HYPOCOERCIVITY AND SUB-EXPONENTIAL LOCAL EQUILIBRIA

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Abstract. Hypocoercivity methods are applied to linear kinetic equations without any space confinement, when local equilibria have a sub-exponential decay. By Nash type estimates, global rates of decay are obtained, which reflect the behavior of the heat equation obtained in the diffusion limit. The method applies to Fokker-Planck and scattering collision operators. The main tools are a weighted Poincaré inequality (in the Fokker-Planck case) and norms with various weights. The advantage of weighted Poincaré inequalities compared to the more classical weak Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential and exponential local equilibria to sub-exponential local equilibria.

1. Introduction

This paper is devoted to a *hypocoercivity* method designed for obtaining decay rates in weighted L^2 norms of the solution to the Cauchy problem

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = \mathsf{L} f \,, \\ f(0,x,v) = f^{\mathrm{in}}(x,v) \,, \end{cases} \tag{1}$$

for a distribution function $f(t,x,v)$, with *position* $x \in \mathbb{R}^d$, *velocity* $v \in \mathbb{R}^d$, and *time* $t \geq 0$. The linear *collision operator* L acts only on the velocity variable and its null space is assumed to be one-dimensional and spanned by the *local equilibrium* F, a probability density of the form

$$F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha} \,, \quad v \in \mathbb{R}^d \,, \qquad \text{with } C_\alpha^{-1} = \int_{\mathbb{R}^d} e^{-\langle v \rangle^\alpha} \, \mathrm{d}v \,, \tag{2}$$

where we use the notation

$$\langle v \rangle := \sqrt{1 + |v|^2} \,.$$

Our results will be concerned with the sub-exponential case $0 < \alpha < 1$, as opposed to the exponential ($\alpha = 1$) and super-exponential ($\alpha > 1$, including the Gaussian with $\alpha = 2$) cases. This specific choice of the form of the equilibrium is for notational convenience in the proofs. The results can easily be extended to more general

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distributions F, satisfying

$$\alpha := \lim_{|v|\to+\infty} \frac{\log(-\log F(v))}{\log|v|} \in (0,1)\,.$$

We shall consider two types of collision operators, either the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(F\,\nabla_v\big(F^{-1}\,f\big)\Big)\,,$$

or the *scattering* operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \mathrm{b}(\cdot,v')\,\Big(f(v')\,F(\cdot) - f(\cdot)\,F(v')\Big)\,\mathrm{d}v'\,.$$

We assume *local mass conservation*

$$\int_{\mathbb{R}^d} \mathsf{L} f\,\mathrm{d}v = 0\,,$$

which always holds for $\mathsf{L} = \mathsf{L}_1$, and also for $\mathsf{L} = \mathsf{L}_2$ under the assumption

$$\int_{\mathbb{R}^d} \Big(\mathrm{b}(v,v') - \mathrm{b}(v',v)\Big)\,F(v')\,\mathrm{d}v' = 0\,. \tag{H1}$$

Note that *micro-reversibility*, *i.e.*, the symmetry of b, is not required.

Further assumptions on the *cross-section* b that will be given below guarantee that the operators L_1 and L_2 are responsible for the same type of asymptotic behavior. As a motivation, the relaxation properties of L_1 can be made transparent by the symmetrizing transformation $f = g\,\sqrt{F}$, leading to the transformed operator

$$g \mapsto \frac{1}{\sqrt{F}}\,\nabla_v \cdot \left(F\,\nabla_v\,\frac{g}{\sqrt{F}}\right) = \Delta_v g - \nu_1(v)\,g$$

with the *collision frequency*

$$\nu_1(v) = \frac{\Delta_v F}{2\,F} - \frac{|\nabla_v F|^2}{4\,F^2} \approx \frac{\alpha^2}{4}\,|v|^{-2\,(1-\alpha)} \qquad \text{as } |v| \to \infty\,. \tag{3}$$

Partially motivated by this, we assume the existence of constants β, $\overline{\mathrm{b}}$, $\underline{\mathrm{b}} > 0$, $\gamma \ge 0$, with $\gamma \le \beta$, $\gamma < d$, such that

$$\underline{\mathrm{b}}\,\langle v\rangle^{-\beta}\,\langle v'\rangle^{-\beta} \le \mathrm{b}(v,v') \le \overline{\mathrm{b}}\,\min\Big\{|v-v'|^{-\beta}, |v-v'|^{-\gamma}\Big\}\,. \tag{H2}$$

The upper bound with the restriction on the exponent γ is a local integrability assumption. Hypotheses (H1) and (H2) allow for the choice $\mathrm{b}(v,v') = \langle v\rangle^{-\beta}\,\langle v'\rangle^{-\beta}$ with arbitrary $\beta > 0$, as well as Boltzmann kernels $\mathrm{b}(v,v') = |v-v'|^{-\beta}$ with $0 < \beta < d$.

As a consequence of (H2) the collision frequency

$$\nu_2(v) = \int_{\mathbb{R}^d} \mathrm{b}(v,v')\,F(v')\,\mathrm{d}v'$$

satisfies

$$\underline{\mathrm{b}}\,\langle v\rangle^{-\beta}\int_{\mathbb{R}^d}\langle v'\rangle^{-\beta}\,F(v')\,\mathrm{d}v'\le\nu_2(v)\le\overline{\mathrm{b}}\int_{|v-v'|<1}|v-v'|^{-\gamma}\,F(v')\,\mathrm{d}v' + \overline{\mathrm{b}}\int_{|v-v'|>1}|v-v'|^{-\beta}\,F(v')\,\mathrm{d}v'\,.$$

It is obvious that the last term is $O(|v|^{-\beta})$ as $|v|\to\infty$, and the first term on the right hand side is asymptotically small compared to that as a consequence of the sub-exponential decay of F. Therefore there exist constants $\overline{\nu}\ge\underline{\nu}>0$ such that

$$\underline{\nu}\,\langle v\rangle^{-\beta}\le\nu_2(v)\le\overline{\nu}\,\langle v\rangle^{-\beta}\qquad\forall\,v\in\mathbb{R}^d\,,\tag{4}$$

and the behavior for large $|v|$ is as in (3) with $\beta=2\,(1-\alpha)$.

Since both collision operators are propagators of Markov processes with the same positive stationary distribution F, they also share the (quadratic) entropy dissipation property

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^d\times\mathbb{R}^d}f^2\,\mathrm{d}x\,\mathrm{d}\mu=\iint_{\mathbb{R}^d\times\mathbb{R}^d}(\mathsf{L}f)\,f\,\mathrm{d}x\,\mathrm{d}\mu\le 0\,,\qquad\text{with }\ \mathrm{d}\mu(v):=\frac{\mathrm{d}v}{F(v)}\,,\tag{5}$$

where the dissipations are given by

$$-\int_{\mathbb{R}^d}(\mathsf{L}_1 f)\,f\,\mathrm{d}\mu=\int_{\mathbb{R}^d}\left|\nabla_v\frac{f}{F}\right|^2F\,\mathrm{d}v\tag{6}$$

and

$$-\int_{\mathbb{R}^d}(\mathsf{L}_2 f)\,f\,\mathrm{d}\mu=\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\mathrm{b}(v,v')\Big(f'F-f\,F'\Big)^2\mathrm{d}\mu\,\mathrm{d}\mu'\,,\tag{7}$$

with the prime denoting evaluation at v'. For a derivation of (7) see, *e.g.*, [8, 21].

Our purpose is to consider solutions of (1) with non-negative initial datum f^{in} and to study their large time behavior. If f^{in} has finite mass, then mass is conserved for any $t\ge 0$. Since there is no stationary state with finite mass, it is expected that $f(t,\cdot,\cdot)$ locally tends to zero as $t\to+\infty$. However, the dissipations (6) and (7) vanish for arbitrary *local equilibria* of the form $f(t,x,v)=\rho(t,x)\,F(v)$, and therefore the analysis of the decay to zero requires an *hypocoercivity* method. Our approach relies on the construction of Lyapunov functionals by modifications of natural entropies or norm as in [10, 11] (also see [13, Lemma 4.1], [14, Lemma 4.1], [25] and [29] for earlier contributions).

For the formulation of our main result, we introduce the norms

$$\|f\|_k:=\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d}f^2\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)^{1/2},\qquad k\in\mathbb{R},\tag{8}$$

as well as the scalar product $\langle f_1,f_2\rangle:=\iint_{\mathbb{R}^d\times\mathbb{R}^d}f_1\,f_2\,\mathrm{d}x\,\mathrm{d}\mu$ on $\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$ with the induced norm $\|f\|^2:=\|f\|_0^2=\langle f,f\rangle$. By $\mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$, we denote the space of nonnegative Lebesgue integrable functions on $\mathbb{R}^d\times\mathbb{R}^d$, and by $\mathcal{D}(\Omega)$ the space of smooth functions with compact support in the open domain Ω where, in practice either $\Omega=\mathbb{R}^d$ or $\Omega=\mathbb{R}^d\times\mathbb{R}^d$.

Theorem 1. *Let $\alpha\in(0,1)$, $\beta>0$, $k>0$ and let F be given by (2). Assume that either $\mathsf{L}=\mathsf{L}_1$ and $\beta=2\,(1-\alpha)$, or $\mathsf{L}=\mathsf{L}_2$ and (H1), (H2) hold. Then there exists a constant $\mathcal{C}>0$ such that any solution f of (1) with initial datum $f^{\rm in}\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)\cap\mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$ satisfies*

$$\|f(t,\cdot,\cdot)\|^2\le\mathcal{C}\,\frac{\|f^{\rm in}\|^2}{(1+\kappa\,t)^\zeta}\qquad\forall\,t\ge0$$

with rate $\zeta=\min\{d/2,k/\beta\}$ and with $\kappa>0$, which is an explicit function of the two quotients $\|f^{\rm in}\|\,/\,\|f^{\rm in}\|_k$ and $\|f^{\rm in}\|\,/\,\|f^{\rm in}\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}$.

The proof relies on the L^2-hypocoercivity approach of [10, 11]. An important ingredient is *microscopic coercivity,* meaning that the entropy dissipation controls the distance to the set of local equilibria. For $\mathsf{L}=\mathsf{L}_1$ and for the exponential and super-exponential cases $\alpha\ge1$, this control is provided by the *Poincaré inequality*

$$\int_{\mathbb{R}^d}|\nabla_v g|^2\,F\;\mathrm{d}v\ge\mathcal{C}_P\int_{\mathbb{R}^d}(g-\overline{g})^2\,F\;\mathrm{d}v\,,\tag{9}$$

with $\overline{g}=\int_{\mathbb{R}^d}g F\,\mathrm{d}v$ and $\mathcal{C}_P>0$ implying, with $g=f/F$,

$$-\,\langle\mathsf{L}_1f,f\rangle\ge\mathcal{C}_P\,\|f-\rho_fF\|^2\,,$$

with $\rho_f=\int_{\mathbb{R}^d}f\,dv$. A result similar to Theorem 1 (with $\alpha=2$, $k=0$ and $\zeta=d/2$) has been proven in [5]. In the sub-exponential case of this work, we shall prove a relaxed version of the above Poincaré inequality.

Lemma 2. *Let F be given by (2) with $0<\alpha<1$. Let either $\mathsf{L}=\mathsf{L}_1$ and $\beta=2\,(1-\alpha)$ or $\mathsf{L}=\mathsf{L}_2$ assuming (H1), (H2). Then there exists $\mathcal{C}>0$ such that*

$$-\,\langle\mathsf{L}f,f\rangle\ge\mathcal{C}\,\|f-\rho_fF\|^2_{-\beta}\qquad\forall\,f\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}^d)\,.$$

Proof. For $\mathsf{L}=\mathsf{L}_1$ the result is a consequence of the *weighted Poincaré inequality*

$$\int_{\mathbb{R}^d}|\nabla_v g|^2F\,\mathrm{d}v\ge\mathcal{C}\int_{\mathbb{R}^d}(g-\overline{g})^2\,\langle v\rangle^{-2\,(1-\alpha)}\,F\,\mathrm{d}v\qquad\forall\,g\in\mathcal{D}(\mathbb{R}^d)\,,\tag{10}$$

which will be proved in Appendix A.

For $\mathsf{L}=\mathsf{L}_2$ we estimate

$$\int_{\mathbb{R}^d}(f-\rho_fF)^2\,\langle v\rangle^{-\beta}\;\mathrm{d}\mu=\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\left(f\,F'-f'F\right)F'\,\mathrm{d}\mu'\right)^2\langle v\rangle^{-\beta}\;\mathrm{d}\mu$$
$$\le\int_{\mathbb{R}^d}F\,\langle v\rangle^\beta\;\mathrm{d}v\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(f\,F'-f'F\right)^2\langle v\rangle^{-\beta}\,\langle v'\rangle^{-\beta}\,\mathrm{d}\mu\,\mathrm{d}\mu'\le-\frac{1}{\mathcal{C}}\int_{\mathbb{R}^d}(\mathsf{L}_2f)\,f\;\mathrm{d}\mu$$

with

$$\mathcal{C}=\frac{\mathrm{b}}{2}\left(\int_{\mathbb{R}^d}F\,\langle v\rangle^\beta\;\mathrm{d}v\right)^{-1}\,.$$

For the first inequality we have used Cauchy-Schwarz and for the second, (7) and the hypothesis (H2). Integration with respect to x completes the proof. □

Apart from proving the weighted Poincaré inequality (10), in Appendix A, we shall also show in Appendix B how it can be used to prove algebraic decay to equilibrium for the spatially homogeneous equation with $\mathsf{L} = \mathsf{L}_1$ and $0 < \alpha < 1$, *i.e.* the Fokker-Planck equation with sub-exponential equilibrium. The loss of information due to the weight $\langle v\rangle^{-2(1-\alpha)}$ has to be compensated by a L^2-bound for the initial datum with a weight $\langle v\rangle^k$, $k > 0$, as in Theorem 1. For this problem, estimates based on *weak Poincaré inequalities* are also very popular in the scientific community of semi-group theory and Markov processes (see [27], [3, Proposition 7.5.10], [17] and Appendix B). Estimates based on weak Poincaré inequalities rely on a uniform bound for the initial data for $\alpha < 1$ which is not needed for $\alpha \geq 1$, while the approach developed in this paper provides a continuous transition from the range $0 < \alpha < 1$ to the range $\alpha \geq 1$ since we may choose $k \searrow 0$ as $\alpha \nearrow 1$. Note that for $\alpha = 1$, the weighted Poincaré inequality (10) reduces to the Poincaré inequality (9).

The proof of Theorem 1 goes along the lines of the hypocoercivity approach (with $\alpha \geq 1$) of [10, 11] (also see [13, 14, 25, 29]) and its extension to cases without confinement as in [5, 6]. It combines information on the *microscopic* and the *macroscopic* dissipation properties. The core of the microscopic results is given in Lemma 2. Since the macroscopic limit of (1) is the heat equation on the whole space, it is natural that for the estimation of the macroscopic dissipation we use *Nash's inequality,*

$$\|u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)} \leq \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(\,\mathrm{d}x)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(\,\mathrm{d}x)}\,, \tag{11}$$

a tool which has been developed for this purpose. The result of Theorem 1 can be interpreted as giving the weaker of the microscopic decay rate $t^{-k/\beta}$ and the macroscopic decay rate $t^{-d/2}$. Only for $k \geq \beta\,d/2$, the decay rate of the macroscopic diffusion limit is recovered.

Related results have been shown in [5, 6, 7]. Results in [7] are somewhat complementary to this work, as they deal with Gaussian local equilibria in the presence of an external potential with sub-exponential growth in the variable x. Also see [30, 19, 16] for various earlier results dealing with external potentials with a growth like $\langle x\rangle^\gamma$, $\gamma < 1$, based on weak Poincaré inequalities, spectral techniques, H^1 hypocoercivity methods, *etc.*

This paper is organized as follows. In Section 2, we prove an hypocoercive estimate relating a modified entropy, which is equivalent to $\|f\|^2$, to an entropy production term involving a microscopic and a macroscopic component. Using weighted L^2-estimates established in Section 3, we obtain a new control by the microscopic component in Lemma 8 while the macroscopic component is estimated as in [5] using Nash's inequality, see Lemma 7. By collecting these estimates in Section 4, we complete the proof of Theorem 1. Two appendices are devoted to $\mathsf{L} = \mathsf{L}_1$: in Appendix A, we provide a new proof of (10) and comment on the interplay with weak Poincaré inequalities, while the spatially homogeneous version of (1) is dealt with in Appendix B and rates of relaxation towards the local equilibrium are discussed using weighted L^2-norms, as an alternative approach to the weak Poincaré inequality method of [17]. The main novelty of our approach is that we use new interpolations in order to exploit the entropy production term. As a consequence, with the appropriate weights, no other norm is needed than weighted L^2-norms. For simplicity, we assume that the distribution function is nonnegative but the extension to sign changing functions is straightforward.

2. An entropy–entropy production estimate

We adapt the strategy of [11, 5], denoting by $\mathsf{T} = v \cdot \nabla_x$ the free streaming operator and by Π the orthogonal projection on $\mathrm{Ker}(\mathsf{L})$ in $\mathrm{L}^2(\mathrm{d}\mu)$, given by

$$\Pi f := \rho_f \, F \qquad \text{where } \rho_f = \int_{\mathbb{R}^d} f \, \mathrm{d}v \, .$$

To build a suitable Lyapunov functional, we introduce the operator

$$\mathsf{A} := \Big(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\Big)^{-1}(\mathsf{T}\Pi)^*$$

and consider

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A}f, f\rangle \, .$$

All computations can be done with functions f in $\mathcal{D}(\mathbb{R}^d \times \mathbb{R}^d)$ and later extended by density to natural functional spaces. It is known from [11, Lemma 1] that A is a bounded operator on $\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)$ with operator norm bounded by $1/2$, such that, for any $\delta \in (0,1)$, $\mathsf{H}[f]$ and $\|f\|^2$ are equivalent in the sense that

$$\frac{1}{2}\,(1-\delta)\,\|f\|^2 \le \mathsf{H}[f] \le \frac{1}{2}\,(1+\delta)\,\|f\|^2 \, . \tag{12}$$

A direct computation shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] = -\,\mathsf{D}[f] \tag{13}$$

with

$$\begin{aligned}\mathsf{D}[f] := &-\langle \mathsf{L}f, f\rangle + \delta\,\langle \mathsf{AT}\Pi f, \Pi f\rangle \\ &+ \delta\,\langle \mathsf{AT}(\mathsf{Id}-\Pi)f, \Pi f\rangle - \delta\,\langle \mathsf{TA}(\mathsf{Id}-\Pi)f, (\mathsf{Id}-\Pi)f\rangle - \delta\,\langle \mathsf{AL}(\mathsf{Id}-\Pi)f, \Pi f\rangle\end{aligned}$$

where we have used that $\langle \mathsf{A}f, \mathsf{L}f\rangle = 0$. With our notation, the result of Lemma 2 reads

$$\langle \mathsf{L}f, f\rangle \le -\,\mathcal{C}\,\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2 \, . \tag{14}$$

It is the essence of the approach of [11] that the second term in $\mathsf{D}[f]$ controls the macroscopic contribution $\|\Pi f\|^2$ and that the first two terms control the remaining ones.

Proposition 3. *Under the assumptions of Theorem 1 and for small enough $\delta > 0$, there exists $\kappa > 0$ such that, for any $f \in \mathrm{L}^2\left(\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,*

$$\mathsf{D}[f] \geq \kappa\left(\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2 + \langle\mathsf{AT}\Pi f, \Pi f\rangle\right).$$

Note that κ does not depend on $k > 0$ (the parameter k appears in the assumptions of Theorem 1). An estimate of $\mathsf{D}[f]$ in terms of $\langle\mathsf{AT}\Pi f, \Pi f\rangle$ and $\|(\mathsf{Id}-\Pi)f\|^2$ has already been derived in [5], but using the weighted norm $\|(\mathsf{Id}-\Pi)f\|_{-\beta}$ is a new idea.

Proof. We have to prove that the three last terms in $\mathsf{D}[f]$ are controlled by the first two. The main difference with [11, 5] is the additional weight $\langle v\rangle^{-\beta}$ in the velocity variable.

- ***Step 1: rewriting*** $\langle\mathsf{AT}\Pi f, \Pi f\rangle$***.*** Let $u = u_f$ be such that

$$u\,F = \left(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\right)^{-1}\Pi f\,.$$

Then u solves $(u - \Theta\,\Delta u)\,F = \Pi f$, that is,

$$u - \Theta\,\Delta u = \rho_f \qquad \text{where } \Theta := \int_{\mathbb{R}^d} |v\cdot\mathsf{e}|^2\,F(v)\,\mathrm{d}v\,, \tag{15}$$

for an arbitrary unit vector e. Since

$$\begin{aligned}\mathsf{AT}\Pi f &= \left(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\right)^{-1}(\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\,\Pi f\\ &= \Big(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\Big)^{-1}\Big(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi) - \mathsf{Id}\Big)\,\Pi f\\ &= \Pi f - \Big(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\Big)^{-1}\Pi f = \Pi f - u\,F = (\rho_f - u)\,F\,,\end{aligned}$$

then by using equation (15), we obtain

$$\langle\mathsf{AT}\Pi f, \Pi f\rangle = \langle\Pi f - u\,F, \Pi f\rangle = \langle -\Theta\,\Delta u\,F, (u - \Theta\,\Delta u)\,F\rangle\,,$$

from which we deduce

$$\langle\mathsf{AT}\Pi f, \Pi f\rangle = \Theta\,\|\nabla u\|^2_{\mathrm{L}^2(\mathrm{d}x)} + \Theta^2\,\|\Delta u\|^2_{\mathrm{L}^2(\mathrm{d}x)} \geq 0\,. \tag{16}$$

Both terms on the right hand side are finite, since (15) defines $\rho_f \mapsto u$ as a bounded map $\mathrm{L}^2(\mathrm{d}x) \to \mathrm{H}^2(\mathrm{d}x)$.

- ***Step 2: a bound on*** $\langle\mathsf{AT}(\mathsf{Id}-\Pi)f, \Pi f\rangle$***.*** If u solves (15), we use the fact that

$$\mathsf{A}^*\Pi f = \mathsf{T}\Pi\,u\,F = \mathsf{T}u\,F \tag{17}$$

to compute

$$\langle\mathsf{AT}(\mathsf{Id}-\Pi)f, \Pi f\rangle = \langle(\mathsf{Id}-\Pi)f, \mathsf{T}^*\mathsf{A}^*\Pi f\rangle = \langle(\mathsf{Id}-\Pi)f, \mathsf{T}^*\mathsf{T}u\,F\rangle\,.$$

Therefore, since $\mathsf{T}^*\mathsf{T}u\,F = -\,v\cdot\nabla_x\,(v\cdot\nabla_x u)\,F$, the Cauchy-Schwarz inequality yields

$$\begin{aligned}|\langle \mathsf{AT}(\mathsf{Id}-\Pi)f,\Pi f\rangle| &\le \|(\mathsf{Id}-\Pi)f\|_{-\beta}\,\left\|\sqrt{F}\,\langle v\rangle^{\frac{\beta}{2}}\,v\cdot\nabla_x\,(v\cdot\nabla_x u)\,\sqrt{F}\right\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)}\\ &\le \Theta_{\beta+4}\,\|(\mathsf{Id}-\Pi)f\|_{-\beta}\,\|\Delta u\|_{\mathrm{L}^2(\mathrm{d}x)}\,,\end{aligned}$$

hence

$$|\langle \mathsf{AT}(\mathsf{Id}-\Pi)f,\Pi f\rangle| \le \mathcal{C}_4\,\|(\mathsf{Id}-\Pi)f\|_{-\beta}\,\langle \mathsf{AT}\Pi f,\Pi f\rangle^{\frac12} \tag{18}$$

where we have used identity (16), $\mathcal{C}_4 = \Theta_{\beta+4}/\Theta$ and

$$\Theta_k := \int_{\mathbb{R}^d}\langle v\rangle^k\,F(v)\,\mathrm{d}v\,.$$

With this convention, note that $\Theta_2 = 1 + d\,\Theta$.

• ***Step 3: estimating*** $\langle \mathsf{TA}(\mathsf{Id}-\Pi)f,(\mathsf{Id}-\Pi)f\rangle$***.*** As noted in [11, Lemma 1], the equation $g = \Pi g = \mathsf{A}f$ is equivalent to

$$\Big(\mathsf{Id} + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)\Big)g = (\mathsf{T}\Pi)^*f$$

which, after multiplying by g and integrating, yields

$$\begin{aligned}\|g\|^2 + \|\mathsf{T}g\|^2 &= \langle g, g + (\mathsf{T}\Pi)^*(\mathsf{T}\Pi)g\rangle\\ &= \langle g,(\mathsf{T}\Pi)^*f\rangle = \langle \mathsf{T}\Pi g, f\rangle = \langle \mathsf{TA}f,f\rangle \le \|(\mathsf{Id}-\Pi)f\|_{-\beta}\,\|\mathsf{TA}f\|_\beta\end{aligned}$$

by the Cauchy-Schwarz inequality. We know that $(\mathsf{T}\Pi)^* = -\,\Pi\mathsf{T}$, so that $\mathsf{A}f = g = w\,F$ is determined by the equation

$$w - \Theta\,\Delta w = -\,\nabla_x\cdot\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\,.$$

After multiplying by w, now a function in $\mathrm{H}^1(\mathrm{d}x)$ by elliptic regularity of the solution of the above equation, and integrating in x, we obtain

$$\begin{aligned}\Theta\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x &\le \int_{\mathbb{R}^d}|w|^2\,\mathrm{d}x + \Theta\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x\\ &\le \left(\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x\right)^{\frac12}\left(\int_{\mathbb{R}^d}\left|\textstyle\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\right|^2\,\mathrm{d}x\right)^{\frac12}\end{aligned}$$

and note that

$$\begin{aligned}\int_{\mathbb{R}^d}\left|\textstyle\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\right|^2\,\mathrm{d}x &= \int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\langle v\rangle^{-\frac{\beta}{2}}\,\frac{(\mathsf{Id}-\Pi)f}{\sqrt{F}}\cdot|v|\,\langle v\rangle^{\frac{\beta}{2}}\,\sqrt{F}\,\mathrm{d}v\right|^2\,\mathrm{d}x\\ &\le \Theta_{\beta+2}\,\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2\end{aligned}$$

by the Cauchy-Schwarz inequality. Hence

$$\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \le \frac{\Theta_{\beta+2}}{\Theta^2}\,\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2$$

and

$$\|\mathsf{TA}f\|_\beta^2 = \left\|\nabla_x w \cdot \left(v\,\langle v\rangle^{\beta/2}\,F\right)\right\|^2 = \Theta_{\beta+2}\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \le \mathcal{C}_2^2\,\|(\mathsf{Id}-\mathsf{\Pi})f\|_{-\beta}^2$$

with $\mathcal{C}_2 := \Theta_{\beta+2}/\Theta$. Since $g = \mathsf{A}f$ so that $\|\mathsf{A}f\|^2 + \|\mathsf{TA}f\|^2 = \|g\|^2 + \|\mathsf{T}g\|^2$, we obtain

$$\langle \mathsf{TA}f, f\rangle = \langle \mathsf{TA}(\mathsf{Id}-\mathsf{\Pi})f, (\mathsf{Id}-\mathsf{\Pi})f\rangle \le \|(\mathsf{Id}-\mathsf{\Pi})f\|_{-\beta}\,\|\mathsf{TA}f\|_\beta \le \mathcal{C}_2\,\|(\mathsf{Id}-\mathsf{\Pi})f\|_{-\beta}^2\,. \tag{19}$$

We also remark that

$$\langle \mathsf{TA}f, f\rangle = \langle (v\cdot\nabla_x w)\,F, f\rangle = \int_{\mathbb{R}^d}\nabla_x w\cdot\left(\int_{\mathbb{R}^d} v\,f\,\mathrm{d}v\right)\mathrm{d}x$$
$$= \int_{\mathbb{R}^d}|w|^2\,\mathrm{d}x + \Theta\int_{\mathbb{R}^d}|\nabla_x w|^2\,\mathrm{d}x \ge 0\,.$$

- ***Step 4: bound for*** $\langle \mathsf{AL}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle$***.*** We use again identity (17) to compute

$$|\langle \mathsf{AL}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle| = |\langle (\mathsf{Id}-\mathsf{\Pi})f, \mathsf{L}^*\mathsf{A}^*\mathsf{\Pi}f\rangle| = |\langle (\mathsf{Id}-\mathsf{\Pi})f, \mathsf{L}^*\mathsf{T}u\,F\rangle|$$
$$\le \|(\mathsf{Id}-\mathsf{\Pi})f\|_{-\beta}\,\|\mathsf{L}^*\mathsf{T}u\,F\|_\beta\,.$$

In case $\mathsf{L} = \mathsf{L}_1$ we remark that

$$\|\mathsf{L}_1^*\mathsf{T}u\,F\|_\beta^2 = \iint_{\mathbb{R}^d\times\mathbb{R}^d}\left|\nabla_v\cdot\Big(F\,\nabla_v\,(v\cdot\nabla_x u)\Big)\right|^2\,\langle v\rangle^\beta\,\mathrm{d}x\,\mathrm{d}\mu$$
$$= \iint_{\mathbb{R}^d\times\mathbb{R}^d}|\nabla_v F\cdot\nabla_x u|^2\,\langle v\rangle^\beta\,\mathrm{d}x\,\mathrm{d}\mu \le \|\nabla_v F\|^2_{\mathrm{L}^2\left(\langle v\rangle^\beta\mathrm{d}\mu\right)}\,\|\nabla_x u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)}\,.$$

In case $\mathsf{L} = \mathsf{L}_2$, note first that

$$(\mathsf{L}_2^*\mathsf{T}u\,F)(v) = \left(\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,(v'-v)\,F(v')\,\mathrm{d}v'\right)\cdot\nabla_x u\,F(v)\,,$$

and thus, by the Cauchy-Schwarz inequality,

$$\|\mathsf{L}_2^*\mathsf{T}u\,F\|_\beta \le \mathcal{B}\,\|\nabla_x u\|_{\mathrm{L}^2(\,\mathrm{d}x)}\,,\qquad \text{with}\quad \mathcal{B} = \left\|\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,(v'-v)\,F'\,F\,\mathrm{d}v'\right\|_{\mathrm{L}^2\left(\langle v\rangle^\beta\mathrm{d}\mu\right)}\,.$$

For proving finiteness of $\mathcal{B}$ we use (H2) in

$$\left|\int_{\mathbb{R}^d}\mathrm{b}(v',v)\,(v'-v)\,F'\,\mathrm{d}v'\right|$$
$$\le \overline{\mathrm{b}}\int_{|v'-v|<1}|v'-v|^{1-\gamma}F'\,\mathrm{d}v' + \overline{\mathrm{b}}\int_{|v'-v|>1}|v'-v|^{1-\beta}F'\,\mathrm{d}v' \le c\left(1+\langle v\rangle^{1-\beta}\right)\,,$$

which implies

$$\mathcal{B}^2 \le c^2\int_{\mathbb{R}^d}\left(1+\langle v\rangle^{1-\beta}\right)^2\langle v\rangle^\beta\,F\,\mathrm{d}v < \infty\,.$$

Combining these estimates with identity (16), we get

$$|\langle \mathsf{AL}(\mathsf{Id}-\mathsf{\Pi})f, \mathsf{\Pi}f\rangle| \le \mathcal{C}_F\,\|(\mathsf{Id}-\mathsf{\Pi})f\|_{-\beta}\,\langle \mathsf{AT\Pi}f, \mathsf{\Pi}f\rangle^{\frac{1}{2}} \tag{20}$$

where $\mathcal{C}_F = \mathcal{B}/\sqrt{\Theta}$.

• ***Step 5: collecting all estimates.*** Altogether, combining (14) and (18), (19) and (20), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \le -\,\mathcal{C}\,\left\|(\mathsf{Id}-\mathsf{\Pi})f\right\|_{-\beta}^2 - \delta\,\langle \mathsf{AT\Pi}f,\mathsf{\Pi}f\rangle + \delta\,(\mathcal{C}_4+\mathcal{C}_F)\,\left\|(\mathsf{Id}-\mathsf{\Pi})f\right\|_{-\beta}\,\langle \mathsf{AT\Pi}f,\mathsf{\Pi}f\rangle^{\frac12} + \delta\,\mathcal{C}_2\,\left\|(\mathsf{Id}-\mathsf{\Pi})f\right\|_{-\beta}^2$$

which by Young's inequality yields the existence of $\kappa>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \le -\,\kappa\left(\left\|(\mathsf{Id}-\mathsf{\Pi})f\right\|_{-\beta}^2 + \langle \mathsf{AT\Pi}f,\mathsf{\Pi}f\rangle\right)$$

for some $\delta\in(0,1)$. Indeed, with $X:=\left\|(\mathsf{Id}-\mathsf{\Pi})f\right\|_{-\beta}$ and $Y:=\langle \mathsf{AT\Pi}f,\mathsf{\Pi}f\rangle^{\frac12}$, it is enough to check that the quadratic form

$$\mathcal{Q}(X,Y):=(\mathcal{C}-\delta\,\mathcal{C}_2)\,X^2-(\mathcal{C}_4+\mathcal{C}_F)\,X\,Y+\delta\,Y^2$$

is positive, *i.e.*, $\mathcal{Q}(X,Y)\ge\kappa\,(X^2+Y^2)$ for some $\kappa=\kappa(\delta)$ and $\delta\in(0,1)$. □

3. Weighted L^2 estimates

In this section, we show the propagation of weighted norms with weights $\langle v\rangle^k$ of arbitrary positive order $k\in\mathbb{R}^+$.

Proposition 4. *Let $k>0$ and f be solution of (1) with $f^{\mathrm{in}}\in\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$. Then there exists a constant $\mathcal{K}_k>1$ such that*

$$\forall\,t\ge0\qquad \|f(t,\cdot,\cdot)\|_k\le\mathcal{K}_k\,\left\|f^{\mathrm{in}}\right\|_k\,.$$

We recall that $\|f\|_k$ is defined by (8). We shall state a technical lemma (Lemma 5 below) before proving a splitting result in Lemma 6, from which the proof of Proposition 4 easily follows (see Section 3.3).

3.1. A technical lemma.

Lemma 5. *If either $\mathsf{L}=\mathsf{L}_1$ or $\mathsf{L}=\mathsf{L}_2$, then there exists $\ell>0$ for which, for any $k\ge0$, there exist $a_k,b_k,R_k>0$ such that*

$$\left\langle \mathsf{L}f,f\,\langle v\rangle^k\right\rangle\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(a_k\,\mathbb{1}_{|v|<R_k}-b_k\,\langle v\rangle^{-\ell}\right)|f|^2\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\,,\tag{21}$$

for any $f\in\mathcal{D}(\mathbb{R}^d\times\mathbb{R}^d)$.

Proof. In the Fokker-Planck case $\mathsf{L}=\mathsf{L}_1$, with $h:=f/F$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\mathsf{L}_1 f\,f\,\langle v\rangle^k\,\mathrm{d}\mu &= -\int_{\mathbb{R}^d}|\nabla_v h|^2\,\langle v\rangle^k\,F\,\mathrm{d}v-\frac12\int_{\mathbb{R}^d}\nabla_v h^2\cdot\nabla_v\,\langle v\rangle^k\,F\,\mathrm{d}\mu\\ &\le \frac12\int_{\mathbb{R}^d}f^2\left(\frac{\nabla_v F}{F}\cdot\nabla_v\,\langle v\rangle^k+\Delta_v\,\langle v\rangle^k\right)\mathrm{d}\mu\\ &= \frac{k}{2}\int_{\mathbb{R}^d}f^2\,\langle v\rangle^{k-4}\left(2-k+(d+k-2)\,\langle v\rangle^2+\alpha\,\langle v\rangle^\alpha-\alpha\,\langle v\rangle^{\alpha+2}\right)\mathrm{d}\mu\\ &\le \frac{k}{2}\int_{\mathbb{R}^d}f^2\,\langle v\rangle^{k-2}\left(c_k-\alpha\,\langle v\rangle^\alpha\right)\mathrm{d}\mu\\ &= \int_{\mathbb{R}^d}\left(a_k\,\mathbb{1}_{|v|<R_k}-b_k\,\langle v\rangle^{-\ell}\right)|f|^2\,\langle v\rangle^k\,\mathrm{d}\mu\\ &\quad+\frac{k}{2}\int_{\mathbb{R}^d}f^2\,\langle v\rangle^{k-2}\left(c_k\left(1-\mathbb{1}_{|v|<R_k}\,\langle v\rangle^2\right)-\frac{\alpha}{2}\,\langle v\rangle^\alpha\right)\mathrm{d}\mu\,,\end{aligned}$$

with $c_k=|k-2|+|d+k-2|+\alpha$, $a_k=c_k\,k/2$, $b_k=\alpha\,k/4$, $\ell=2-\alpha$. The choice $R_k=(2\,c_k/\alpha)^{1/\alpha}$ makes the last term negative, which completes the proof.

In the case of the scattering operator $\mathsf{L}=\mathsf{L}_2$, with $h:=f/F$, we have

$$\begin{aligned}2\int_{\mathbb{R}^d}f\,\mathsf{L}_2 f\,\langle v\rangle^k\,\mathrm{d}\mu &= 2\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\,(h'-h)\,h\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\ &= \iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\left(2\,h'h-h^2\right)\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\ &\quad-\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v',v)\,h^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\,,\end{aligned}$$

where we have used (H1). Swapping v and v' in the last integral gives

$$\begin{aligned}2\int_{\mathbb{R}^d}f\,\mathsf{L}_2 f\,\langle v\rangle^k\,\mathrm{d}\mu &= -\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\,(h-h')^2\,\langle v\rangle^k\,F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\ &\quad+\iint_{\mathbb{R}^d\times\mathbb{R}^d}\mathrm{b}(v,v')\,(h')^2\left(\langle v\rangle^k-\langle v'\rangle^k\right)F\,F'\,\mathrm{d}v\,\mathrm{d}v'\\ &\le\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}\mathrm{b}(v',v)\left(\langle v'\rangle^k-\langle v\rangle^k\right)F'\,\mathrm{d}v'\right)f^2\,\mathrm{d}\mu\,,\end{aligned}$$

with another swap $v\leftrightarrow v'$ in the last step. Now we use (H2) and its consequence (4):

$$\begin{aligned}\int_{\mathbb{R}^d}\mathrm{b}(v',v)\left(\langle v'\rangle^k-\langle v\rangle^k\right)F'\,\mathrm{d}v' &= \int_{\mathbb{R}^d}\mathrm{b}(v',v)\,\langle v'\rangle^k\,F'\,\mathrm{d}v'-\langle v\rangle^k\,\nu_2(v)\\ &\le 2\,a_k\,\langle v\rangle^{-\beta}-\underline{\nu}\,\langle v\rangle^{k-\beta}\,,\end{aligned}$$

where the estimation of the first term is analogous to the derivation of (4). This implies

$$\begin{aligned}\int_{\mathbb{R}^d}\mathsf{L}_2 f\,f\,\langle v\rangle^k\,\mathrm{d}\mu &\le \int_{\mathbb{R}^d}\left(a_k\,\mathbb{1}_{|v|<R_k}-b_k\,\langle v\rangle^{-\ell}\right)|f|^2\,\langle v\rangle^k\,\mathrm{d}\mu\\ &\quad+\int_{\mathbb{R}^d}f^2\,\langle v\rangle^k\left(a_k\left(\langle v\rangle^{-\beta-k}-\mathbb{1}_{|v|<R_k}\right)+b_k\,\langle v\rangle^{-\ell}-\frac{\underline{\nu}}{2}\,\langle v\rangle^{-\beta}\right)\mathrm{d}v\,.\end{aligned}$$

The last term is made negative by the choices $\ell=\beta$, $b_k=\underline{\nu}/4$, $R_k=(4\,a_k/\underline{\nu})^{1/k}$. $\square$

3.2. **A splitting result.** As in [12, 17, 23], we write $\mathsf{L}-\mathsf{T}$ as a dissipative part C and a bounded part B such that $\mathsf{L}-\mathsf{T}=\mathsf{B}+\mathsf{C}$.

Lemma 6. *With the notation of Lemma 5, let $k>0$, $k_1>k+2\,\ell$, $a=\max\{a_k,a_{k_1}\}$, $R=\max\{R_k,R_{k_1}\}$, $\mathsf{C}=a\,\mathbb{1}_{|v|<R}$ and $\mathsf{B}=\mathsf{L}-\mathsf{T}-\mathsf{C}$. For any $t\ge0$ we have:*

(i) $\|\mathsf{C}\|_{\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}\mu)\to\mathrm{L}^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le a\,\langle R\rangle^{k_1/2}$,

(ii) $\|e^{t\mathsf{B}}\|_{\mathrm{L}^2\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^2\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le 1$,

(iii) $\|e^{t\mathsf{B}}\|_{\mathrm{L}^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^2\left(\langle v\rangle^{k}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le C\,(1+t)^{-\frac{k_1-k}{2\,\ell}}$ *for some $C>0$.*

Proof. Property (i) is an obvious consequence of the definition of C. Lemma 5 and $\int_{\mathbb{R}^d} f\,\mathsf{T}f\,\mathrm{d}x=0$ imply

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,\mathsf{B}f\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\le\iint_{\mathbb{R}^d\times\mathbb{R}^d}\left(a_k\,\mathbb{1}_{|v|<R_k}-a\,\mathbb{1}_{|v|<R}-b_k\,\langle v\rangle^{-\ell}\right)|f|^2\,\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu$$
$$\le -\,b_k\,\|f\|_{k-\ell}^2\,, \tag{22}$$

which proves (ii) and, analogously,

$$\|e^{t\mathsf{B}}\|_{\mathrm{L}^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^2\left(\langle v\rangle^{k_1}\,\mathrm{d}x\,\mathrm{d}\mu\right)}\le 1\,. \tag{23}$$

Now we consider $f=e^{t\mathsf{B}}\,f^{\mathrm{in}}$ in (22) and use Hölder's inequality

$$\|f\|_k^2\le\|f\|_{k-\ell}^{\frac{2\,(k_1-k)}{k_1-k+\ell}}\;\|f\|_{k_1}^{\frac{2\,\ell}{k_1-k+\ell}}$$

as well as (23):

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|f\|_k^2\le-\,b_k\,\|f\|_k^{2\left(1+\frac{\ell}{k_1-k}\right)}\,\left\|f^{\mathrm{in}}\right\|_{k_1}^{-\frac{2\,\ell}{k_1-k}}\,.$$

The Bihari-LaSalle inequality [4, 18], a nonlinear version of Grönwall's lemma, implies

$$\|f\|_k^2\le\left(\left\|f^{\mathrm{in}}\right\|_k^{-\frac{2\,\ell}{k_1-k}}+\tfrac{2\,\ell\,b_k\,t}{k_1-k}\left\|f^{\mathrm{in}}\right\|_{k_1}^{-\frac{2\,\ell}{k_1-k}}\right)^{-\frac{k_1-k}{\ell}}\le\left(\tfrac{k_1-k}{k_1-k+2\,\ell\,b_k\,t}\right)^{\frac{k_1-k}{\ell}}\left\|f^{\mathrm{in}}\right\|_{k_1}^2\,,$$

thus completing the proof. □

3.3. **Proof of Proposition 4.** With the notation of Lemma 6, *i.e.*, $\mathsf{B}=\mathsf{L}-\mathsf{T}-\mathsf{C}$, integration of the identity

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{(t-s)\mathsf{B}}\,e^{s(\mathsf{L}-\mathsf{T})}\right)=e^{(t-s)\mathsf{B}}\,(\mathsf{L}-\mathsf{T}-\mathsf{B})\,e^{s(\mathsf{L}-\mathsf{T})}=e^{(t-s)\mathsf{B}}\,\mathsf{C}e^{s(\mathsf{L}-\mathsf{T})}\,,$$

gives

$$e^{t(\mathsf{L}-\mathsf{T})}=e^{t\mathsf{B}}+\int_0^t e^{(t-s)\mathsf{B}}\,\mathsf{C}\,e^{s(\mathsf{L}-\mathsf{T})}ds\,.$$

The entropy dissipation inequality (5) implies

$$\|e^{t(\mathsf{L}-\mathsf{T})}\|_{\mathrm{L}^2(\,\mathrm{d}x\,\mathrm{d}\mu)\to\mathrm{L}^2(\,\mathrm{d}x\,\mathrm{d}\mu)}\le 1\,,$$

and therefore, since $k > 0$,

$$\|e^{t(\mathsf{L}-\mathsf{T})}\|_{\mathrm{L}^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^2(\,\mathrm{d}x\,\mathrm{d}\mu)} \le 1\,.$$

Combining this with the results of Lemma 6 leads to

$$\left\|e^{t(\mathsf{L}-\mathsf{T})}\right\|_{\mathrm{L}^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)\to\mathrm{L}^2\left(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu\right)} \le 1 + a\,\langle R\rangle^{k_1/2}\,C\int_0^t (1+s)^{-\frac{k_1-k}{2\,\ell}}\,\mathrm{d}s\,,$$

which completes the proof, since the right hand side is bounded uniformly in t by $k_1 > k + 2\,\ell$. □

4. Proof of Theorem 1

The control of the macroscopic part Πf by $\langle \mathsf{AT}\Pi f, \Pi f\rangle$ is achieved as in [5]. We sketch a proof for the sake of completeness.

Lemma 7. *Under the assumptions of Theorem 1, for any* $f \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu)\cap\mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)$*, we have*

$$\langle \mathsf{AT}\Pi f, \Pi f\rangle \ge \Phi\left(\|\Pi f\|^2\right)$$

with

$$\Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}\quad,\quad \mathsf{c} = \Theta\,\mathcal{C}_{\mathrm{Nash}}^{-\frac{d+2}{d}}\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}\,,$$

where $\mathcal{C}_{\mathrm{Nash}}$ *is the constant in Nash's inequality* (11) *and* Θ *is defined in* (15).

Proof. Equations (15), (16), Nash's inequality (11), and $\|\Pi f\| = \|\rho_f\|_{\mathrm{L}^2(\mathrm{d}x)}$ imply

$$\begin{aligned}\|\Pi f\|^2 &= \|u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)} + 2\,\Theta\,\|\nabla u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)} + \Theta^2\,\|\Delta u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)} \le \|u\|^2_{\mathrm{L}^2(\,\mathrm{d}x)} + 2\,\langle \mathsf{AT}\Pi f, f\rangle\\ &\le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(\,\mathrm{d}x)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(\,\mathrm{d}x)} + 2\,\langle \mathsf{AT}\Pi f, f\rangle\,.\end{aligned}$$

Again from (15) and (16), we deduce

$$\|u\|_{\mathrm{L}^1(\mathrm{d}x)} = \|\rho_f\|_{\mathrm{L}^1(\mathrm{d}x)} = \|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}\,,\qquad \|\nabla u\|^2_{\mathrm{L}^2(\mathrm{d}x)} \le \frac{1}{\Theta}\,\langle \mathsf{AT}\Pi f, f\rangle\,,$$

which completes the proof. □

The control of $(\mathsf{Id}-\Pi)f$ by the entropy production term relies on a simple, new estimate.

Lemma 8. *Under the assumptions of Theorem 1, for any solution* f *of* (1) *with initial datum* $f^{\mathrm{in}} \in \mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)\cap\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$*, we have*

$$\|(\mathsf{Id}-\Pi)f(t,\cdot,\cdot)\|^2_{-\beta} \ge \Psi\left(\|(\mathsf{Id}-\Pi)f(t,\cdot,\cdot)\|^2\right)$$

for any $t \ge 0$*, where* $\mathcal{K}_k$ *is as in Proposition 4 and*

$$\Psi(y) := C_0\,y^{1+\beta/k}\,,\quad C_0 := \left(\mathcal{K}_k\left(1+\Theta_k\right)\|f^{\mathrm{in}}\|_k\right)^{-\frac{2\,\beta}{k}}\,.$$

Proof. Hölder's inequality

$$\|(\mathsf{Id}-\Pi)f\| \le \|(\mathsf{Id}-\Pi)f\|_{-\beta}^{\frac{k}{k+\beta}}\;\|(\mathsf{Id}-\Pi)f\|_{k}^{\frac{\beta}{k+\beta}}$$

and

$$\|(\mathsf{Id}-\Pi)f\|_k \le \|f\|_k + \Theta_k\,\|\rho\|_{\mathrm{L}^2(\,\mathrm{d}x)} \le (1+\Theta_k)\,\|f\|_k \le \mathcal{K}_k\,(1+\Theta_k)\,\left\|f^{\rm in}\right\|_k\,,$$

where the last inequality holds by Proposition 4, provide us with the estimate. □

Proof of Theorem 1. Using the estimates of Lemma 7 and Lemma 8, we obtain

$$\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2 + \langle \mathsf{AT}\Pi f,\Pi f\rangle \ge \Psi\left(\|(\mathsf{Id}-\Pi)f\|^2\right) + \Phi\left(\|\Pi f\|^2\right)\,.$$

Using (13) and the fact that $\mathsf{D}[f]\ge 0$ by Proposition 3, we know that

$$\|(\mathsf{Id}-\Pi)f\|^2 \le z_0 \quad\text{and}\quad \|\Pi f\|^2 \le z_0 \quad\text{where}\quad z_0 := \|f^{\rm in}\|^2\,.$$

Thus, from

$$\Phi^{-1}(y) = 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}} \le \left(C_1^{-1}\,y\right)^{\frac{d}{d+2}} \quad\text{with}\quad C_1 := \left(2\,\Phi(z_0)^{\frac{2}{d+2}} + \mathsf{c}^{-\frac{d}{d+2}}\right)^{-\frac{d+2}{d}}\,,$$

as long as $y \le \Phi(z_0)$, we obtain

$$\Phi\left(\|\Pi f\|^2\right) \ge C_1\,\|\Pi f\|^{2\,\frac{d+2}{d}}\,,$$

since $\|\Pi f\|^2 \le z_0$. As a consequence,

$$\begin{aligned}\|(\mathsf{Id}-\Pi)f\|_{-\beta}^2 + \langle \mathsf{AT}\Pi f,\Pi f\rangle &\ge C_0\,\|(\mathsf{Id}-\Pi)f\|^{2\,\frac{k+\beta}{k}} + C_1\,\|\Pi f\|^{2\,\frac{d+2}{d}}\\ &\ge \min\left\{C_0\,z_0^{\frac{\beta}{k}-\frac{1}{\zeta}},\,C_1\,z_0^{\frac{2}{d}-\frac{1}{\zeta}}\right\}\|f\|^{2+\frac{2}{\zeta}}\end{aligned}$$

where $1/\zeta = \max\{2/d,\beta/k\}$, *i.e.*, $\zeta = \min\{d/2,k/\beta\}$. Collecting terms, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{H}[f] \le -\,C\,\zeta\,\mathsf{H}[f]^{1+\frac{1}{\zeta}}$$

using (12), (13) and Proposition 3, with

$$C := \frac{\kappa}{\zeta}\,\min\left\{C_0\,z_0^{\frac{\beta}{k}-\frac{1}{\zeta}},\,C_1\,z_0^{\frac{2}{d}-\frac{1}{\zeta}}\right\}\left(\tfrac{2}{1+\delta}\right)^{1+\frac{1}{\zeta}}\,.$$

Then the result of Theorem 1 follows from the Bihari-LaSalle estimate

$$\mathsf{H}[f(t,\cdot,\cdot)] \le \mathsf{H}[f^{\rm in}]\left(1 + C\,\mathsf{H}[f^{\rm in}]^{\frac{1}{\zeta}}\,t\right)^{-\zeta}\,.$$

The expression of C can be explicitly computed in terms of $C_0\,z_0^{\frac{\beta}{k}-\frac{1}{\zeta}}\,\mathsf{H}[f^{\rm in}]^{\frac{1}{\zeta}}$, which is proportional to $(\|f^{\rm in}\|\,/\,\|f^{\rm in}\|_k)^{\frac{2\beta}{k}}$, and in terms of $C_1\,z_0^{\frac{2}{d}-\frac{1}{\zeta}}\,\mathsf{H}[f^{\rm in}]^{\frac{1}{\zeta}}$ which is a function of $(\|f^{\rm in}\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}\,/\,\|f^{\rm in}\|)^{4/(d+2)}$. To see this, one has to take into account the expressions of C_0, C_1 and c in terms of the initial datum $f^{\rm in}$. □

As a concluding remark, we emphasize that a control of the solution in the space $\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}x\,\mathrm{d}\mu)$, based on Proposition 4, is enough to prove Theorem 1. In particular, there is no need of a uniform bound on f. This observation is new in L^2 hypocoercive methods, and consistent with the homogeneous case (see Appendix B).

Appendix A. Weighted Poincaré inequalities

The goal of this appendix is to provide a proof of (10). Inequality (10) is not a standard weighted Poincaré inequality because the average in the right-hand side of the inequality involves the measure of the left-hand side, so that the right-hand side cannot be interpreted as a variance. Section A.1 is devoted to a reformulation of a spectral gap issue associated with Poincaré inequalities with weights into spectral considerations for a Schrödinger operator. We establish a criterion for Poincaré inequalities which is well adapted to the weights in (10). The average, however, corresponds to a standard variance. In Section A.2, we establish the result for the average which appears in (10).

A.1. **Continuous spectrum and weighted Poincaré inequalities.** Let us consider two probability measures on $\mathbb{R}^d$

$$\mathrm{d}\xi = e^{-\phi}\,\mathrm{d}v \quad \text{and} \quad \mathrm{d}\nu = \psi\,\mathrm{d}\xi\,,$$

where ϕ and $\psi > 0$ are two measurable functions, and the *weighted Poincaré inequality*

$$\forall\,h \in \mathcal{D}(\mathbb{R}^d)\,, \quad \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\xi \ge \mathcal{C}_\star \int_{\mathbb{R}^d} \left|h - \widehat{h}\right|^2 \mathrm{d}\nu \tag{24}$$

where $\widehat{h} = \int_{\mathbb{R}^d} h\,\mathrm{d}\nu$. The question we address here is: *on which conditions on ϕ and ψ do we know that* (24) *holds for some constant $\mathcal{C}_\star > 0$?* Our key example is

$$\phi(v) = \langle v\rangle^\alpha + \log Z_\alpha \quad \text{and} \quad \psi(v) = c_{\alpha,\beta}^{-1}\,\langle v\rangle^{-\beta} \tag{25}$$

with $\alpha > 0$, $\beta > 0$, $Z_\alpha = \int_{\mathbb{R}^d} e^{-\langle v\rangle^\alpha}\,\mathrm{d}v$ and $c_{\alpha,\beta} = Z_\alpha^{-1} \int_{\mathbb{R}^d} \langle v\rangle^{-\beta}\,e^{-\langle v\rangle^\alpha}\,\mathrm{d}v$.

Here we use a spectral property of Schrödinger type operators, which goes as follows. Let us consider a measurable function Φ on $\mathbb{R}^d$ such that

$$\sigma = \lim_{r\to+\infty} \inf_{w\in\mathcal{D}(B_r^c)\setminus\{0\}} \frac{\int_{\mathbb{R}^d} \left(|\nabla w|^2 + \Phi\,|w|^2\right)\mathrm{d}v}{\int_{\mathbb{R}^d} |w|^2\,\mathrm{d}v} > 0\,,$$

where $B_r^c := \left\{v \in \mathbb{R}^d\,:\,|v| > r\right\}$ and $\mathcal{D}(B_r^c)$ denotes the space of smooth functions on $\mathbb{R}^d$ with compact support in B_r^c. According to Persson's result [26, Theorem 2.1], the lower end $\sigma_\star$ of the continuous spectrum of the Schrödinger operator $-\,\Delta + \Phi$ is such that

$$\sigma_\star \ge \sigma \ge \lim_{r\to+\infty} \operatorname*{ess\,inf}_{v\in B_r^c} \Phi(v)\,.$$

If we replace $\int_{\mathbb{R}^d} |w|^2\,\mathrm{d}v$ by the weighted integral $\int_{\mathbb{R}^d} |w|^2\,\psi\,\mathrm{d}v$ for some measurable function ψ, we have the modified result that the operator $\mathcal{L} = \psi^{-1}\,(-\,\Delta + \Phi)$ on

$\mathrm{L}^2(\mathbb{R}^d, \psi\, \mathrm{d}v)$, associated with the quadratic form

$$w \mapsto \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \Phi\, |w|^2\right) \mathrm{d}v$$

has only discrete eigenvalues in the interval $(-\infty, \sigma)$ where

$$\sigma = \lim_{r\to+\infty} \inf_{w\in\mathcal{D}(B_r^c)\setminus\{0\}} \frac{\int_{\mathbb{R}^d} \left(|\nabla w|^2 + \Phi\, |w|^2\right) \mathrm{d}v}{\int_{\mathbb{R}^d} |w|^2\, \psi\, \mathrm{d}v} > 0\,.$$

To prove it, it is enough to observe that 0 is the lower end of the continuous spectrum of $\mathcal{L} - \sigma_\star\, \psi$, where $\sigma_\star$ is again defined as the lower end of the continuous spectrum of $\mathcal{L}$, and to apply [26, Theorem 2.1]. It is also straightforward to check that $\sigma_\star$ is such that

$$\sigma_\star \ge \sigma \ge \lim_{r\to+\infty} \mathsf{q}(r) =: \sigma_0 \qquad \text{where} \quad \mathsf{q}(r) := \operatorname*{ess\,inf}_{B_r^c} \frac{\Phi}{\psi}\,. \tag{26}$$

Note that σ_0 is either finite or infinite.

Relating the weighted Poincaré inequality (24) with the spectrum of $\mathcal{L}$ is then classical. With

$$\Phi = \tfrac{1}{4}\, |\nabla\phi|^2 - \tfrac{1}{2}\, \Delta\phi\,, \tag{27}$$

the spectral gap of $\mathcal{L}$ is equal to the optimal constant in the Poincaré inequality. Indeed, let $h = w\, e^{\phi/2}$ and observe that

$$\int_{\mathbb{R}^d} |\nabla h|^2\, \mathrm{d}\xi = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \Phi\, |w|^2\right) \mathrm{d}v\,,$$
$$\int_{\mathbb{R}^d} \left|h - \widehat{h}\right|^2 \mathrm{d}\nu = \int_{\mathbb{R}^d} |w - \widetilde{w}|^2\, \psi\, \mathrm{d}v\,,$$

where $\widetilde{w} = e^{-\phi/2} \int_{\mathbb{R}^d} w\, \psi\, e^{-\phi/2}\, \mathrm{d}v$ is the orthogonal projection of w, with respect to $\mathrm{L}^2(\psi\, \mathrm{d}v)$, onto the kernel of $\mathcal{L}$. The kernel of $\mathcal{L}$ is generated by the constant functions. With $\sigma_\star > 0$, we know that the interval $(0, \sigma_\star)$ contains only eigenvalues, with finite dimensional eigenspaces, which may eventually accumulate, but only with $\sigma_\star$ as the adherence value. As a consequence, there is a lowest positive eigenvalue of $\mathcal{L}$, which is positive and determines the spectral gap.

Proposition 9. *Let ϕ and $\psi > 0$ be two measurable functions. Let Φ and σ_0 be defined respectively by (27) and (26) and assume that σ_0 is nonnegative. Then inequality (24) holds for some positive, finite, optimal constant $\mathcal{C}_\star \ge \sigma_0$ if σ_0 is positive. Otherwise, if $\sigma_0 = 0$, then inequality (24) does not hold.*

Proof. By construction, σ is nonnegative and the infimum of the Rayleigh quotient

$$w \mapsto \frac{\int_{\mathbb{R}^d} \left(|\nabla w|^2 + \Phi\, |w|^2\right) \mathrm{d}v}{\int_{\mathbb{R}^d} |w|^2\, \psi\, \mathrm{d}v}$$

is achieved by $h \equiv \widehat{h} = 1$, that is, by $w = \widetilde{w} = e^{-\phi/2}$, which moreover generates the kernel of $\mathcal{L}$. Hence we can interpret $\mathcal{C}_\star$ as the first positive eigenvalue, if there is any in the interval $(0, \sigma_\star)$, or $\mathcal{C}_\star = \sigma_\star$ if there is none. Notice that in the latter case $\sigma_\star$

is finite. If $\sigma_0 = 0$, it is easy to construct a sequence of functions which shows that inequality (24) holds only if $\mathcal{C}_\star = 0$. □

In the case of (25), we have $\frac{\Phi}{\psi} \sim \frac{1}{4}\,\alpha^2\,c_{\alpha,\beta}\,\langle v\rangle^{2\,(\alpha-1)+\beta}$ as $|v| \to \infty$. Thus the condition $\beta \ge 2\,(1-\alpha)$ is necessary and sufficient for the inequality (24) to hold. The threshold case $\beta = 2\,(1-\alpha)$ is remarkable: inequality (24) can be rewritten for any $\alpha \in (0,1)$ as the following *weighted Poincaré inequality* :

$$\forall\,h \in \mathcal{D}(\mathbb{R}^d)\,,\quad \int_{\mathbb{R}^d} |\nabla h|^2\,e^{-\langle v\rangle^\alpha}\,\mathrm{d}v \ge \mathcal{C}_\star \int_{\mathbb{R}^d} \frac{\big|h-\widehat{h}\big|^2\,e^{-\langle v\rangle^\alpha}}{\left(1+|v|^2\right)^{1-\alpha}}\,\mathrm{d}v\,, \tag{28}$$

for some constant $\mathcal{C}_\star \in (0,\sigma_0)$. The above computation shows that $\sigma_0 = \alpha^2/4$ and

$$\widehat{h} := \frac{1}{z_\alpha}\int_{\mathbb{R}^d} \frac{h\,e^{-\langle v\rangle^\alpha}}{\left(1+|v|^2\right)^{1-\alpha}}\,\mathrm{d}v\,,\quad z_\alpha = \int_{\mathbb{R}^d} \frac{e^{-\langle v\rangle^\alpha}}{\left(1+|v|^2\right)^{1-\alpha}}\,\mathrm{d}v\,.$$

Notice that (28) differs from (10), as the average in the right-hand side is not taken with respect to the same measure in both inequalities. The purpose of the next subsection is to deduce (10) from (28).

A.2. **A weighted Poincaré inequality with a non-classical average.**

Corollary 10. *Let the assumptions of Proposition 9 hold with $\sigma_0 > 0$ and let, additionally, ψ be bounded, $\psi^{-1} \in \mathrm{L}^1(\mathrm{d}\xi)$ and such that $\lim_{R\to+\infty} R^2 \inf_{B_{2R}} \psi = +\infty$. Then the inequality*

$$\forall\,h \in \mathcal{D}(\mathbb{R}^d)\,,\quad \int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\xi \ge \mathcal{C}\int_{\mathbb{R}^d} \big|h-\widetilde{h}\big|^2\,\mathrm{d}\nu \tag{29}$$

holds for some optimal constant $\mathcal{C} \in (0,\mathcal{C}_\star]$, where $\widetilde{h} := \int_{\mathbb{R}^d} h\,\mathrm{d}\xi$. Here $\mathcal{C}_\star$ denotes the optimal constant in (24).

Notice that (29) is similar to (24), except that the average is computed with respect to the measure of the left-hand side. We emphasize that in (29), the right-hand side is not the *variance* of h with respect to the measure $\mathrm{d}\nu$, as we subtract the average with respect to the measure $\mathrm{d}\xi$. In the case $\phi(v) = \langle v\rangle^\alpha + \log Z_\alpha$, which corresponds to (25), Inequality (29) is precisely (10). Inequality (10) has been established in [17, inequality (1.12)] by a different method, based on the strategy of [1, 2]. Also see Appendix B.1 for further details. As we shall see in the proof, our method provides an explicit lower bound $\mathcal{C}$ in terms of $\mathcal{C}_\star$.

Proof. Without loss of generality, we can assume that $\widetilde{h} = \int_{\mathbb{R}^d} h\,\mathrm{d}\xi = 0$ up to the replacement of h by $h - \widetilde{h}$. We use the IMS decomposition method (see [24, 28]), which goes as follows. Let χ be a truncation function on $\mathbb{R}_+$ with the following properties: $0 \le \chi \le 1$, $\chi \equiv 1$ on $[0,1]$, $\chi \equiv 0$ on $[2,+\infty)$ and $\chi'^2/\left(1-\chi^2\right) \le \kappa$ for some $\kappa > 0$. Next, we define $\chi_R(v) = \chi\big(|v|/R\big)$, $h_{1,R} = h\,\chi_R$ and $h_{2,R} = h\,\sqrt{1-\chi_R^2}$, so that $h_{1,R}$ is supported in the ball B_{2R} of radius $2R$ centered at $v=0$ and $h_{2,R}$ is supported in $B_R^c=\mathbb{R}^d\setminus B_R$. Elementary computations show that $h^2=h_{1,R}^2+h_{2,R}^2$ and $|\nabla h|^2=|\nabla h_{1,R}|^2+|\nabla h_{2,R}|^2-h^2\,|\nabla\chi|^2/\left(1-\chi^2\right)$, so that $\Big||\nabla h|^2-|\nabla h_{1,R}|^2-|\nabla h_{2,R}|^2\Big|\le \kappa\,h^2/R^2$.

Since $h_{2,R}$ is supported in B_R^c, we know that

$$\int_{\mathbb{R}^d}|\nabla h_{2,R}|^2\,\mathrm{d}\xi\ge\mathsf{q}(R)\int_{\mathbb{R}^d}|h_{2,R}|^2\,\mathrm{d}\nu$$

for any $R>0$, where q is the quotient involved in the definition (26) of σ_0. We recall that $\lim_{r\to+\infty}\mathsf{q}(r)=\sigma_0>0$. Using the method of the Holley-Stroock lemma (see [15] and [9] for a recent presentation), we deduce from inequality (24) that

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla h_{1,R}|^2\,\mathrm{d}\xi&\ge\mathcal{C}_\star\int_{\mathbb{R}^d}\left|h_{1,R}-\widehat{h}_{1,R}\right|^2\mathrm{d}\nu\\&\ge\mathcal{C}_\star\int_{B_{2R}}\left|h_{1,R}-\widehat{h}_{1,R}\right|^2\,\psi\,\mathrm{d}\xi\\&\ge\mathcal{C}_\star\inf_{B_{2R}}\psi\,\min_{c\in\mathbb{R}}\int_{B_{2R}}|h_{1,R}-c|^2\,\mathrm{d}\xi\\&\ge\mathsf{Q}(R)\int_{\mathbb{R}^d}|h_{1,R}|^2\,\mathrm{d}\nu-\mathcal{C}_\star\frac{\inf_{B_{2R}}\psi}{\xi(B_{2R})}\left(\int_{\mathbb{R}^d}h_{1,R}\,\mathrm{d}\xi\right)^2\end{aligned}$$

where $\mathsf{Q}(R):=\mathcal{C}_\star\inf_{B_{2R}}\psi/\sup_{B_{2R}}\psi$. By the assumption $\widetilde{h}=0$, we know that

$$\int_{B_R}h\,\mathrm{d}\xi=-\int_{B_R^c}h\,\mathrm{d}\xi\,,$$

from which we deduce that

$$\left(\int_{\mathbb{R}^d}h_{1,R}\,\mathrm{d}\xi\right)^2=\left(\int_{B_R}h\,\mathrm{d}\xi+\int_{B_R^c}\chi_R\,h\,\mathrm{d}\xi\right)^2\le\left(\int_{B_R^c}|h|\,\mathrm{d}\xi\right)^2\le\int_{\mathbb{R}^d}h^2\,\mathrm{d}\nu\int_{B_R^c}\psi^{-1}\,\mathrm{d}\xi$$

where the last inequality is simply a Cauchy-Schwarz inequality. Let

$$\varepsilon(R):=\mathcal{C}_\star\frac{\inf_{B_{2R}}\psi}{\xi(B_{2R})}\int_{B_R^c}\psi^{-1}\,\mathrm{d}\xi\,.$$

By the assumption that $\psi^{-1}\in\mathrm{L}^1(\mathbb{R}^d,\mathrm{d}\xi)$, we know that

$$\lim_{R\to+\infty}\varepsilon(R)=0\quad\text{and}\quad\lim_{R\to+\infty}\frac{\varepsilon(R)}{\mathsf{Q}(R)}=0\,.$$

Collecting all our assumptions, we have

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla h|^2\,\mathrm{d}\xi&\ge\int_{\mathbb{R}^d}\left(|\nabla h_{1,R}|^2+|\nabla h_{2,R}|^2-\frac{\kappa}{R^2}\,h^2\right)\mathrm{d}\xi\\&\ge\left(\min\Big\{\mathsf{Q}(R),\mathsf{q}(R)\Big\}-\varepsilon(R)-\frac{\kappa}{R^2}\right)\int_{\mathbb{R}^d}|h|^2\,\mathrm{d}\nu\end{aligned}$$

where $\min\Big\{\mathsf{Q}(R),\mathsf{q}(R)\Big\}-\varepsilon(R)-\kappa/R^2$ is positive for $R>0$, large enough, as follows from the assumptions on ψ.

Finally, let us notice that, for any $c \in \mathbb{R}$, we have

$$\int_{\mathbb{R}^d} |h - c|^2 \, d\nu = \int_{\mathbb{R}^d} h^2 \, d\nu - 2\, c \int_{\mathbb{R}^d} h \, d\nu + c^2 \geq \int_{\mathbb{R}^d} \left| h - \widehat{h} \right|^2 d\nu$$

with equality if and only if $c = \widehat{h} = \int_{\mathbb{R}^d} h \, d\nu$. As a special case corresponding to $c = \widetilde{h} = \int_{\mathbb{R}^d} h \, d\xi$, we have

$$\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 d\nu \geq \int_{\mathbb{R}^d} \left| h - \widehat{h} \right|^2 d\nu \, .$$

This proves that $\mathcal{C}_\star \geq \mathcal{C}$. □

In the special case of (25), the assumptions of Corollary 10 are not difficult to check. It is also possible to give a slightly shorter proof using the Poincaré inequality on B_R when (25) holds: see [22, Chapter 6].

Appendix B. Algebraic decay rates for the Fokker-Planck equation

Here we consider simple estimates of the decay rates in the spatially homogeneous case of equation (1), that is, $f(t, x, v) = g(t, v)$ solving the Fokker-Planck equation

$$\partial_t g = \mathsf{L}_1 g \, . \tag{30}$$

After summarizing the standard approach based on the *weak Poincaré inequality* (see for instance [17]) in Section B.1, we introduce a new method which relies on *weighted* L^2 *estimates*. As already mentioned, the advantage of weighted Poincaré inequalities is that the description of the convergence rates to the local equilibrium does not require extra regularity assumptions to cover the transition from super-exponential ($\alpha > 1$) and exponential ($\alpha = 1$) local equilibria to sub-exponential local equilibria, with $\alpha \in (0, 1)$.

B.1. **Weak Poincaré inequality.** We assume $\alpha \in (0, 1)$ and $\eta \in \big(0, \beta\big)$ with $\beta = 2\,(1 - \alpha)$. By a simple Hölder inequality, with $(\tau + 1)/\tau = \beta/\eta$, we obtain

$$\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 d\xi = \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\eta} \, \langle v \rangle^{\eta} \, d\xi$$
$$\leq \left(\int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, d\xi \right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \left\| h - \widetilde{h} \right\|^2_{\mathrm{L}^\infty(\mathbb{R}^d)} \langle v \rangle^{\beta\,\tau} \, d\xi \right)^{\frac{1}{1+\tau}} \, .$$

We assume that $d\xi = Z_\alpha^{-1} e^{-\langle v \rangle^\alpha} dv$ as in (25) and take $\widetilde{h} := \int_{\mathbb{R}^d} h \, d\xi$. Using (10), we end up with

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 d\xi \leq \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2 \, d\xi \right)^{\frac{\tau}{1+\tau}} \left\| h - \widetilde{h} \right\|^{\frac{2}{1+\tau}}_{\mathrm{L}^\infty(\mathbb{R}^d)} \, , \tag{31}$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$. We learn from (6) that

$$\frac{d}{dt} \int_{\mathbb{R}^d} \left| h(t, \cdot) - \widetilde{h} \right|^2 d\xi = -\, 2 \int_{\mathbb{R}^d} |\nabla_v h|^2 \, d\xi$$

if $g = h\,F$ solves (30), and we also know that $\widetilde{h}$ does not depend on t. By a strategy that goes back at least to [20, Theorem 2.2] and, according to the author of [20], due to D. Stroock, we obtain

$$\int_{\mathbb{R}^d} \left|h(t,\cdot)-\widetilde{h}\right|^2 \mathrm{d}\xi \le \left(\left(\int_{\mathbb{R}^d} \left|h(0,\cdot)-\widetilde{h}\right|^2 \mathrm{d}\xi\right)^{-\frac{1}{\tau}} + \frac{2\,\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau}\,\mathcal{M}}\,t\right)^{-\tau}$$

with $\mathcal{M} = \sup_{s\in(0,t)} \|h(s,\cdot)-\widetilde{h}\|^{2/\tau}_{\mathrm{L}^\infty(\mathbb{R}^d)}$, where the Bihari-LaSalle inequality has been employed again. The limitation is of course that we need to restrict the initial conditions in order to have $\mathcal{M}$ uniformly bounded with respect to t. Since η can be chosen arbitrarily close to β, the exponent τ can be taken arbitrarily large but to the price of a constant $\mathcal{C}_{\alpha,\tau}$ which explodes as $\eta \to \beta_-$.

Note that, with the denomination used in [27, (1.6)], Formula (31) is equivalent to the *weak Poincaré inequality*

$$\forall\, h \in \mathcal{D}(\mathbb{R}^d)\,,\quad \mathcal{C}_{\alpha,\tau}^{-1}\int_{\mathbb{R}^d} \left|h-\widetilde{h}\right|^2 \mathrm{d}\xi \le \tfrac{\tau}{(1+\tau)^{1+1/\tau}}\,r^{-1/\tau}\int_{\mathbb{R}^d} |\nabla h|^2\,\mathrm{d}\xi + r\,\left\|h-\widetilde{h}\right\|^2_{\mathrm{L}^\infty(\mathbb{R}^d)}\,,$$

for all $r>0$. The equivalence of this inequality and (31) is easily recovered by optimizing on $r>0$. It is worth to remark that here we consider $\|h-\widetilde{h}\|_{\mathrm{L}^\infty(\mathbb{R}^d)}$ while various other quantities like, *e.g.*, the median can be used in weak Poincaré inequalities.

B.2. **Weighted L^2 estimates.** As an alternative approach to the *weak Poincaré inequality* method of Appendix B.1, we can consider for some arbitrary $k>0$ the evolution according to equation (30) of $\int_{\mathbb{R}^d} |h(t,v)|^2\,\langle v\rangle^k\,\mathrm{d}\xi = \int_{\mathbb{R}^d} |h(t,v)|^2\,\langle v\rangle^k\,F\,\mathrm{d}v$ where $\mathrm{d}\xi$ is as in (25) and $h := g/F$ solves

$$\partial_t h = F^{-1}\,\nabla_v\cdot\left(F\,\nabla_v h\right).$$

Let us compute

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |h(t,v)|^2\,\langle v\rangle^k\,F\,\mathrm{d}v + 2\int_{\mathbb{R}^d} |\nabla_v h|^2\,\langle v\rangle^k\,F\,\mathrm{d}v = -\int_{\mathbb{R}^d} \nabla_v(h^2)\cdot\left(\nabla_v\,\langle v\rangle^k\right)F\,\mathrm{d}v$$

and observe with $\ell = 2-\alpha$ that

$$\begin{aligned}\nabla_v\cdot\left(F\,\nabla_v\,\langle v\rangle^k\right) &= \frac{k}{\langle v\rangle^4}\left(d+(k+d-2)\,|v|^2-\alpha\,\langle v\rangle^\alpha\,|v|^2\right)F\,\langle v\rangle^k\\ &\le \left(a-b\,\langle v\rangle^{-\ell}\right)F\,\langle v\rangle^k\,,\end{aligned}$$

for some $a\in\mathbb{R}$, $b\in(0,+\infty)$. This estimate corresponds to Lemma 5 for the spatially inhomogeneous equation. From here the same proof as in Proposition 4 shows that there exists a constant $\mathcal{K}_k>0$ such that

$$\forall\, t\ge 0\quad \|h(t,\cdot)\|_{\mathrm{L}^2\left(\langle v\rangle^k\,\mathrm{d}\xi\right)} \le \mathcal{K}_k\,\left\|h^{\mathrm{in}}\right\|_{\mathrm{L}^2\left(\langle v\rangle^k\,\mathrm{d}\xi\right)}\,.$$

Hence, if $g = hF$ solves (30) with initial value $h^{\rm in}$, we can use (10) to write

$$\frac{\mathrm d}{\mathrm dt}\int_{\mathbb R^d}\left|h(t,\cdot)-\widetilde h\right|^2\mathrm d\xi = -\,2\int_{\mathbb R^d}|\nabla_v h|^2\,\mathrm d\xi \le -\,2\,\mathcal C\int_{\mathbb R^d}\left|h-\widetilde h\right|^2\,\langle v\rangle^{-\beta}\,\mathrm d\xi$$

with $\beta = 2\,(1-\alpha)$ and $\widetilde h = \fint_{\mathbb R^d} h\,\mathrm d\xi$. With $\theta = k/\big(k+\beta\big)$, Hölder's inequality

$$\int_{\mathbb R^d}\left|h-\widetilde h\right|^2\mathrm d\xi \le \left(\int_{\mathbb R^d}\left|h-\widetilde h\right|^2\,\langle v\rangle^{-\beta}\,\mathrm d\xi\right)^{\theta}\left(\int_{\mathbb R^d}\left|h-\widetilde h\right|^2\,\langle v\rangle^{k}\,\mathrm d\xi\right)^{1-\theta}$$

allows us to estimate the right hand side and obtain the following result.

Proposition 11. *Let $\alpha \in (0,1)$, let $g^{\rm in} \in \mathrm L^1_+(\mathrm d\mu)\cap \mathrm L^2(\langle v\rangle^k\,\mathrm d\mu)$ for some $k > 0$, and consider the solution g to (30) with initial datum $g^{\rm in}$. With $\mathcal C$ as in (10), if $\overline g = \big(\fint_{\mathbb R^d} g\,\mathrm dv\big)\,F$ where F is given by (2), then*

$$\int_{\mathbb R^d}|g(t,\cdot)-\overline g|^2\,\mathrm d\mu \le \left(\left(\int_{\mathbb R^d}\left|g^{\rm in}-\overline g\right|^2\mathrm d\mu\right)^{-\beta/k} + \frac{2\,\beta\,\mathcal C}{k\,\mathcal K^{\beta/k}}\,t\right)^{-k/\beta}$$

with $\beta = 2\,(1-\alpha)$ and $\mathcal K := \mathcal K_k^2\,\|g^{\rm in}\|^2_{\mathrm L^2\left(\langle v\rangle^k\,\mathrm d\mu\right)} + \Theta_k\,\big(\fint_{\mathbb R^d} g^{\rm in}\,\mathrm dv\big)^2$.

We recall that $g = h\,F$, $\overline g = \widetilde h\,F$ and $F\,\mathrm d\mu = \mathrm dv = F^{-1}\,\mathrm d\xi$. We note that arbitrarily large decay rates can be obtained under the condition that $k > 0$ is large enough. We recover that when $k < d\,\beta/2$, the rate of relaxation to the equilibrium is slower than $(1+t)^{-d/2}$ and responsible for the limitation that appears in Theorem 1. However, the rate of the heat flow is recovered in Theorem 1 for a weight of order k with an arbitrarily small but fixed $k > 0$, if α is taken close enough to 1.

Proof. Using

$$\int_{\mathbb R^d}\left|h-\widetilde h\right|^2\,\langle v\rangle^k\,\mathrm d\xi \le \int_{\mathbb R^d}|h|^2\,\langle v\rangle^k\,\mathrm d\xi + \Theta_k\,\widetilde h^2 = \mathcal K\,,$$

we obtain that $y(t) := \fint_{\mathbb R^d}|g(t,\cdot)-\overline g|^2\,\mathrm d\mu$ obeys to $y' \le -\,2\,\mathcal C\,\mathcal K^{1-1/\theta}\,y^{1/\theta}$ and conclude by the Bihari-LaSalle inequality. □

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