Erratum of “On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients”

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Abstract

In “On the continuity of the time derivative of the solution to the parabolic obstacle problem with variable coefficients” our statement on Harnack's inequality is incorrect. This statement was used to establish *a priori* estimates. In this erratum we give a direct proof of these *a priori* estimates.

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Consider a domain D of $\mathbb{R}^2$ and denote by $\mathcal{H}^\alpha(D)$ the set of function $f \in \mathcal{C}^0 \cap L^\infty(D)$ such that

$$\sup_{\substack{(x,t),(y,s)\in D\\(x,t)\neq(y,s)}} \frac{|f(x,t)-f(y,s)|}{\left(\,|x-y|^2+|t-s|\,\right)^{\alpha/2}} < \infty\,,$$

with $\alpha \in (0,1)$, and by $W^{2,1;q}_{x,t}(D)$ the set of functions $u \in L^q(D)$ such that $\frac{\partial u}{\partial x}$, $\frac{\partial^2 u}{\partial x^2}$, and $\frac{\partial u}{\partial t}$ are in $L^q(D)$, with $q \in [1,\infty]$. To any point $P_0 = (x_0,t_0) \in \mathbb{R}^2$ and $R > 0$, we associate the parabolic cylinder $Q_R(P_0) := \{\,(x,t) \in \mathbb{R}^2 \,:\, |x-x_0| < R \text{ and } |t-t_0| < R^2\,\}$.

Let a, b and c and f be given functions. We consider the non-negative solutions in $W^{2,1;1}_{x,t}(Q_R(P_0))$ to

$$Lu(x,t) = f(x,t)\,\mathbb{1}_{\{u>0\}}(x,t) \quad (x,t) \in Q_R(P_0) \text{ a.e.} \tag{1}$$

with $Lu(x,t) := a(x,t)\,\frac{\partial^2 u}{\partial x^2} + b(x,t)\,\frac{\partial u}{\partial x} + c(x,t)\,u - \frac{\partial u}{\partial t}$.

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Theorem 1 *Assume that a, b, c and f belong to $\mathcal{H}^\alpha(Q_R(P_0))$ for some $\alpha \in (0,1)$, and that there exists a constant $\delta_0 > 0$ such that for any $(x,t) \in Q_R(P_0)$, $a(x,t) \geq \delta_0$ and $f(x,t) \geq \delta_0$. Consider a nonnegative solution u of (1). For all $R' < R$, u is bounded in $W^{2,1;\infty}_{x,t}(Q_{R'}(P_0))$.*

The proof of these *a priori* estimates in [1, Theorem 2.1] uses Harnack's inequality and the Schauder interior estimates. However the statement of Harnack's inequality [1, Lemma 2.2] is not correct. In the constant coefficients case, the proof of Theorem 1 can be found in [3], as a consequence of [3, Theorem 4.1]. This method also applies to our case once the following result has been established.

Lemma 2 *Under the assumptions of Theorem 1, if $0 \in Q_R(P_0) \cap \partial\{u = 0\}$, then there exists a positive constant C such that*

$$\sup_{Q_r(0)} u \leq C\, r^2$$

for any $r > 0$ such that $Q_r(0) \subset Q_R(P_0)$.

Proof. The first part of the proof goes as for [3, Lemma 4.2], which was itself adapted from [2].

Up to a scaling, we can assume that $Q_r(0) \subset Q_R(P_0)$ if and only if $r \leq 1$. We introduce $S_k(u) := \sup_{Q_{2^{-k}}(0)} u$ and $N(u) := \{k \in \mathbb{N} \,:\, 2^2 S_{k+1}(u) \geq S_k(u)\}$. Let $M := \sup_{Q_R(P_0)} u$. If there exists $C_0 > 0$ such that $S_{k+1}(u) \leq C_0\, M\, 2^{-2k}$ for any $k \in N(u)$, then we also have $S_{k+1}(u) \leq C_0\, M\, 2^{-2k}$ for any $k \in \mathbb{N}$. The result then holds with $C := 16\, M\, C_0$. Assume therefore that there is no such C_0: for any $j \in \mathbb{N}$, there exists $k_j \in N(u)$ such that

$$S_{k_j+1}(u) \geq j\, 2^{-2k_j}\ . \tag{2}$$

We define $L^j v(x,t) := a(2^{-k_j} x, 2^{-2k_j} t)\, \frac{\partial^2 v}{\partial x^2} + 2^{-k_j}\, b(2^{-k_j} x, 2^{-2k_j} t)\, \frac{\partial v}{\partial x} + 2^{-2k_j}\, c(2^{-k_j} x, 2^{-2k_j} t)\, v - \frac{\partial v}{\partial t}$ and

$$u_j(x,t) := \frac{1}{S_{k_j+1}(u)}\, u(2^{-k_j} x, 2^{-2k_j} t) \quad \text{for all } (x,t) \in Q_1(0)\ .$$

By regularity of a, b, c and f and by (2), the functions u_j satisfy $\lim_{j\to\infty} \sup_{Q_1(0)} |L^j u_j| = 0$, $\sup_{Q_1(0)} u \leq 4$ and $\sup_{Q_{1/2}(0)} u_j = 1$. Moreover u_j is non-negative and $u_j(0,0) = 0$. By L^p parabolic estimates, up to the extraction of a sub-sequence, $(u_j)_{j\in\mathbb{N}}$ converges to a function u_∞ locally uniformly on compact sets.

In [3] the authors use Caffarelli's monotonicity formula to obtain a contradiction. This monotonicity formula is not valid for the variable coefficients case. Here the sign condition on the solution allows to conclude directly by the maximum principle in the following way: the function u_∞ is a bounded non-negative function such that $a(0,0)\frac{\partial^2 u_\infty}{\partial x^2} - \frac{\partial u_\infty}{\partial t} = 0$ in $Q_1(0)$ and achieves its minimum in 0. By the strong maximum principle u_∞ is constant and egal to zero which contradicts $\sup_{Q_{1/2}(0)} u_\infty = 1$. □

Finally, let us mention that in [1, page 375], just before Theorem 1.4, the backward heat kernel has to be defined as $G(x,t) = (4\pi\,(-t))^{-1/2}\, \exp(-x^2/(-4t))$.

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