta^{\frac{k}{k-\eta}}\,X_k^{\frac{\eta}{\eta-k}}$$

and the fact that X_k is uniformly bounded in t by a positive constant depending only on the initial datum, we obtain

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \gtrsim \delta\, Y^{1+\frac{\alpha}{d}} + \delta\, X_0^{1-\frac{\eta}{k}} - \delta^{\frac{k-\beta}{\eta-\beta}} - \tfrac{\delta\,\varepsilon}{1+\varepsilon\,t}\,\mathcal{H}\,,$$

and we can still assume that the inequality has a positive r.h.s. at $t=0$ without loss of generality. It is now clear that

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \gtrsim \delta\left(\mathcal{H}^{1+\kappa} - \delta^{\frac{k-\eta}{\eta-\beta}} - \tfrac{\varepsilon}{1+\varepsilon\,t}\,\mathcal{H}\right) \quad \text{with} \quad \kappa = \max\left\{\frac{\alpha}{d}, -\frac{\eta}{k}\right\}.$$

Up to a multiplication by a constant, we can actually fix the mutiplicative constant to a given value $\tau>0$ that will be chosen below (with the corresponding redefinition of ε and δ_0) so that the differential inequality is

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \ge \tau\,\delta\left(3\,\mathcal{H}^{1+\kappa} - \delta^{\frac{k-\eta}{\eta-\beta}} - \tfrac{\varepsilon}{1+\varepsilon\,t}\,\mathcal{H}\right).$$

Now, let us fix $\delta_0>0$ and $\varepsilon>0$ small enough so that

$$\delta_0^{\frac{k-\eta}{\eta-\beta}} + \varepsilon\,\mathcal{H}_0 \le \mathcal{H}_0^{1+\kappa}$$

with $\mathcal{H}_0 = \mathcal{H}(t=0)$. We have to check that this condition is stable under the evolution, that is,

$$\forall\, t\ge 0\,,\quad \delta(t)^{\frac{k-\eta}{\eta-\beta}} \le \mathcal{H}(t)^{1+\kappa} \quad \text{and} \quad \varepsilon\,\mathcal{H}(t) \le \mathcal{H}(t)^{1+\kappa}\,. \tag{19}$$

Keeping track of the coefficients is paid by unnecessary complications, so that we are going to make some simplifying assumptions, in order to emphasize the key idea of the estimate. Up to a change of variable $t\mapsto\varepsilon\,t$, we can choose $\varepsilon=1$ and also take $\delta_0=1$ and $\mathcal{H}_0=1$ without loss of generality, so that, in particular,

$$\forall\, t\ge 0\,,\quad \delta(t) = (1+t)^{-\mathsf{a}}\,.$$

As a result, let us consider the differential inequality

$$-\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} \ge \tau\,\delta\left(3\,\mathcal{H}^{1+\kappa} - \delta^{\frac{k-\eta}{\eta-\beta}} - \tfrac{\mathcal{H}}{1+t}\right).$$

We aim at showing that

$$\mathcal{H}(t) \le \overline{\mathcal{H}}(t) := (1+t)^{-\,\tau} \tag{20}$$

where $\overline{\mathcal{H}}$ solves

$$\frac{\mathrm{d}\overline{\mathcal{H}}'}{\mathrm{d}t} = -\,\tau\,\delta\,\overline{\mathcal{H}}^{1+\kappa} \quad \text{with} \quad \tau = \frac{1-\mathsf{a}}{\kappa}\,.$$

As a consequence of

$$\delta^{\frac{k-\eta}{\eta-\beta}} = (1+t)^{-\mathsf{a}\,\frac{k-\eta}{\eta-\beta}} \quad \text{and} \quad \tfrac{\overline{\mathcal{H}}}{1+t} = (1+t)^{-\frac{1-\mathsf{a}}{\kappa}-1}\,,$$

we learn that

$$\forall\,t\ge 0\,,\quad \delta(t)^{\frac{k-\eta}{\eta-\beta}} \le \overline{\mathcal{H}}(t)^{1+\kappa} \quad \text{and} \quad \tfrac{\overline{\mathcal{H}}}{1+t} \le \overline{\mathcal{H}}(t)^{1+\kappa}$$

under the condition that

$$-\,\mathsf{a}\,\frac{k-\eta}{\eta-\beta} \le -\,(1-\mathsf{a})\,\frac{1+\kappa}{\kappa} \quad \text{and} \quad \mathsf{a}\ge 0\,. \tag{21}$$

Since $-\,\frac{\mathrm{d}\mathcal{H}'}{\mathrm{d}t} \le -\,\frac{\mathrm{d}\overline{\mathcal{H}}'}{\mathrm{d}t}$ if $\mathcal{H}(t)=\overline{\mathcal{H}}(t)$, it is then clear that (19) holds and $\mathcal{H}(t)\le \overline{\mathcal{H}}(t)$ for any $t\ge 0$. In other words, $\overline{\mathcal{H}}$ is a barrier function and (20) holds for any $t\ge 0$. The result is also true for the generic case.

With the choice $\mathsf{a}=(\beta-\eta)/\beta$, Condition (21) is satisfied if $\beta\le\eta\le 0\le k$ and $\kappa=|\eta|/k$, with $\tau=k/|\beta|$ if $d\ge 2$ because

$$\alpha = \frac{\gamma-\beta}{1-\beta} \le \max\left\{1, \frac{2\,\gamma}{1+\gamma}\right\}$$

and because $\kappa=|\eta|/k>\alpha/d$ for an appropriate choice of $\eta\in(-\,\gamma,0)$ and $k\in(0,|\eta|\,d/\alpha)$. The same argument applies if $d=1$ and $\gamma\le 1$.

If $d=1$, in the range $1\le\gamma\le|\beta|$, we have $\alpha>1$ and distinguish two cases.
• Either $\kappa=\alpha>|\eta|/k$: with $|\eta|/\alpha<k<\gamma$ and $\eta>-\,\gamma$, we find that $\tau=\tau_\star(\eta,k)$ and $a=a_\star(\eta,k)$ where

$$\tau_\star(\eta,k) := \frac{k-\eta}{\alpha\,(k-\beta)+\eta-\beta} \quad \text{and} \quad a_\star(\eta,k) := \frac{(1+\alpha)\,(\eta-\beta)}{\alpha\,(k-\beta)+\eta-\beta}\,.$$

Using $\frac{\partial\tau_\star}{\partial k}>0$ and $\frac{\partial\tau_\star}{\partial\eta}<0$, the largest admissible value of $\tau_\star$ is achieved by

$$\lim_{(\eta,k)\to(-\,\gamma,\gamma)} \tau_\star(\eta,k) = \frac{2\,\gamma}{\alpha\,(\gamma-\beta)+|\beta+\gamma|}.$$

• Or $\alpha\le|\eta|/k=\kappa$: in that case, we can take $\mathsf{a}=(\beta-\eta)/\beta$, Condition (21) is satisfied if $-\,\gamma\le\eta\le 0\le k\le|\eta|/\alpha<\gamma/\alpha$, $\tau=k/|\beta|$ and $\kappa=|\eta|/k$. This is possible as soon as $k<\gamma/\alpha$ since this condition is then verified if we take any $\eta\in(-\,\gamma,-\,k\,\alpha)$.

This completes the proof of Theorems 1, 2 and 3 with $\tau=k/|\beta|$, except if $d=1$, $1\le\gamma\le|\beta|$ and $k\in[\gamma/\alpha,\gamma)$, where the rate can be chosen arbitrarily close to $\tau_\star(-\,\gamma,k)$.

8.2. The case of a flat torus. As in [11], the case of the flat d-dimensional torus $\mathbb{T}^d$ (with position $x \in \mathbb{T}^d$ and velocity $v \in \mathbb{R}^d$) follows from our method without additional efforts. In that case, Equation (1) admits a global equilibrium given by $f_\infty = \rho_\infty F$ with $\rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f^{\rm in} \, \mathrm{d}x \, \mathrm{d}v$, and the rate of convergence to the equilibrium is just given by the microscopic dynamics

$$\|f - f_\infty\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \lesssim e^{-\lambda\,t} \, \|f^{\rm in} - f_\infty\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \qquad \text{if} \quad 0 \le \beta < \gamma\,,$$
$$\|f - f_\infty\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)} \lesssim (1+t)^{-k/|\beta|} \, \|f^{\rm in} - f_\infty\|_{\mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)} \qquad \text{if} \quad \beta < 0\,,$$

with $k \in (0, \gamma)$. In particular, if $f = f(t, v)$ does not depend on x, then (1) is reduced to the homogeneous equation $\partial_t f = \mathsf{L} f$ and we recover the rate of convergence of f to F in the norm $\mathrm{L}^2(\mathrm{d}\mu)$, as in Section 1. This is coherent with the results in [3,41,40,17,28,2]. Moreover, we point out that our result is a little bit stronger than some of those results, because it relies on a finite $\|f^{\rm in}\|_{\mathrm{L}^2(\langle v\rangle^k \mathrm{d}x\,\mathrm{d}\mu)}$ norm for the initial condition, which is a weaker condition than the usual boundedness condition on $\|f^{\rm in} F^{-1}\|_{\mathrm{L}^\infty(\mathrm{d}x\,\mathrm{d}v)}$, or H^1-type estimates as in [2, Section 6], where $\beta = 0$ in Case (b). Remark however that weighted L^2 norms already appear in the homogeneous case in [27,28].

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